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ON SOME GENERALIZATION OF ABSOLUTE CESÀRO SUMMABILITY FACTORS

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Abstract. In this paper, we improve the result of Saxsena [3] concerning generalization of absolute Cesàro summablity factors of infinite series.

1. Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n), and let (φ_n) be a sequence of positive real numbers. By (t_n) we denote the n-th $(C,1)$ means of the sequence (na_n). The series $\sum a_n$ is said to be summable $|C,1|_k$, if (see [1])

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty \tag{1}$$

and it is summable $\varphi - |C,1|_k$, $k \geqslant 1$, if (see [4])

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |t_n|^k < \infty. \tag{2}$$

Clearly, $\varphi - |C,1|_k$ summability reduces $|C,1|_k$ for $\varphi = n$.

A positive sequence $\gamma = (\gamma_n)$ is said to be a quasi-f-power increasing sequence, if (see [6]) there exists a constant $K = K(\gamma, f) \geqslant 1$ such that

$$K f_n \gamma_n \geqslant f_m \gamma_m \tag{3}$$

holds for all $n \geqslant m \geqslant 1$. Every non-decreasing sequence is quasi-f-power increasing but the converse is not true.

The following result is due to Mazhar [2]

THEOREM 1.1. *If*

$$\lambda_m = O(1), \quad m \to \infty, \tag{4}$$

$$\sum_{n=1}^{m} n \log n \left|\Delta^2 \lambda_n\right| = O(1), \tag{5}$$

$$\sum_{v=1}^{m} \frac{1}{v} |t_v|^k = O(\log m), \ \textit{as } m \to \infty, \tag{6}$$

then the series $\sum a_n \lambda_n$ *is summable* $|C,1|_k$, $k \geqslant 1$.

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Özarslan [5] in his role generalized the above theorem by giving the following

THEOREM 1.2. *Let* (φ_n) *be a sequence of positive real numbers and conditions (4) and (5) of Theorem 1.1 are satisfied. If*

$$\sum_{v=1}^{m} \frac{\varphi_v^{k-1}}{v^k} |t_v|^k = O(\log m), \text{ as } m \to \infty, \tag{7}$$

$$\sum_{n=v}^{m} \frac{\varphi_n^{k-1}}{n^{k+1}} = O\left(\frac{\varphi_v^{k-1}}{v^k}\right), \tag{8}$$

then the series $\sum a_n \lambda_n$ *is summable* $\varphi - |C,1|_k,\ k \geqslant 1$.

By weakening the conditions, Saxsena [3], presented the following

THEOREM 1.3. *Let* (φ_n) *be a sequence of positive real numbers and condition (4) of Theorem 1.1 and condition (8) of Theorem 1.2 are satisfied. Let* (X_n) *be a positive non-decreasing sequence and* (λ_n) *a sequence such that*

$$|\lambda_n| X_n = O(1), \text{ as } n \to \infty \tag{9}$$

$$\sum_{n=1}^{m} n \left|\Delta^2 \lambda_n\right| X_n = O(1), \quad m \to \infty, \tag{10}$$

$$\sum_{v=1}^{m} \frac{\varphi_v^{k-1}}{v^k} |t_v|^k = O(X_m \mu_m), \text{ as } m \to \infty, \tag{11}$$

where (μ_m) *is a positive non-decreasing such that*

$$n X_n \mu_n \Delta\left(\frac{1}{\mu_n}\right) = O(1), \quad m \to \infty, \tag{12}$$

then the series $\sum a_n \lambda_n / \mu_n$ *is summable* $\varphi - |C,1|_k,\ k \geqslant 1$.

2. Lemmas

LEMMA 2.1. *Let* (X_n) *be a positive non-decreasing sequence, and let* (λ_n) *be a sequence of number satisfying (4), (9) and (10), then these conditions does not imply neither* $\sum_{n=1}^{\infty} \frac{|\lambda_n|}{n} < \infty$, *nor* $\sum_{n=1}^{\infty} |\lambda_n| < \infty$.

Proof. As $\sum_{n=1}^{\infty} |\lambda_n| > \sum_{n=1}^{\infty} \frac{|\lambda_n|}{n}$, it is sufficient to prove the first part. The following counter example give the proof.

Let $\lambda_n = (\log n)^{-1}$, $X_n = (\log n)^{\alpha}$, $0 < \alpha < 1$. Clearly $\Delta^2(\lambda_n) = O\left(\frac{1}{(n \log n)^2}\right)$, and the three conditions of the lemma are satisfied, but $\sum_{n=1}^{\infty} \frac{|\lambda_n|}{n} = \infty$. □

We name the conditions

$$n^{1+\beta}(\log n)^{\gamma} X_n \mu_n \Delta\left(\frac{1}{\mu_n}\right) = O(1), \quad n \to \infty, \tag{13}$$

$$\lambda_n \to 0, \text{ as } n \to \infty, \tag{14}$$

$$\sum_{n=1}^{\infty} n^{\beta+1}(\log n)^{\gamma} X_n \left|\Delta^2 \lambda_n\right| < \infty \tag{15}$$

$$\sum_{n=2}^{m} \frac{\varphi_n^{k-1}|t_n|^k}{n^k\left(n^{\beta}(\log n)^{\gamma} X_n\right)^{k-1}} = O\left(m^{\beta}(\log m)^{\gamma} X_m \mu_m\right), \quad m \to \infty. \tag{16}$$

LEMMA 2.2. *Let (X_n) be a quasi-f-power increasing sequence, $f = (f_n)$, $f_n = n^{\beta}(\log n)^{\gamma}$, $0 < \beta \leqslant 1$, $\gamma \geqslant 0$. Then conditions (14) and (15) imply*

$$m^{\beta+1}(\log m)^{\gamma} X_m |\Delta\lambda_m| = O(1), \quad m \to \infty, \tag{17}$$

$$\sum_{n=1}^{\infty} n^{\beta}(\log n)^{\gamma} X_n |\Delta\lambda_n| = O(1), \tag{18}$$

and

$$n^{\beta}(\log n)^{\gamma} X_n |\lambda_n| = O(1), \quad n \to \infty. \tag{19}$$

Proof. As $\Delta\lambda_n \to 0$, we have

$$\begin{aligned} n^{\beta+1}(\log n)^{\gamma} X_n |\Delta\lambda_n| &= n^{\beta+1}(\log n)^{\gamma} X_n \sum_{v=n}^{\infty} \Delta|\Delta\lambda_v| \\ &= O(1) \sum_{v=n}^{\infty} v^{\beta+1}(\log v)^{\gamma} X_v \left|\Delta^2\lambda_v\right| \\ &= O(1). \end{aligned}$$

This proves (16). To prove (17), we observe that

$$\begin{aligned} \sum_{n=1}^{\infty} n^{\beta}(\log n)^{\gamma} X_n |\Delta\lambda_n| &= \sum_{n=1}^{\infty} n^{\beta}(\log n)^{\gamma} X_n \sum_{v=n}^{\infty} \Delta|\Delta\lambda_v| \\ &\leqslant \sum_{v=1}^{\infty} |\Delta|\Delta\lambda_v|| \sum_{n=1}^{v} n^{\beta}(\log n)^{\gamma} X_n \\ &= O(1) \sum_{v=1}^{\infty} v^{\beta+1}(\log v)^{\gamma} X_v \left|\Delta^2\lambda_v\right| \\ &= O(1). \end{aligned}$$

Finally,

$$\begin{aligned} n^{\beta}(\log n)^{\gamma} X_n |\lambda_n| &= n^{\beta}(\log n)^{\gamma} X_n \sum_{v=n}^{\infty} \Delta|\lambda_v| \\ &\leqslant \sum_{v=n}^{\infty} v^{\beta}(\log v)^{\gamma} X_v |\Delta\lambda_v| \\ &= O(1), \text{ by } (17). \quad \square \end{aligned}$$

LEMMA 2.3. *Condition (15) is weaker than (11).*

Proof. If (11) holds, then as $\left(n^{\beta}(\log n)^{\gamma} X_n\right)$ is non-decreasing, we have

$$\sum_{n=2}^{m} \frac{\varphi_n^{k-1}|t_n|^k}{n^k\left(n^{\beta}(\log n)^{\gamma} X_n\right)^{k-1}} = O\left(\frac{1}{\left(2^{\beta}(\log 2)^{\gamma} X_2\right)^{k-1}}\right)\sum_{n=1}^{m} \frac{\varphi_n^{k-1}|t_n|^k}{n^k} = O\left(X_m \mu_m\right),$$

while if (15) is satisfied then,

$$\begin{aligned}
\sum_{n=2}^{m} \frac{\varphi_n^{k-1}|t_n|^k}{n^k} &= \sum_{n=2}^{m} \frac{\varphi_n^{k-1}|t_n|^k}{n^k\left(n^{\beta}(\log n)^{\gamma} X_n\right)^{k-1}}\left(n^{\beta}(\log n)^{\gamma} X_n\right)^{k-1} \\
&= O\left(m^{\beta}(\log m)^{\gamma} X_m\right)^{k-1} \sum_{n=2}^{m} \frac{\varphi_n^{k-1}|t_n|^k}{n^k\left(n^{\beta}(\log n)^{\gamma} X_n\right)^{k-1}} \\
&= O\left(m^{\beta}(\log m)^{\gamma} X_m\right)^{k-1} O\left(m^{\beta}(\log m)^{\gamma} X_m \mu_m\right) \\
&= O\left(\left(m^{\beta}(\log m)^{\gamma} X_m\right)^{k} \mu_m\right) \\
&\neq O\left(X_m \mu_m\right).
\end{aligned}$$

Therefore (11) implies (15) but not conversely. □

3. Main result

THEOREM 3.1. *Let (φ_n) be a sequence of positive real numbers. Let (X_n) be a quasi-f-power increasing sequence, $f = (f_n)$, $f_n = n^{\beta}(\log n)^{\gamma}$, $0 < \beta \leqslant 1$, $\gamma \geqslant 0$, and let $(\lambda_n), (\mu_n)$ be sequences of numbers such that (μ_n) is positive non-decreasing and all satisfying (8), (15), (14), (13), (16) and the following*

$$\sum_{n=1}^{\infty} \frac{|\lambda_n|}{n} < \infty, \tag{20}$$

$$\mu_n \Delta^2\left(\frac{1}{\mu_n}\right) = O\left(\frac{|\Delta\lambda_n|}{n|\lambda_{n+1}|}\right), \tag{21}$$

then the series $\sum a_n \lambda_n/\mu_n$ is summable $\varphi - |C,1|_k$, $k \geqslant 1$.

REMARK 3.2. 1. It may be mentioned that there exists two mistakes in proof of Theorem 1.3. The author consider the two series $\sum_{n=1}^{\infty} \frac{|\lambda_n|}{n}$ and $\sum_{n=1}^{\infty} |\lambda_n|$ are convergent via conditions (4), (9) and (10). But this is not true (see Lemma 2.1).

2. In this paper we are giving the corrected proof via adding the conditions (20) and (21).

3. We also reducing conditions (4) and (9) to one condition which is (14), (see Lemma 2.2).

4. We also replaced condition (11) by a weaker one which is (16), (see Lemma 2.3).

Proof of Theorem 3.1. Let T_n be the n-th $(C,1)$ mean of the sequence $(na_n\lambda_n/\mu_n)$. Then we have

$$\begin{aligned}
T_n &= \frac{1}{n+1}\sum_{v=1}^{n}\frac{va_v\lambda_v}{\mu_v}\\
&= \frac{1}{n+1}\left(\sum_{v=1}^{n-1}\left(\sum_{r=1}^{v} ra_r\right)\Delta\left(\frac{\lambda_v}{\mu_v}\right)+\left(\frac{\lambda_n}{\mu_n}\right)\sum_{v=1}^{n} va_v\right)\\
&= \frac{1}{n+1}\left(\sum_{v=1}^{n-1}(v+1)t_v\left(\Delta\left(\frac{1}{\mu_v}\right)\lambda_v+\frac{\Delta\lambda_v}{\mu_{v+1}}\right)\right)+\frac{t_n\lambda_n}{\mu_n}\\
&= \frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)t_v\Delta\left(\frac{1}{\mu_v}\right)\lambda_v+\frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)t_v\frac{\Delta\lambda_v}{\mu_{v+1}}+\frac{t_n\lambda_n}{\mu_n}\\
&= T_{n1}+T_{n2}+T_{n3}.
\end{aligned}$$

In order to prove the theorem, by Minkowski's inequality, it is sufficient to prove that

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^k}\left|T_{nj}\right|^k<\infty,\quad j=1,2,3.$$

Applying Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^k}|T_{n1}|^k &= \sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^k}\left|\frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)t_v\Delta\left(\frac{1}{\mu_v}\right)\lambda_v\right|^k\\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^{2k}}\sum_{v=1}^{n-1}v^k|t_v|^k\Delta\left(\frac{1}{\mu_v}\right)|\lambda_v|^k\left(\sum_{v=1}^{n-1}\Delta\left(\frac{1}{\mu_v}\right)\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^{2k}}\sum_{v=1}^{n-1}v^k|t_v|^k\Delta\left(\frac{1}{\mu_v}\right)|\lambda_v|^k\\
&= O(1)\sum_{v=1}^{m}v|t_v|^k\Delta\left(\frac{1}{\mu_v}\right)|\lambda_v|^k\sum_{n=v}^{m}\frac{\varphi_n^{k-1}}{n^{k+1}}\\
&= O(1)\sum_{v=1}^{m}\frac{v|t_v|^k}{v^k\left(v^\beta(\log v)^\gamma X_v\right)^{k-1}}\Delta\left(\frac{1}{\mu_v}\right)|\lambda_v|\varphi_v^{k-1}\left(|\lambda_v|v^\beta(\log v)^\gamma X_v\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\frac{v|t_v|^k}{v^k\left(v^\beta(\log v)^\gamma X_v\right)^{k-1}}\Delta\left(\frac{1}{\mu_v}\right)|\lambda_v|\varphi_v^{k-1}\\
&= O(1)\sum_{v=1}^{m}\frac{\varphi_v^{k-1}|t_v|^k}{v^k\left(v^\beta(\log v)^\gamma X_v\right)^{k-1}}v|\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\frac{\varphi_r^{k-1}|t_r|^k}{r^k\left(r^\beta(\log r)^\gamma X_r\right)^{k-1}}\right)\Delta\left(v|\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)\right)\\
&\quad+O(1)\left(\sum_{v=1}^{m}\frac{\varphi_v^{k-1}|t_v|^k}{v^k\left(v^\beta(\log v)^\gamma X_v\right)^{k-1}}\right)m|\lambda_m|\Delta\left(\frac{1}{\mu_m}\right)\\
&= O(1)\sum_{v=1}^{m-1}v^\beta(\log v)^\gamma X_v\mu_v\left(|\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)+(v+1)|\Delta\lambda_v|\Delta\left(\frac{1}{\mu_v}\right)\right.\\
&\quad\left.+(v+1)|\lambda_{v+1}|\Delta^2\left(\frac{1}{\mu_v}\right)\right)+O(1)m^{\beta+1}(\log m)^\gamma X_m\mu_m|\lambda_m|\Delta\left(\frac{1}{\mu_m}\right)\\
&= O(1)\sum_{v=1}^{m-1}\frac{|\lambda_v|}{v}+O(1)\sum_{v=1}^{m-1}v^\beta(\log v)^\gamma X_v|\Delta\lambda_v|\\
&\quad+O(1)\sum_{v=1}^{m-1}v^\beta(\log v)^\gamma X_v|\Delta\lambda_v|+O(|\lambda_m|)\\
&= O(1).
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^k}|T_{n2}|^k &= \sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^k}\left|\frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)t_v\frac{\Delta\lambda_v}{\mu_{v+1}}\right|^k\\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^{2k}}\sum_{v=1}^{n-1}v^k\frac{|t_v|^k}{\left(v^\beta(\log v)^\gamma X_v\right)^{k-1}}\frac{\Delta\lambda_v}{\mu_{v+1}^k}\left(\sum_{v=1}^{n-1}v^\beta(\log v)^\gamma X_v\Delta\lambda_v\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}v^k\frac{|t_v|^k}{\left(v^\beta(\log v)^\gamma X_v\right)^{k-1}}\frac{|\Delta\lambda_v|}{\mu_{v+1}^k}\sum_{n=v}^{m}\frac{\varphi_n^{k-1}}{n^{2k}}\\
&= O(1)\sum_{v=1}^{m}v\frac{|t_v|^k}{\left(v^\beta(\log v)^\gamma X_v\right)^{k-1}}\frac{\Delta\lambda_v}{\mu_{v+1}^k}\sum_{n=v}^{m}\frac{\varphi_n^{k-1}}{n^{k+1}}\\
&= O(1)\sum_{v=1}^{m}\frac{|t_v|^k\varphi_v^{k-1}}{v^k\left(v^\beta(\log v)^\gamma X_v\right)^{k-1}}\frac{v|\Delta\lambda_v|}{\mu_v}\\
&= O(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\frac{|t_r|^k\varphi_r^{k-1}}{r^k\left(r^\beta(\log r)^\gamma X_r\right)^{k-1}}\right)\Delta\left(\frac{v|\Delta\lambda_v|}{\mu_v}\right)\\
&\quad+\left(\sum_{v=1}^{m}\frac{|t_v|^k\varphi_v^{k-1}}{v^k\left(v^\beta(\log v)^\gamma X_v\right)^{k-1}}\right)\frac{m|\Delta\lambda_m|}{\mu_m}\\
&= O(1)\sum_{v=1}^{m-1}v^\beta(\log v)^\gamma X_v\mu_v\left(\Delta\left(\frac{1}{\mu_v}\right)\left(v|\Delta\lambda_v|+\frac{1}{\mu_{v+1}}\left(|\Delta\lambda_v|+(v+1)\left|\Delta^2\lambda_v\right|\right)\right)\right)\\
&\quad+O(1)mX_m|\Delta\lambda_m|\\
&= O(1)\sum_{v=1}^{m-1}v^\beta(\log v)^\gamma X_v|\Delta\lambda_v|+O(1)\sum_{v=1}^{m-1}v^\beta(\log v)^\gamma X_v|\Delta\lambda_v|
\end{aligned}
$$

$$
\begin{aligned}
&+O(1)\sum_{v=1}^{m-1} v^{\beta+1}(\log v)^{\gamma} X_v \left|\Delta^2\lambda_v\right| + O(1) \\
&= O(1).
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}|T_{n3}|^k &= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}\left|\frac{t_n\lambda_n}{\mu_n}\right|^k \\
&= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}|t_n|^k}{n^k\left(n^\beta(\log n)^\gamma X_n\right)^{k-1}}\frac{|\lambda_n|}{\mu_n^k}\left(n^\beta(\log n)^\gamma X_n|\lambda_n|\right)^{k-1} \\
&= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}|t_n|^k}{n^k\left(n^\beta(\log n)^\gamma X_n\right)^{k-1}}\frac{|\lambda_n|}{\mu_n} \\
&= O(1)\sum_{n=1}^{m-1}\left(\sum_{v=1}^{n}\frac{|t_v|^k\varphi_v^{k-1}}{v^k\left(v^\beta(\log v)^\gamma X_v\right)^{k-1}}\right)\Delta\left(\frac{|\lambda_n|}{\mu_n}\right) \\
&\quad+\left(\sum_{n=1}^{m}\frac{\varphi_n^{k-1}|t_n|^k}{n^k\left(n^\beta(\log n)^\gamma X_n\right)^{k-1}}\right)\frac{|\lambda_m|}{\mu_m} \\
&= O(1)\sum_{n=1}^{m-1}n^\beta(\log n)^\gamma X_n\mu_n\left(\Delta\left(\frac{1}{\mu_n}\right)|\lambda_n|+\frac{|\Delta\lambda_n|}{\mu_{n+1}}\right)+O(1)X_m|\lambda_m| \\
&= O(1)\sum_{n=1}^{m-1}\frac{|\lambda_n|}{n}+O(1)\sum_{n=1}^{m-1}n^\beta(\log n)^\gamma X_n|\Delta\lambda_n|+O(1) \\
&= O(1) \quad \square
\end{aligned}
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