Journal of
Classical
Analysis
Volume 1, Number 1 (2012), 85–90
doi:10.7153/jca-01-09

SUFFICIENT CONDITIONS FOR STARLIKENESS

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Abstract. The purpose of the present paper is to consider some sufficient conditions for analytic functions in the open unit disk to be starlike. Here we establish three theorems by using Jack's lemma and a simple result contained in Lemma 2.2. Our theorems provide improvements of the results about sufficient conditions for starlike functions given earlier by Lewandowski et al. [2], Li and Owa [3], Nunokawa et al. [5] and Ramesha et al. [6].

1. Introduction

Let $\mathscr{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are *analytic* in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

A function f belonging to the class $\mathscr{A}$ is said to be *starlike* of order α in $\mathbb{D}$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{D}; 0 \leqslant \alpha < 1) \tag{1}$$

We denote by $S^*(\alpha)$ the class of all functions in $\mathscr{A}$ which are starlike of order α in $\mathbb{D}$. Also we write $S^*(0)$ simply by S^*.

Several results appeared previously about sufficient conditions of starlikeness. The result discussed in Theorems A, B, C, D, E and F (given below) were obtained earlier by Lewandowski et al. [2], Ramesha et al. [6], Li and Owa [3], and Nunokawa et al. [5] respectively.

THEOREM A. ([2]) *Let* $f(z) \in \mathscr{A}$ *satisfy the condition*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0 \quad (z \in \mathbb{D}),$$

then $f(z) \in S^*$.

THEOREM B. ([6]) *Let* $f(z) \in \mathscr{A}$ *satisfy the condition*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right)\right\} > 0 \quad (z \in \mathbb{D}; \alpha \geqslant 0),$$

then $f(z) \in S^*$.

Mathematics subject classification (2010): 30C55.
Keywords and phrases: Analytic functions, starlike functions.

Paper JCA-01-09

THEOREM C. ([3]) *Let* $f(z) \in \mathscr{A}$ *satisfy the condition*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{\alpha}{2} \quad (z \in \mathbb{D}; \alpha \geqslant 0),$$

then $f(z) \in S^*$.

THEOREM D. ([3]) *Let* $f(z) \in \mathscr{A}$ *satisfy the condition*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{\alpha^2}{4}(1-\alpha) \quad (z \in \mathbb{D}; 0 \leqslant \alpha < 2),$$

then $f(z) \in S^*\left(\frac{\alpha}{2}\right)$.

THEOREM E. ([3]) *Let* $f(z) \in \mathscr{A}$ *satisfy the condition*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > 0 \quad (z \in \mathbb{D}),$$

then $f(z) \in S^*\left(\frac{1}{2}\right)$.

THEOREM F. ([5]) *Let* $f(z) \in \mathscr{A}$ *satisfy the condition*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{\alpha}{2}\left\{1+3\left(\operatorname{Im}\frac{zf'(z)}{f(z)}\right)^2\right\} \quad (z \in \mathbb{D}; \alpha \geqslant 0),$$

then $f(z) \in S^*$.

Obviously Theorems C, D and E are improvements of the Theorems A and B while Theorem F is an improvement of Theorem C. The aim of this paper is to provide improvements of all these theorems.

2. Main results

In order to establish our main results, we require the following lemmas:

LEMMA 2.1. (Jack's lemma [1]) *Let* $w(z)$ *be a non-constant analytic function in* $\mathbb{D}$ *with* $w(0) = 0$. *If* $|w(z)|$ *attains its maximum value on the circle* $|z| = r < 1$ *at* z_0, *then*

$$z_0 w'(z_0) = k w(z_0),$$

where $k \geqslant 1$ *is a real number.*

LEMMA 2.2. *Let* $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *be analytic in* $\mathbb{D}$ *and suppose that there exists a point* $z_0 \in \mathbb{D}$ *such that*

$$\operatorname{Re}\{p(z)\} > 0 \quad \text{for } |z| < |z_0| \tag{2}$$

and

$$\operatorname{Re}\{p(z_0)\} = 0. \tag{3}$$

Then we have

$$z_0 p'(z_0) \leqslant -\frac{1}{2}\left(1+|p(z_0)|^2\right). \tag{4}$$

Proof. Let us put

$$\varphi(z) = \frac{1-p(z)}{1+p(z)}. \tag{5}$$

Then we have that

$$\varphi(0)=0, |\varphi(z)|<1 \quad \text{for} \quad |z|<|z_0| \quad \text{and} \quad |\varphi(z_0)|=1.$$

From (2), (5) and Lemma 2.1, we find that

$$\frac{z_0\varphi'(z_0)}{\varphi(z_0)} = \frac{-2z_0p'(z_0)}{1-\{p(z_0)\}^2} = \frac{-2z_0p'(z_0)}{1+|p(z_0)|^2} \geqslant 1.$$

This shows that

$$z_0 p'(z_0) \leqslant -\frac{1}{2}\left(1+|p(z_0)|^2\right),$$

and $z_0p'(z_0)$ is a negative real number.

THEOREM 2.3. *Let* $f(z)\in\mathscr{A}$ *satisfies* $f(z)f'(z)\neq 0$ *in* $0<|z|<1$ *and*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{\alpha}{2}\left\{1+3\left|\frac{zf'(z)}{f(z)}\right|^2\right\} \quad (z\in\mathbb{D};\alpha\geqslant 0),$$

then $f(z)\in S^*$.

Proof. Let

$$p(z)=\frac{zf'(z)}{f(z)}, \tag{6}$$

then $p(z)$ is analytic in $\mathbb{D}$ and $p(0)=1$.

Suppose that there exists a point $z_0\in\mathbb{D}$ which satisfies the conditions (2) and (3) of Lemma 2.2.

Now using (6), it follows that

$$\begin{aligned}\operatorname{Re}&\left\{\frac{z_0f'(z_0)}{f(z_0)}\left(1+\alpha\frac{z_0f''(z_0)}{f'(z_0)}\right)\right\}\\ &=\operatorname{Re}\left\{\alpha z_0p'(z_0)+\alpha\left(p(z_0)\right)^2-\alpha p(z_0)+p(z_0)\right\}\end{aligned} \tag{7}$$

Since the function $p(z)$ and the point z_0 satisfy all conditions Lemma 2.2, therefore in view of (3) and (4), equation (7) gives

$$\begin{aligned}\operatorname{Re}\left\{\frac{z_0f'(z_0)}{f(z_0)}\left(1+\alpha\frac{z_0f''(z_0)}{f'(z_0)}\right)\right\} &\leqslant -\frac{\alpha}{2}\left(1+|p(z_0)|^2\right)-\alpha|p(z_0)|^2\\ &\leqslant -\frac{\alpha}{2}\left(1+3|p(z_0)|^2\right)\\ &\leqslant -\frac{\alpha}{2}\left(1+3\left|\frac{z_0f'(z_0)}{f(z_0)}\right|^2\right). \quad \text{(by(4))}\end{aligned}$$

This is a contradiction and therefore proof of the Theorem 2.3 is completed.

The result discussed in Theorem 2.3 is an improvement of the result contained in Theorem F.

REMARK.. We have supposed the condition $f(z)f'(z) \neq 0$ in $0 < |z| < 1$ while Li and Owa [3] did not put this condition in their result, but in fact they applied it during the proof of their result and so, it is not a weak point of our paper.

THEOREM 2.4. *Let $f(z) \in \mathscr{A}$ satisfies $f(z)f'(z) \neq 0$ in $0 < |z| < 1$ and*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{1}{2}\left|\frac{zf'(z)}{f(z)}\right|^2 \quad (z \in \mathbb{D}),$$

then $S^*\left(\frac{1}{2}\right)$.

Proof. Let

$$p(z) = 2\left(\frac{zf'(z)}{f(z)} - \frac{1}{2}\right), \quad p(0) = 1 \tag{8}$$

Now using (8), we have

$$\mathrm{Re}\left\{\frac{z_0 f'(z_0)}{f(z_0)}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right)\right\} = \mathrm{Re}\left\{\frac{1}{2}z_0 p'(z_0) + \frac{1}{4}(1+p(z_0))^2\right\} \tag{9}$$

where $z_0 \in \mathbb{D}$ is a point which satisfies the conditions (2) and (3) of Lemma 2.2.

Under the conditions (3) and (4) of Lemma 2.2, it follows that

$$\begin{aligned}\mathrm{Re}\left\{\frac{z_0 f'(z_0)}{f(z_0)}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right)\right\} &\leqslant -\frac{1}{4}\left(1+|p(z_0)|^2\right) - \frac{1}{4}|p(z_0)|^2 + \frac{1}{4} \\ &\leqslant -\frac{1}{2}|p(z_0)|^2 \\ &\leqslant -\frac{1}{2}\left|\frac{z_0 f'(z_0)}{f(z_0)}\right|^2\end{aligned}$$

which contradicts the hypothesis of Theorem 2.3 and therefore, we have

$$\mathrm{Re}\{p(z)\} > 0 \quad (z \in \mathbb{D})$$

or

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \quad (z \in \mathbb{D}).$$

Theorem 2.4 is an improvement of Theorem E.

THEOREM 2.5. *Let $f(z) \in \mathscr{A}$ satisfies $f(z)f'(z) \neq 0$ in $0 < |z| < 1$ and*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{\alpha}{4}(2-\alpha)(1-\alpha) \qquad (z \in \mathbb{D};\ 0 \leqslant \alpha < 2)$$

then $f(z) \in S^*\left(\frac{\alpha}{2}\right)$.

Proof. Let $p(z)$ is defined by

$$\frac{zf'(z)}{f(z)} = \left(1-\frac{\alpha}{2}\right)p(z)+\frac{\alpha}{2}, \quad p(0)=1.$$

Suppose that there exists a point $z_0 \in \mathbb{D}$ such that it satisfies the conditions (2) and (3) of Lemma 2.2. Now, we have

$$\begin{aligned}\operatorname{Re}&\left\{\frac{z_0 f'(z_0)}{f(z_0)}\left(1+\alpha\frac{z_0 f''(z_0)}{f'(z_0)}\right)\right\}\\ &=\operatorname{Re}\Big\{\alpha\left(1-\frac{\alpha}{2}\right)z_0 p'(z_0)+\alpha\left(1-\frac{\alpha}{2}\right)^2(p(z_0))^2\\ &\quad+\left(1-\frac{\alpha}{2}\right)\left(\alpha^2+1-\alpha\right)p(z_0)+\frac{\alpha^3}{4}+\frac{\alpha}{2}(1-\alpha)\Big\}\end{aligned} \tag{10}$$

Using the conditions (3) and (4) in (10), we find that

$$\begin{aligned}\operatorname{Re}&\left\{\frac{z_0 f'(z_0)}{f(z_0)}\left(1+\alpha\frac{z_0 f''(z_0)}{f'(z_0)}\right)\right\}\\ &\leqslant -\frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)\left(1+|p(z_0)|^2\right)-\alpha\left(1-\frac{\alpha}{2}\right)^2|p(z_0)|^2\\ &\quad+\frac{\alpha^3}{4}+\frac{\alpha}{2}(1-\alpha)\\ &\leqslant -\frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)+\frac{\alpha^3}{4}+\frac{\alpha}{2}(1-\alpha)\\ &\leqslant -\frac{\alpha}{4}(2-\alpha)(1-\alpha).\end{aligned}$$

This is a contradiction and therefore

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{\alpha}{2} \quad (z \in \mathbb{D}).$$

Thus proof of Theorem 2.4 is completed.

REMARK. Theorem 2.4 improves Theorem D because

$$0 > -\frac{\alpha^2}{4}(1-\alpha) > -\frac{\alpha}{4}(1-\alpha)(2-\alpha) \quad \text{when } 0 \leqslant \alpha < 1,$$

$$-\frac{\alpha}{4}(1-\alpha) = -\frac{\alpha}{4}(1-\alpha)(2-\alpha) \quad \text{when } \alpha = 1,$$

$$-\frac{\alpha^2}{4}(1-\alpha) > -\frac{\alpha}{4}(1-\alpha)(2-\alpha) > 0 \quad \text{when } 1 < \alpha < 2.$$

Acknowledgement. The authors are thankful to the worthy referee for his valuable suggestions regarding the improvement of the paper. The second author (S. P. G.) is thankful to CSIR, New Delhi, India for awarding Emeritus Scientistship, under scheme number 21(084)/10/EMR-II.

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