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INTEGRAL REPRESENTATION OF SCHLÖMILCH SERIES

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Abstract. Certain integral representations are derived for the Schlömilch series of Bessel functions of the first kind J_ν, using newly derived integral representation of first kind Kapteyn–type series.

1. Introduction

Oscar Xavier Schlömilch introduced in 1857 in his article [18, pp. 155–158] the series of the form

$$\mathfrak{S}_\nu(z) := \sum_{n=1}^{\infty} \alpha_n J_\nu\left((\nu+n)z\right), \quad z \in \mathbb{C}, \tag{1.1}$$

where ν, α_n are constants and J_ν stands for the Bessel function of the first kind of order ν. This kind series are known as *Schlömilch series* (of the order ν [1]). Rayleigh [17] has showed that such series play serious roles in physics, because for $\nu = 0$ they are useful in investigation of a periodic transverse vibrations uniformly distributed in direction through the two dimensions of the membrane. Also, Schlömilch series present various features of purely mathematical interest and it is remarkable that a null–function can be represented by such series in which the coefficients are not all zero [20, p. 634].

It is worth of mention, that Schlömilch [18] proved that there exists a series $\mathfrak{S}_0^f(x)$ associated with any analytic function f. Namely, according to Watson's contemporary formulation [20, p. 619]: let $f(x)$ be an arbitrary function, with a derivative $f'(x)$ which is continuous in the interval $(0,\pi)$ and which has limited total fluctuation in this interval. Then $f(x)$ admits of the expansion

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m J_0(mx) =: \mathfrak{S}_0^f(x), \tag{1.2}$$

where

$$\begin{aligned} a_0 &= 2f(0) + \frac{2}{\pi}\int_0^\pi \int_0^{\frac{1}{2}\pi} u f'(u\sin\phi)\,\mathrm{d}\phi\,\mathrm{d}u, \\ a_m &= \frac{2}{\pi}\int_0^\pi \int_0^{\frac{1}{2}\pi} u f'(u\sin\phi)\cos(mu)\,\mathrm{d}\phi\,\mathrm{d}u, \qquad m \in \mathbb{N} \end{aligned}$$

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[1] O.X. Schlömilch considered only cases $\nu = 0, 1$.

and this expansion is valid, and the series converges in $(0,\pi)$.

We point out that this Schlömilch's result may be generalized by replacing the expansion (1.2) of order zero by $\mathfrak{S}_\nu^f(x)$ of arbitrary order ν, see [14], [20, Ch. XIX.], [19] and [2].

The next generalization is suggested by the theory of Fourier series, and the functions which naturally come under consideration are Bessel functions of the second kind and Struve's functions. The types of series to be considered may be written in the forms:

$$\frac{\frac12 a_0}{\Gamma(\nu+1)} + \sum_{m=1}^\infty \frac{a_m J_\nu(mx) + b_m Y_\nu(mx)}{\left(\frac12 mx\right)^\nu},$$
$$\frac{\frac12 a_0}{\Gamma(\nu+1)} + \sum_{m=1}^\infty \frac{a_m J_\nu(mx) + b_m \mathbf{H}_\nu(mx)}{\left(\frac12 mx\right)^\nu}.$$

Such series, with $\nu = 0$ have been considered in 1886 by Coates [3], but his proof of expanding an arbitrary functions $f(x)$ into this kind of series seems to be invalid except in some trivial case in which $f(x)$ is defined to be periodic (with period 2π) and to tend to zero as $x \to \infty$. Also for further subsequent generalizations consult e.g. Bondarenko's recent article [2] and the references therein and Miller's multidimensional expansion [8].

The series of much greater interest are direct generalization of trigonometrical series and they are called *generalized Schlömilch series*. Nielsen studied such kind of series in his memoirs consecutively in 1899 [9, 10, 11], 1900 [12] and finally in 1901 [13, 14]. He has given the forms for the coefficients in the generalized Schlömilch expansion of arbitrary function and he has investigated the construction of Schlömlich series which represent null–functions [15, p. 348]. Filon also investigated the possibility of expanding an arbitrary function into a generalized Schlömilch series for $\nu = 0$ [4]. Using Filon's method for finding coefficients in the generalized Schlömlich expansion, Watson proved a similar fashion expansion result.

THEOREM A. [20, p. 629] *Let ν be a number such that $-\frac12 < \nu < \frac12$; and let $f(x)$ be defined arbitrarily in the interval $(-\pi,\pi)$ subject to the following conditions:* **(i)** *the function $h(x) = 2\nu f(x) + xf'(x) \in \mathrm{C}^1(-\pi,\pi)$ and it has limited total fluctuation in the interval $(-\pi,\pi)$, and* **(ii)** *the integral*

$$\int_0^\Delta \frac{\mathrm{d}}{\mathrm{d}x}\left(|x|^{2\nu}\{f(x)-f(0)\}\right)\mathrm{d}x \qquad \nu \in (-1/2, 0)$$

is absolutely convergent when Δ is a (small) number either positive or negative. Then $f(x)$ admits of the expansion

$$f(x) = \frac{\frac12 a_0}{\Gamma(\nu+1)} + \sum_{m=1}^\infty \frac{a_m J_\nu(mx) + b_m \mathbf{H}_\nu(mx)}{\left(\frac12 mx\right)^\nu},$$

where

$$a_m = \frac{1}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\nu)} \int_{-\pi}^{\pi} \int_0^{\frac{1}{2}\pi} \sec^{2\nu+1}\phi \, \frac{\mathrm{d}}{\mathrm{d}\phi} \left(\{f(u\sin\phi) - f(0)\} \sin^{2\nu}\phi \right) \cos(mu)\, \mathrm{d}\phi\, \mathrm{d}u\,, \tag{1.3}$$

$$b_m = \frac{1}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\nu)} \int_{-\pi}^{\pi} \int_0^{\frac{1}{2}\pi} \sec^{2\nu+1}\phi \, \frac{\mathrm{d}}{\mathrm{d}\phi} \left(\{f(u\sin\phi) - f(0)\} \sin^{2\nu}\phi \right) \sin(mu)\, \mathrm{d}\phi\, \mathrm{d}u\,,$$

when $m > 0$*; the value of* a_0 *is obtained by inserting an additional term*

$$2\Gamma(\nu+1) f(0)$$

on the right in the first equation of the system (1.3).

2. Integral representation of Schlömilch series

In this section we will derive the double definite integral representation of the special kind of Schlömilch series

$$\mathfrak{S}_\nu^\mu(z) := \sum_{n=1}^{\infty} \alpha_n J_\nu\left((\mu+n)z\right), \quad z \in \mathbb{C}, \tag{2.1}$$

using an integral representation of Kapteyn–type series. So, let us first introduce Kapteyn series.

A *Kapteyn series of the first kind* is a series of the form

$$\mathfrak{K}_\nu(z) := \sum_{n=1}^{\infty} \alpha_n J_{\nu+n}\left((\nu+n)z\right), \quad z \in \mathbb{C}, \tag{2.2}$$

where ν, α_n are constants and J_ν stands for the Bessel function of the first kind of order ν.

It is worth to mention that very recently Baricz *et. al.* derived integral representations for such kind of series in [1]. Here, we would need integral representation of Kapteyn–type series

$$\widetilde{K}_{\nu,\beta}^{\mu}(z) := \sum_{n=1}^{\infty} \alpha_n J_{\nu+\beta n}\big((\mu+n)z\big), z \in \mathbb{C} \tag{2.3}$$

where ν, α_n, μ are constants and $\beta > 0$.

First, let us introduce some symbols and formulae, which we will need below.

Symbols $[a]$ and $\{a\} = a - [a]$ denote the integer and fractional part of some $a \in \mathbb{R}$, respectively.

Also, for the real–valued function $x \mapsto a_x = a(x)$, where $a \in C^1[k,m]$, $k,m \in \mathbb{Z}$, $k < m$, using the operator

$$\mathfrak{d}_x := 1 + \{x\} \frac{\mathrm{d}}{\mathrm{d}x},$$

we have the following condensed form of the Euler–Maclaurin summation formula [16, p. 2365]:

$$\sum_{j=k+1}^{m} a_j = \int_k^m \big(a(x) + \{x\}a'(x)\big)\mathrm{d}x = \int_k^m \mathfrak{d}_x a(x)\mathrm{d}x. \tag{2.4}$$

Now, we are ready to formulate our first main result.

THEOREM 1. *Let* $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$, $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\in\mathbb{N}}$ *and assume*

$$\mathscr{C} = \limsup_{n\to\infty} |\alpha_n|^{1/n} < 1.$$

Then, for all $\beta > 0$, $2(\nu+\beta)+1 > 0$ *and for all*

$$x \in \mathscr{I}_{\alpha,\beta} := \Big(0, 2\min\big(1, \beta\,(\mathrm{e}\,\mathscr{C}^{\frac{1}{\beta}})^{-1}\big)\Big)$$

we have the integral representation

$$\widetilde{K}^{\mu}_{\nu,\beta}(x) = -\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\left(\frac{\Gamma(\beta u+\nu+1/2)}{(\mu+u)^{\beta u+\nu}} J_{\beta u+\nu}\big((\mu+u)x\big)\right) \times \mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+1/2)}\right)\mathrm{d}u\,\mathrm{d}s. \tag{2.5}$$

Proof. Let us first establish the convergence conditions for the Kapteyn series $\widetilde{K}^{\mu}_{\nu,\beta}(x)$. For this purpose we use Landau's bounds [7] for the first kind Bessel function J_ν:

$$|J_\nu(x)| \leqslant b_L\, \nu^{-1/3}, \qquad b_L = \sqrt[3]{2}\sup_{x\in\mathbb{R}_+} \mathrm{Ai}(x),$$
$$|J_\nu(x)| \leqslant c_L\, |x|^{-1/3}, \qquad c_L = \sup_{x\in\mathbb{R}_+} x^{1/3} J_0(x),$$

where $\mathrm{Ai}(\cdot)$ stands for the familiar Airy function

$$\mathrm{Ai}(x) := \frac{\pi}{3}\sqrt{\frac{x}{3}}\left[J_{-1/3}\big(2(x/3)^{3/2}\big) + J_{1/3}\big(2(x/3)^{3/2}\big)\right].$$

Now it holds

$$\left|\widetilde{K}^{\mu}_{\nu,\beta}(x)\right| \leqslant \sum_{n=1}^{\infty} |\alpha_n| \max\left(\frac{b_L}{(\nu+\beta n)^{1/3}}, \frac{c_L}{((\mu+n)|x|)^{1/3}}\right)$$
$$\leqslant \max\left(\frac{b_L}{(\nu+\beta)^{1/3}}, \frac{c_L}{((\mu+1)|x|)^{1/3}}\right)\sum_{n=1}^{\infty} |\alpha_n|,$$

and thus the series (2.3) absolutely converges being $\mathscr{C} < 1$.

In the following we would need the integral representation of the Bessel function [5, p. 902]

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+1/2)} \int_{-1}^{1} \mathrm{e}^{\mathrm{i}zt}(1-t^2)^{\nu-1/2}\mathrm{d}t, \quad z \in \mathbb{C},\, \mathrm{Re}(\nu) > -1/2, \tag{2.6}$$

and thus, having in mind the definition of $\widetilde{K}^\mu_{\nu,\beta}(x)$ it has to be $2(\nu+\beta)+1>0$. Substituting (2.6) into (2.3) we get

$$\widetilde{K}^\mu_{\nu,\beta}(x) = \sqrt{\frac{x}{2\pi}} \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\mu xt} \left(\frac{x(1-t^2)}{2}\right)^{\nu-1/2} \mathscr{D}_\alpha(t)\mathrm{d}t, \qquad x>0, \tag{2.7}$$

where $\mathscr{D}_\alpha(t)$ is the Dirichlet series

$$\mathscr{D}_\alpha(t) := \sum_{n=1}^\infty \frac{\alpha_n(\mu+n)^{\nu+\beta n}}{\Gamma(\nu+\beta n+1/2)} \exp\left(-n\ln\left(\frac{2}{\mathrm{e}^{\mathrm{i}xt/\beta}x(1-t^2)}\right)^\beta\right). \tag{2.8}$$

For the convergence of (2.8) we find that the related radius of convergence equals

$$\rho = \left(\frac{\beta}{\mathrm{e}}\right)^\beta \left(\limsup_{n\to\infty} |\alpha_n|^{1/n}\right)^{-1} = \frac{\beta^\beta}{\mathrm{e}^\beta \mathscr{C}}.$$

Now, because of $|t|<1$, there holds

$$\left|\mathrm{e}^{\mathrm{i}xt}\left(\frac{x(1-t^2)}{2}\right)^\beta\right| \leqslant \left|\frac{x}{2}\right|^\beta < \rho\,,$$

hence the convergence domain of $\mathscr{D}_\alpha(t)$ is

$$|x| < 2\rho^{1/\beta} = \frac{2\beta}{\mathrm{e}}\mathscr{C}^{-\frac{1}{\beta}}. \tag{2.9}$$

Moreover, the Dirichlet series' parameter needs to have positive real part [6, 16], i.e.

$$\mathrm{Re}\left(\ln\frac{2^\beta}{\mathrm{e}^{\mathrm{i}tx}x^\beta(1-t^2)^\beta}\right) = \beta\ln\frac{2}{x(1-t^2)} > \beta\ln\frac{2}{x} > 0, \qquad |t|<1\,,$$

so, this additional convergence range is $x\in(0,2)$. Collecting all these estimates, we deduce that the desired integral expression exists for $x \in \mathscr{I}_{\alpha,\beta}$.

Expressing (2.8) first as the Laplace–integral, then transforming it by the condensed Euler–Maclaurin formula, we get

$$\begin{aligned}\mathscr{D}_\alpha(t) &= \ln\frac{2^\beta}{\mathrm{e}^{\mathrm{i}xt}\,(x(1-t^2))^\beta}\int_0^\infty \left(\mathrm{e}^{\mathrm{i}xt}\left(\frac{x(1-t^2)}{2}\right)^\beta\right)^u \sum_{n=1}^{[u]} \frac{\alpha_n(\mu+n)^{\nu+\beta n}}{\Gamma(\nu+\beta n+1/2)}\mathrm{d}u \\ &= -\int_0^\infty\int_0^{[u]} \left(\mathrm{e}^{\mathrm{i}xt}\left(\frac{x(1-t^2)}{2}\right)^\beta\right)^u \ln\frac{\mathrm{e}^{\mathrm{i}xt}\,(x(1-t^2))^\beta}{2^\beta}\,\mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+1/2)}\right)\mathrm{d}u\mathrm{d}s\,.\end{aligned} \tag{2.10}$$

Combination of (2.7) and (2.10) yields

$$\widetilde{K}^{\mu}_{\nu,\beta}(x) = -\sqrt{\frac{x}{2\pi}}\int_0^\infty\int_0^{[u]}\mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+1/2)}\right) \times\left(\int_{-1}^1 \mathrm{e}^{\mathrm{i}x(\mu+u)t}\left(\frac{x(1-t^2)}{2}\right)^{\nu+\beta u-1/2}\ln\frac{\mathrm{e}^{\mathrm{i}xt}\left(x(1-t^2)\right)^\beta}{2^\beta}\,\mathrm{d}t\right)\mathrm{d}u\mathrm{d}s. \tag{2.11}$$

In the following, we will simplify the t–integral

$$\mathscr{I}_x(u) := \int_{-1}^1 \mathrm{e}^{\mathrm{i}(\mu+u)xt}\left(\frac{x(1-t^2)}{2}\right)^{\nu+\beta u-1/2}\ln\frac{\mathrm{e}^{\mathrm{i}xt}\left(x(1-t^2)\right)^\beta}{2^\beta}\,\mathrm{d}t\,.$$

We have

$$\int\mathscr{I}_x(u)\mathrm{d}u = \int_{-1}^1 \mathrm{e}^{\mathrm{i}(\mu+u)xt}\left(\frac{x(1-t^2)}{2}\right)^{\nu+\beta u-1/2}\mathrm{d}t = \sqrt{\frac{2\pi}{x}}\frac{\Gamma(\nu+\beta u+1/2)}{(\mu+u)^{\nu+\beta u}}J_{\beta u+\nu}\left((\mu+u)x\right),$$

that is

$$\mathscr{I}_x(u) = \sqrt{\frac{2\pi}{x}}\frac{\partial}{\partial u}\left(\frac{\Gamma(\nu+\beta u+1/2)}{(\mu+u)^{\nu+\beta u}}J_{\beta u+\nu}\left((\mu+u)x\right)\right). \tag{2.12}$$

Now, by virtue of (2.11) and (2.12) we immediately get the integral representation (2.5).

Now, it is easy to establish a connection between Schlömilch series (2.1) and Kapteyn–type series (2.3):

$$\mathfrak{S}^{\mu}_{\nu}(x) = \lim_{\beta\to 0}\widetilde{K}^{\mu}_{\nu,\beta}(x)\,. \tag{2.13}$$

Using that equality, we have the following result, from the Theorem 1:

COROLLARY 1. *Let* $\alpha\in \mathrm{C}^1(\mathbb{R}_+)$ *such that the function*

$$\kappa(u,s) := \frac{\partial}{\partial u}\left(\frac{\Gamma(\beta u+\nu+1/2)}{(\mu+u)^{\beta u+\nu}}J_{\beta u+\nu}((\mu+u)x)\right)\mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+1/2)}\right), \qquad \beta>0$$

is integrable. Let $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\in\mathbb{N}}$ *and assume that* $\mathscr{C}<1$. *Then, for all* $2\nu+1>0$ *and* $x\in(0,2) =: \mathscr{I}_{\alpha,0}$ *we have the integral representation*

$$\mathfrak{S}^{\mu}_{\nu}(x) = -\int_1^\infty\int_0^{[u]}\frac{\partial}{\partial u}\left(\frac{J_\nu((\mu+u)x)}{(\mu+u)^\nu}\right)\mathfrak{d}_s\left(\alpha(s)(\mu+s)^\nu\right)\mathrm{d}u\mathrm{d}s. \tag{2.14}$$

Proof. It is enough to establish the behavior of the convergence domain $\mathscr{I}_{\alpha,\beta}$ when β vanishes. Having in mind that $\mathscr{C} < 1$ we have that

$$\lim_{\beta\to 0_+} \beta\mathscr{C}^{-\frac{1}{\beta}} = -\ln\mathscr{C} \lim_{\beta\to 0_+} \mathscr{C}^{-\frac{1}{\beta}} = +\infty,$$

and $\mathscr{I}_{\alpha,0} = (0,2)$. Thus the statement (2.14) immediately follows from Theorem 1, relation (2.13) and Lebesgue dominated convergence theorem.

3. Another integral representation of Schlömilch series

In this section we would derive integral representations for Schlömilch series (1.1), using Bessel differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0. \tag{3.1}$$

It is well known that the Bessel functions of the first kind are particular solutions of the previous equation, i.e. it holds

$$x^2J_\nu''(x) + xJ_\nu'(x) + (x^2 - \nu^2)J_\nu(x) = 0\,.$$

Now, taking $x \mapsto (\nu+n)x$ we obtain

$$x^2(\nu+n)^2J_\nu''((\nu+n)x) + x(\nu+n)J_\nu'((\nu+n)x) + (x^2(\nu+n)^2 - \nu^2)J_\nu((\nu+n)x) = 0\,. \tag{3.2}$$

Multiplying (3.2) by α_n, and then summing up that expression in $n \in \mathbb{N}$ we get the following equality

$$x^2\mathfrak{S}_\nu''(x) + x\mathfrak{S}_\nu'(x) + (x^2 - \nu^2)\mathfrak{S}_\nu(x) \tag{3.3}$$

$$= \sum_{n=1}^{\infty} \big(1-(\nu+n)^2\big)x^2\alpha_nJ_\nu\big((\nu+n)x\big) =: \mathfrak{T}_\nu(x); \tag{3.4}$$

the right side expression $\mathfrak{T}_\nu(x)$ defines the so–called *Schlömilch series of Bessel functions associated to* $\mathfrak{S}_\nu(x)$.

Now, we can derive the following theorem:

THEOREM 2. *Schlömilch series* (1.1) *is the solution of the nonhomogeneous Bessel–type differential equation*

$$x^2\mathfrak{y}'' + x\mathfrak{y}' + (x^2 - \nu^2)\mathfrak{y} = \mathfrak{T}_\nu(x)\,, \tag{3.5}$$

where $\mathfrak{T}_\nu(x)$ *is given with* (3.4)*. Moreover, if we assume that* $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$, $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\in\mathbb{N}}$ *and that the series* $\sum_{n=1}^{\infty} n^{5/3}\alpha_n$ *absolutely converges, then for all* $x \in \mathscr{I}_{\alpha,0}$ *we have the integral representation*

$$\mathfrak{T}_\nu(x) = -x^2\int_1^\infty\int_0^{[u]} \frac{\partial}{\partial u}\left(\frac{J_\nu\big((\nu+u)x\big)}{(\nu+u)^\nu}\right)\mathfrak{d}_s\Big(\alpha(s)\left(1-(\nu+s)^2\right)(\nu+s)^\nu\Big)\,\mathrm{d}u\mathrm{d}s\,. \tag{3.6}$$

Proof. We already showed, in the first lines of this section, that Schlömilch series (1.1) is a solution of (3.5).

Further, from (3.4) we have

$$\mathfrak{T}_\nu(x) = x^2 \mathfrak{S}_\nu(x) - x^2 \sum_{n=1}^{\infty} (\nu+n)^2 \alpha_n J_\nu((\nu+n)x)\,, \tag{3.7}$$

and from Landau's bound it follows that the second series converges absolutely when the series $\sum\limits_{n=1}^{\infty} n^{5/3}|\alpha_n| < \infty$ absolutely converges. Using the Corollary with $\alpha_n \mapsto \big(1-(\nu+n)^2\big)\alpha_n$, we get the integral expression (3.6).

Below, we will derive a new integral representation of the Schlömilch series (1.1), using the Bessel differential equation (3.3).

THEOREM 3. *Let $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$, $\alpha|_{\mathbb{N}} = \{\alpha_n\}_{n\in\mathbb{N}}$ and assume that series $\sum\limits_{n=1}^{\infty} n^{5/3}\alpha_n$ absolutely converges. Then, for all $\nu > -1/2$ and $x \in \mathscr{I}_{\alpha,0}$ we have*

$$\begin{aligned}\mathfrak{S}_\nu(x) &= \frac{J_\nu(x)}{2}\int \frac{1}{xJ_\nu^2(x)}\left(\int \frac{J_\nu(x)\mathfrak{T}_\nu(x)}{x}\,\mathrm{d}x\right)\mathrm{d}x \\ &\quad + \frac{Y_\nu(x)}{2}\int \frac{1}{xY_\nu^2(x)}\left(\int \frac{Y_\nu(x)\mathfrak{T}_\nu(x)}{x}\,\mathrm{d}x\right)\mathrm{d}x\,, \end{aligned} \tag{3.8}$$

where $\mathfrak{T}_\nu$ is the Schlömilch series, given with (3.4).

Proof. We would need the *Bessel function of the second kind of order* ν (or MacDonald function) Y_ν which is defined by

$$Y_\nu(x) = \mathrm{cosec}(\pi\nu)\,(J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x))\,, \qquad \nu \notin \mathbb{Z},\ |\arg(z)| < \pi.$$

The homogeneous solution of the Bessel differential equation is given with

$$y_h(x) = C_1 Y_\nu(x) + C_2 J_\nu(x)\,,$$

where J_ν and Y_ν are independent solutions of the Bessel differential equation.

As J_ν is a solution, we search for the particular solution y_p in the form $y_p(x) = J_\nu(x)w(x)$. Substituting this form into (3.3), we have

$$x^2(J_\nu'' w + 2J_\nu' w' + J_\nu w'') + x(J_\nu' w + J_\nu w') + (x^2-\nu^2)J_\nu w = \mathfrak{T}_\nu(x).$$

If we write previous equation in the following form

$$w\big(x^2 J_\nu'' + xJ_\nu' + (x^2-\nu^2)J_\nu\big) + w'(2x^2 J_\nu' + xJ_\nu) + w''(x^2 J_\nu) = \mathfrak{T}_\nu(x),$$

using the fact that J_ν is a solution of the homogeneous Bessel differential equation, we get the solution

$$w = \int \frac{1}{xJ_\nu^2}\left(\int \frac{\mathfrak{T}_\nu J_\nu}{x}\,\mathrm{d}x\right)\mathrm{d}x + C_3\frac{\pi}{2}\frac{Y_\nu}{J_\nu} + C_4\,,$$

because

$$\int \frac{1}{xJ_\nu^2} \mathrm{d}x = \frac{\pi}{2} \frac{Y_\nu}{J_\nu}.$$

So, we have the particular solution

$$\mathfrak{S}_\nu(x) = J_\nu(x)w(x) = J_\nu(x) \int \frac{1}{xJ_\nu^2} \left(\int \frac{\mathfrak{T}_\nu J_\nu}{x} \mathrm{d}x \right) \mathrm{d}x + C_3 \frac{\pi}{2} Y_\nu(x) + C_4 J_\nu(x).$$

Using the fact that y_h is formed by independent functions J_ν and Y_ν, that functions do not contribute to the particular solution y_p and the constants C_3, C_4 can be taken to be zero.

Analogously, taking particular solution in the form $\mathfrak{y}_p(x) = Y_\nu(x)w(x)$ and using the equality

$$\int \frac{1}{xY_\nu^2} \mathrm{d}x = -\frac{\pi}{2} \frac{J_\nu}{Y_\nu}$$

we get

$$\mathfrak{S}_\nu(x) = Y_\nu(x)w(x) = Y_\nu(x) \int \frac{1}{xY_\nu^2} \left(\int \frac{\mathfrak{T}_\nu Y_\nu}{x} \mathrm{d}x \right) \mathrm{d}x - C_5 \frac{\pi}{2} J_\nu(x) + C_6 Y_\nu(x).$$

Again, choosing $C_5 = C_6 = 0$, we get the integral representation (3.7).

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