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# ON A CERTAIN CLASS OF MEROMORPHIC FUNCTIONS ASSOCIATED WITH A DIFFERENTIAL OPERATOR

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*Abstract.* In this paper, we introduce and investigate a new class of meromorphic functions defined by a differential operator. For this class, we obtain coefficient inequality, distortion inequality, radius of close-to-convex, starlikeness and convexity, extreme points and integral means inequality.

## 1. Introduction

Let $\sum^c$ denote the class of functions of the form

$$f(z) = \frac{a_0}{z-c} + \sum_{n=1}^{\infty} a_n z^n \quad (a_0 > 0,\ 0 \leqslant c < 1), \tag{1.1}$$

which are analytic in $U^* = \{z : c < |z| < 1\}$.

Also, by $C_\eta^\varepsilon$ $(\eta \in R,\ \varepsilon \in \{0,1\})$, we denote the class of functions $f \in \sum^c$ of the form (1.1) for which (see [1])

$$\arg(a_n) = \varepsilon\pi - (n+1)\eta \quad (n \in N = \{1,2,\cdots\}). \tag{1.2}$$

For $\eta = 0$, we obtain the classes $C_0^0$ and $C_0^1$ of functions with positive coefficients and negative coefficients respectively.

For two functions $f(z)$ and $g(z)$ which are analytic in $U$, we say that the function $f(z)$ is subordinate to $g(z)$ in $U$, and write as

$$f(z) \prec g(z), \quad z \in U,$$

if there exists a Schwarz function $\omega(z)$, which is analytic in $U$ with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \quad (z \in U)$$

such that

$$f(z) = g(\omega(z)) \quad (z \in U).$$

Various subclasses of $\sum^c$ when $c = 0$ were introduced and studied by many authors (see, [2]–[12]). In recent years, some subclasses of meromorphic functions associated with several families of integral operators and derivative operators were introduced

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and investigated (see, for example [13,14] and [15]; see also [16] and [17]). The first differential operator for meromoephic function was introduced by Frasin and Darus [18]. In [19], Ghanim and Darus introduced a differential operator:

$$\begin{aligned} I^0 f(z) &= f(z), \\ I^1 f(z) &= zf'(z) + \frac{a_0(2z-c)}{(z-c)^2}, \\ I^2 f(z) &= z(I^1 f(z))' + \frac{a_0(2z-c)}{(z-c)^2}, \\ I^k f(z) &= z(I^{(k-1)} f(z))' + \frac{a_0(2z-c)}{(z-c)^2}, \end{aligned}$$

where $k \in N_0 = N \cup \{0\}$, $z \in U^*$.

If $f$ is given by (1.1), then from the definition of the operator $I^k$, it is easy to see that

$$I^k f(z) = \frac{a_0}{z-c} + \sum_{n=1}^{\infty} n^k a_n z^n. \tag{1.3}$$

By using the operator $I^k$, some authors have established many subclasses of meromorphic functions, for example [19]–[22]. But second positive coefficient of some subclasses were fixed in these papers. In this paper we avoid the situation effectively. With the help of the differential operator $I^k$ and making use of the method in [23], we define the following new class of analytic functions and obtain some interesting results.

Let $K_{i,j}(\alpha, A, B)$ denote the subclass of $\sum^c$ consisting of function $f(z)$ which satisfies the following condition:

$$\frac{I^i f(z)}{I^j f(z)} - \alpha \left| \frac{I^i f(z)}{I^j f(z)} - 1 \right| \prec \frac{1+Az}{1+Bz}, \tag{1.4}$$

where $\alpha \geqslant 0$, $-1 \leqslant B < A \leqslant 1$, $i \in N$, $j \in N_0 = N \cup \{0\}$, $z \in U^*$.

Moreover, let us define

$$CK_\eta^\varepsilon(i, j; \alpha, A, B) = C_\eta^\varepsilon \bigcap K_{i,j}(\alpha, A, B) \tag{1.5}$$

The object of the present paper is to investigate the class $CK_\eta^\varepsilon(i, j; \alpha, A, B)$. We provide coefficient inequality, distortion inequality, radius of close-to-convex, starlikeness and convexity, extreme points and integral means inequality for $CK_\eta^\varepsilon(i, j; \alpha, A, B)$.

## 2. Coefficient inequality for the class

THEOREM 1. *A function* $f(z) \in \sum^{c}$ *is in the class* $CK^{\varepsilon}_{\eta}(i, j;\alpha,A,B)$ *if and only if*

$$\sum_{n=1}^{\infty} \phi_{i,j}(n,c,\alpha,A,B)|a_n| \leqslant (A-B)a_0, \tag{2.1}$$

*where*

$$\phi_{i,j}(n,c,\alpha,A,B) = [(1+(1+|B|)\alpha)|n^i - n^j| + |Bn^i - An^j|](1+c) \tag{2.2}$$

$$(\alpha \geqslant 0,\ -1 \leqslant B < A \leqslant 1,\ i \in N,\ j \in N_0 = N \cup \{0\}).$$

*Proof.* Suppose that (2.1) is true for $\alpha \geqslant 0, -1 \leqslant B < A \leqslant 1$. For $f(z) \in \sum^{c}$, let us define the function $p(z)$ by

$$p(z) = \frac{I^i f(z)}{I^j f(z)} - \alpha\left|\frac{I^i f(z)}{I^j f(z)} - 1\right|.$$

It suffices to show that

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| < 1 \quad (z \in U). \tag{2.3}$$

We note that

$$\begin{aligned}
\left|\frac{p(z)-1}{A-Bp(z)}\right| &= \left|\frac{I^i f(z) - \alpha e^{i\theta}|I^i f(z) - I^j f(z)| - I^j f(z)}{AI^j f(z) - B(I^i f(z) - \alpha e^{i\theta}|I^i f(z) - I^j f(z)|)}\right| \\
&= \left|\frac{\sum_{n=1}^{\infty}(n^i - n^j)a_n z^n - \alpha e^{i\theta}|\sum_{n=1}^{\infty}(n^i - n^j)a_n z^n|}{(A-B)\frac{a_0}{z-c} - \sum_{n=1}^{\infty}(Bn^i - An^j)a_n z^n + B\alpha e^{i\theta}|\sum_{n=1}^{\infty}(n^i - n^j)a_n z^n|}\right| \\
&= \left|\frac{\sum_{n=1}^{\infty}(n^i - n^j)a_n(z^{n+1} - cz^n) - \alpha e^{i\theta}e^{i\psi}|\sum_{n=1}^{\infty}(n^i - n^j)a_n(z^{n+1} - cz^n)|}{(A-B)a_0 - \sum_{n=1}^{\infty}(Bn^i - An^j)a_n(z^{n+1} - cz^n) + B\alpha e^{i\theta}e^{i\psi}|\sum_{n=1}^{\infty}(n^i - n^j)a_n(z^{n+1} - cz^n)|}\right| \\
&\leqslant \frac{\sum_{n=1}^{\infty}|n^i - n^j||a_n|(|z|^{n+1} + c|z|^n) + \alpha|e^{i(\theta+\psi)}|\sum_{n=1}^{\infty}|n^i - n^j||a_n|(|z|^{n+1} + c|z|^n)}{(A-B)a_0 - \sum_{n=1}^{\infty}|Bn^i - An^j||a_n|(|z|^{n+1} + c|z|^n) - |B|\alpha|e^{i(\theta+\psi)}|\sum_{n=1}^{\infty}|n^i - n^j||a_n|(|z|^{n+1} + c|z|^n)} \\
&\leqslant \frac{(1+c)(1+\alpha)\sum_{n=1}^{\infty}|n^i - n^j||a_n|}{(A-B)a_0 - \sum_{n=1}^{\infty}|Bn^i - An^j||a_n|(1+c) - |B|\alpha\sum_{n=1}^{\infty}|n^i - n^j||a_n|(1+c)}.
\end{aligned}$$

The last expression is bounded above by 1, if

$$\begin{aligned}
&(1+c)(1+\alpha)\sum_{n=1}^{\infty}|n^i - n^j||a_n| \\
&\leqslant (A-B)a_0 - \sum_{n=1}^{\infty}|Bn^i - An^j||a_n|(1+c) - |B|\alpha\sum_{n=1}^{\infty}|n^i - n^j||a_n|(1+c)
\end{aligned}$$

which is equivalent to our condition (2.1).

Conversely, we need only show that each function $f(z)$ of the class $CK_\eta^\varepsilon(i,j;\alpha,A,B)$ satisfies the coefficient inequality (2.1). Let $f(z)\in CK_\eta^\varepsilon(i,j;\alpha,A,B)$, then by (2.3) and (1.1), we obtain

$$\left|\frac{\sum_{n=1}^{\infty}(n^i-n^j)a_n(z^{n+1}-cz^n)-\alpha e^{i\theta}e^{i\psi}\left|\sum_{n=1}^{\infty}(n^i-n^j)a_n(z^{n+1}-cz^n)\right|}{(A-B)a_0-\sum_{n=1}^{\infty}(Bn^i-An^j)a_n(z^{n+1}-cz^n)+B\alpha e^{i\theta}e^{i\psi}\left|\sum_{n=1}^{\infty}(n^i-n^j)a_n(z^{n+1}-cz^n)\right|}\right|<1.$$

Therefore putting $z=re^{i\gamma}(0\leqslant r<1)$ we have

$$\frac{\sum_{n=1}^{\infty}|n^i-n^j||a_n|(r^{n+1}+cr^n)+\alpha\sum_{n=1}^{\infty}|n^i-n^j||a_n|(r^{n+1}+cr^n)|}{(A-B)a_0-\sum_{n=1}^{\infty}|Bn^i-An^j||a_n|(r^{n+1}+cr^n)-|B|\alpha\sum_{n=1}^{\infty}|n^i-n^j||a_n|(r^{n+1}+cr^n)}<1.$$

It is clear that the denominator of the left hand said cannot vanish for $0\leqslant r<1$. Thus, we obtain

$$\sum_{n=1}^{\infty}[(1+(1+|B|)\alpha)|n^i-n^j|+|Bn^i-An^j|](r^{n+1}+cr^n)|a_n|\leqslant(A-B)a_0,$$

which upon letting $r\to1^-$, readily yields the assertion (2.1). This completes the proof of our theorem. □

From Theorem 1 we have the following result.

COROLLARY 1. *If* $f(z)\in CK_\eta^\varepsilon(i,j;\alpha,A,B)$ *and*

$$\phi_{i,j}(n,c,\alpha,A,B)\geqslant\phi_{i,j}(1,c,\alpha,A,B), \tag{2.4}$$

*then we have*

$$\sum_{n=1}^{\infty}|a_n|\leqslant\frac{a_0}{1+c}.$$

*Moreover, if*

$$\phi_{i,j}(n,c,\alpha,A,B)\geqslant n\phi_{i,j}(1,c,\alpha,A,B), \tag{2.5}$$

*then we have*

$$\sum_{n=1}^{\infty}n|a_n|\leqslant\frac{a_0}{1+c}.$$

## 3. Distortion theorem

THEOREM 2. *If $f(z)$ defined by (1.1) is in the class $CK_\eta^\varepsilon(i,j;\alpha,A,B)$ for $|z| = r < 1$, and $\phi_{i,j}(n,c,\alpha,A,B)$ defined by (2.2) satisfies (2.4), then we have*

$$\frac{a_0}{|z-c|} - r\frac{a_0}{1+c} \leqslant |f(z)| \leqslant \frac{a_0}{|z-c|} + r\frac{a_0}{1+c}. \tag{3.1}$$

*Moreover, if (2.5) holds, then*

$$\frac{a_0}{|z-c|^2} - \frac{a_0}{1+c} \leqslant |f'(z)| \leqslant \frac{a_0}{|z-c|^2} + \frac{a_0}{1+c}. \tag{3.2}$$

*Proof.* Let $f(z)$ be given by (1.1). For $|z| = r < 1$, we have

$$|f(z)| \leqslant \frac{a_0}{|z-c|} + \sum_{n=1}^{\infty} |a_n||z|^n \leqslant \frac{a_0}{|z-c|} + |z|\sum_{n=1}^{\infty} |a_n| = \frac{a_0}{|z-c|} + r\sum_{n=1}^{\infty} |a_n|$$

and

$$|f(z)| \geqslant \frac{a_0}{|z-c|} - \sum_{n=1}^{\infty} |a_n||z|^n \geqslant \frac{a_0}{|z-c|} - |z|\sum_{n=1}^{\infty} |a_n| = \frac{a_0}{|z-c|} - r\sum_{n=1}^{\infty} |a_n|.$$

Then by Corollary 1 we get (3.1). Analogously we can prove (3.2). This completes the proof of our theorem. □

## 4. Radius of close-to-convexity, Starlikeness and Convexity

We concentrate upon getting the radius of close-to-convexity, Starlikeness and Convexity.

THEOREM 3. *Let the function $f(z)$ defined by (1.1) be in the class $CK_\eta^\varepsilon(i,j;\alpha,A,B)$. Then $f(z)$ is close-to-convex of order $\mu$ $(0 \leqslant \mu < 1)$ in $|z-c| < |z| < r_1$, where*

$$r_1 = \inf_n \left\{ \frac{(1-\mu)\phi_{i,j}(n,c,\alpha,A,B)}{n(A-B)} \right\}^{\frac{1}{n+1}} \quad (n \geqslant 1)$$

*and $\phi_{i,j}(n,c,\alpha,A,B)$ is given by (2.2).*

*Proof.* We must show that

$$\left| \frac{(z-c)^2}{a_0} f'(z) + 1 \right| < 1-\mu \quad (0 \leqslant \mu < 1) \tag{4.1}$$

for $|z| < r_1$. Note that

$$\left| \frac{(z-c)^2}{a_0} f'(z) + 1 \right| = \left| \sum_{n=1}^{\infty} \frac{na_n}{a_0} z^{n-1}(z-c)^2 \right| \leqslant \sum_{n=1}^{\infty} \frac{n|a_n|}{|a_0|} |z|^{n-1}|z-c|^2.$$

Thus for $|z-c|<|z|<r$, (4.1) holds true if

$$\sum_{n=1}^{\infty}\frac{nr^{n+1}}{a_0(1-\mu)}|a_n|\leqslant 1. \tag{4.2}$$

By Theorem1, we have

$$\sum_{n=1}^{\infty}\frac{\phi_{i,j}(n,c,\alpha,A,B)}{(A-B)a_0}|a_n|\leqslant 1. \tag{4.3}$$

Hence (4.2) will be true, if

$$\frac{n}{a_0(1-\mu)}r^{n+1}\leqslant\frac{\phi_{i,j}(n,c,\alpha,A,B)}{(A-B)a_0},$$

or equivalently if

$$r\leqslant\left\{\frac{(1-\mu)\phi_{i,j}(n,c,\alpha,A,B)}{n(A-B)}\right\}^{\frac{1}{n+1}}\quad(n\geqslant 1). \tag{4.4}$$

The theorem follows from (4.4). □

THEOREM 4. *Let the function $f(z)$ defined by (1.1) be in the class $CK_{\eta}^{\varepsilon}(i,j;\alpha,A,B)$. Then $f(z)$ is starlike of $\eta$ $(0\leqslant\eta<1)$ in $|z-c|<|z|<r_2$, where*

$$r_2=\inf_{n}\{\frac{(1-\eta)\phi_{i,j}(n,c,\alpha,A,B)}{(2+n-\eta)(A-B)}\}^{\frac{1}{n+1}}\quad(n\geqslant 1)$$

*and $\phi_{i,j}(n,c,\alpha,A,B)$ is given by (2.2).*

*Proof.* It suffices to show that

$$\left|\frac{(z-c)f'(z)}{f(z)}+1\right|<1-\eta\quad(0\leqslant\eta<1) \tag{4.5}$$

for $|z|<r_2$. Note that

$$\begin{aligned}\left|\frac{(z-c)f'(z)}{f(z)}+1\right|&=\left|\frac{(z-c)(-\frac{a_0}{(z-c)^2}+\sum_{n=1}^{\infty}na_nz^{n-1})+\frac{a_0}{z-c}+\sum_{n=1}^{\infty}a_nz^n}{\frac{a_0}{z-c}+\sum_{n=1}^{\infty}a_nz^n}\right|\\&=\left|\frac{\sum_{n=1}^{\infty}na_nz^{n-1}(z-c)^2+\sum_{n=1}^{\infty}a_nz^n(z-c)}{a_0+\sum_{n=1}^{\infty}a_nz^n(z-c)}\right|\\&\leqslant\frac{\sum_{n=1}^{\infty}n|a_n||z|^{n-1}|z-c|^2+\sum_{n=1}^{\infty}|a_n||z|^n|z-c|}{a_0-\sum_{n=1}^{\infty}|a_n||z|^n|z-c|}.\end{aligned}$$

Thus for $|z-c|<|z|<r$, (4.5) holds true if

$$\sum_{n=1}^{\infty}(n+1)|a_n|r^{n+1} \leqslant 1-\eta\,(a_0-\sum_{n=1}^{\infty}|a_n|r^{n+1})$$

or

$$\sum_{n=1}^{\infty}\frac{(2+n-\eta)r^{n+1}}{(1-\eta)a_0}|a_n| \leqslant 1.$$

By using (4.3), (4.5), we have

$$\frac{2+n-\eta}{(1-\eta)a_0}r^{n+1} \leqslant \frac{\phi_{i,j}(n,c,\alpha,A,B)}{(A-B)a_0},$$

or equivalently

$$r \leqslant \left\{\frac{(1-\eta)\phi_{i,j}(n,c,\alpha,A,B)}{(2+n-\eta)(A-B)}\right\}^{\frac{1}{n+1}} \quad (n \geqslant 1). \tag{4.6}$$

The theorem follows from (4.6). □

THEOREM 5. *Let the function $f(z)$ defined by (1.1) be in the class $CK_{\eta}^{\varepsilon}(i,j;\alpha,A,B)$. Then $f(z)$ is convex of order $\xi$ $(0 \leqslant \xi < 1)$ in $|z-c|<|z|<r_3$, where*

$$r_3 = \inf_n\left\{\frac{(1-\xi)\phi_{i,j}(n,c,\alpha,A,B)}{(n^2+2n-n\xi)(A-B)}\right\}^{\frac{1}{n+1}} \quad (n \geqslant 1)$$

*and $\phi_{i,j}(n,c,\alpha,A,B)$ is given by (2.2).*

*Proof.* It suffices to show that

$$\left|\frac{(z-c)f''(z)}{f'(z)}+2\right| < 1-\xi \quad (0 \leqslant \xi < 1) \tag{4.7}$$

for $|z|<r_3$. Note that

$$\left|\frac{(z-c)f''(z)}{f'(z)}+2\right| = \left|\frac{\sum_{n=1}^{\infty}n(n-1)a_nz^{n-2}(z-c)+\sum_{n=1}^{\infty}2na_nz^{n-1}}{-\frac{a_0}{(z-c)^2}+\sum_{n=1}^{\infty}na_nz^{n-1}}\right|$$
$$\leqslant \frac{\sum_{n=1}^{\infty}n(n-1)|a_n||z|^{n-2}|z-c|^3+\sum_{n=1}^{\infty}2n|a_n||z|^{n-1}|z-c|^2}{a_0-\sum_{n=1}^{\infty}n|a_n||z|^{n-1}|z-c|^2}.$$

Thus for $|z-c|<|z|<r$, (4.7) holds true if

$$\sum_{n=1}^{\infty}n(n-1)|a_n|r^{n+1}+\sum_{n=1}^{\infty}2n|a_n|r^{n+1} \leqslant (1-\xi)(a_0-\sum_{n=1}^{\infty}n|a_n|r^{n+1})$$

or

$$\sum_{n=1}^{\infty}\frac{(n^2+2n-n\xi)r^{n+1}}{(1-\xi)a_0}|a_n|\leqslant 1. \tag{4.8}$$

By using (4.3), we get that (4.8) is true if

$$\frac{n^2+2n-n\xi}{(1-\xi)a_0}r^{n+1}\leqslant\frac{\phi_{i,j}(n,c,\alpha,A,B)}{(A-B)a_0}$$

or equivalently

$$r\leqslant\left\{\frac{(1-\xi)\phi_{i,j}(n,c,\alpha,A,B)}{(n^2+2n-n\xi)(A-B)}\right\}^{\frac{1}{n+1}}\quad(n\geqslant 1).$$

And this completes the proof of the theorem. □

## 5. Extreme points

THEOREM 6. *Let the function $f(z)$ be defined by (1.1). We define*

$$f_0(z)=\frac{a_0}{z-c},$$

$$f_n(z)=\frac{a_0}{z-c}+\frac{(A-B)a_0}{\phi_{i,j}(n,c,\alpha,A,B)}z^n\quad(n=1,2,3,\cdots), \tag{5.1}$$

*where $\phi_{i,j}(n,c,\alpha,A,B)$ is defined by (2.2), then $f(z)\in CK_\eta^\xi(i,j;\alpha,A,B)$ if and only if it can be expressed in the form*

$$f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z),$$

*where $\lambda_n>0$ and $\sum_{n=0}^{\infty}\lambda_n=1$.*

*Proof.* Suppose that

$$f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z)=\frac{a_0}{z-c}+\sum_{n=1}^{\infty}\lambda_n\frac{(A-B)a_0}{\phi_{i,j}(n,c,\alpha,A,B)}z^n.$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty}\phi_{i,j}(n,c,\alpha,A,B)\left|\lambda_n\frac{(A-B)a_0}{\phi_{i,j}(n,c,\alpha,A,B)}\right| &= (A-B)a_0\sum_{n=1}^{\infty}\lambda_n\\ &= (A-B)a_0(1-\lambda_0)\\ &< (A-B)a_0.\end{aligned}$$

Thus $f(z)\in CK_\eta^\xi(i,j;\alpha,A,B)$.

Conversely, suppose that $f(z) \in CK_{\eta}^{\varepsilon}(i,j;\alpha,A,B))$. By using Theorem 1, we get

$$|a_n| \leqslant \frac{(A-B)a_0}{\phi_{i,j}(n,c,\alpha,A,B)} \quad (n=1,2,3,\cdots),$$

we may set

$$\lambda_n = \frac{\phi_{i,j}(n,c,\alpha,A,B)}{(A-B)a_0}|a_n| \quad (n=1,2,3,\cdots),$$

and

$$\lambda_0 = 1-\sum_{n=1}^{\infty}\lambda_n.$$

Then

$$f(z) = \sum_{n=0}^{\infty}\lambda_n f_n(z).$$

This completes the proof of the theorem. □

## 6. Integral means inequality

LEMMA. (Littlewood [24]) *If $f(z)$ and $g(z)$ are analytic in $U$ with $f(z) \prec g(z)$, then for $\mu > 0$ and $z = re^{i\theta}(0 < r < 1)$*

$$\int_0^{2\pi} |f(z)|^{\mu}d\theta \leqslant \int_0^{2\pi} |g(z)|^{\mu}d\theta.$$

THEOREM 7. *Let $f(z) \in CK_{\eta}^{\varepsilon}(i,j;\alpha,A,B)$ and suppose that $f_n(z)$ is defined by (5.1). If there exists an analytic function $\omega(z)$ given by*

$$\{\omega(z)\}^n = \frac{\phi_{i,j}(n,c,\alpha,A,B)}{(A-B)a_0}\sum_{n=1}^{\infty}a_n z^n,$$

*then for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(re^{i\theta})|^{\mu}d\theta \leqslant \int_0^{2\pi} |f_n(re^{i\theta})|^{\mu}d\theta.$$

*Proof.* For $z = re^{i\theta}$ $(0 < r < 1)$, we must show that

$$\int_0^{2\pi}\left|\frac{a_0}{z-c}+\sum_{n=1}^{\infty}a_n z^n\right|^{\mu}d\theta \leqslant \int_0^{2\pi}\left|\frac{a_0}{z-c}+\frac{A-B}{\phi_{i,j}(n,c,,\alpha,A,B)}z^n\right|^{\mu}d\theta(\mu>0).$$

Thus by applying Littlewood's subordination theorem, it would suffice to show that

$$\frac{a_0}{z-c}+\sum_{n=1}^{\infty}a_nz^n \prec \frac{a_0}{z-c}+\frac{(A-B)a_0}{\phi_{i,j}(n,c,\alpha,A,B)}z^n.$$

By setting

$$\frac{a_0}{z-c}+\sum_{n=1}^{\infty}a_nz^n = \frac{a_0}{z-c}+\frac{(A-B)a_0}{\phi_{i,j}(n,c,\alpha,A,B)}\{\omega(z)\}^n,$$

we find that

$$\{\omega(z)\}^n=\frac{\phi_{i,j}(n,c,\alpha,A,B)}{(A-B)a_0}\sum_{n=1}^{\infty}a_nz^n,$$

which readily yields $\omega(0)=0$. Therefore, we have

$$\begin{aligned}|\omega(z)|^n &= \left|\frac{\phi_{i,j}(n,c,\alpha,A,B)}{(A-B)a_0}\sum_{n=1}^{\infty}a_nz^n\right| \\ &\leqslant \frac{\phi_{i,j}(n,c,\alpha,A,B)}{(A-B)a_0}\sum_{n=1}^{\infty}|a_n||z|^n \\ &\leqslant |z| \\ &< 1\end{aligned}$$

by means of the hypothesis of the Theorem 7. □

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