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ON CERTAIN PROPERTIES FOR A SUBCLASS OF CLOSE–TO–CONVEX FUNCTIONS

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Abstract. In the present paper, we introduce and investigate an interesting subclass $\mathscr{K}_s^{\lambda}(h)$ of analytic and close-to-convex functions in the open unit disk $\mathbb{U}$. For functions belonging to the class $\mathscr{K}_s^{\lambda}(h)$, we derive several properties as the convolution, the coefficient bounds, the covering theorem, the inclusion relationships as well as distortion theorem. The various results presented here would generalize many known recent results.

1. Introduction, definition and preliminaries

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ \ (|a_n| \geqslant 0) \tag{1}$$

which are analytic and univalent in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathscr{S}^*(\alpha)$ be the subclass of $\mathscr{A}$ consisting of functions which satisfy the condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \ (z \in \mathbb{U}; 0 \leqslant \alpha < 1).$$

$\mathscr{S}^*(\alpha)$ is known as subclass of *starlike functions of order* α.

Further a function $f(z)$ is said to belongs to the class $\mathscr{K}(\alpha)$ of *close-to-convex function of order* α in $\mathbb{U}$ if $g(z) \in \mathscr{S}^*(\alpha)$ and satisfy following inequality

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha \ (z \in \mathbb{U}; 0 \leqslant \alpha < 1).$$

Obviously $\mathscr{S}^*(0) := \mathscr{S}^*$ and $\mathscr{K}(0) := \mathscr{K}$.
In many earlier investigations, various interesting subclasses of analytic functions class $\mathscr{A}$ and the univalent functions class S have been studied from a number of different view points. In particular, Gao and Zhou [1] introduced a subclass $\mathscr{K}_s$ of analytic functions, which is indeed a subclass of close-to-convex functions.

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DEFINITION 1. (see [1]) Let the function $f(z)$ be analytic in $\mathbb{U}$ and defined by (1). We say that $f \in \mathscr{K}_s$, if there exists function $g \in \mathscr{S}^*(1/2)$ such that

$$\Re\left(-\frac{z^2 f'(z)}{g(z)g(-z)}\right) > 0, \quad (z \in \mathbb{U}). \tag{2}$$

In a paper Kowalczyk and Les-Bomba [2] extended Definition 1 by introducing the following subclass of analytic functions.

DEFINITION 2. (see [2]) Let the function $f(z)$ be analytic in $\mathbb{U}$ and defined by (1). We say that $f \in \mathscr{K}_s(\alpha)$, if there exists function $g \in \mathscr{S}^*(1/2)$ such that

$$\Re\left(-\frac{z^2 f'(z)}{g(z)g(-z)}\right) > \alpha, \quad (z \in \mathbb{U}, 0 \leqslant \alpha < 1). \tag{3}$$

DEFINITION 3. (see e.g. [3]) For two functions f and g, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, then above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Using above definition of subordination, Wang and Chen [5] introduced and studied a following subclass of analytic functions

DEFINITION 4. A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{K}_s(\lambda, A, B)$ if there exists a function $g \in \mathscr{S}^*(1/2)$ such that

$$-\frac{z^2 f'(z) + \lambda z^3 f''(z)}{g(z)g(-z)} \prec \frac{1 + Az}{1 + Bz}, \quad (z \in \mathbb{U})$$

where $0 \leqslant \lambda \leqslant 1$ and $-1 \leqslant B < A \leqslant 1$.

Motivated by aforementioned works we now introduce the following subclass of analytic functions :

DEFINITION 5. Let

$$h : \mathbb{U} \longrightarrow \mathbb{C} \text{ (the set of complex numbers)}$$

be a convex function such that

$$h(0) = 1 \text{ and } h(\bar{z}) = \overline{h(z)} \ (z \in \mathbb{U}; \Re(h(z)) > 0).$$

Suppose also that the function h satisfies the following condition for $r \in (0,1)$:

$$\begin{cases} \min_{|z|=r} |h(z)| = \min\{h(r), h(-r)\} & (0 < r < 1) \\ \max_{|z|=r} |h(z)| = \max\{h(r), h(-r)\} & (0 < r < 1) \end{cases} \tag{4}$$

Let the function $f(z)$ be analytic in $\mathbb{U}$ and defined by (1). We say that $f \in \mathscr{K}_s^{\lambda}(h)$ if these exists a function $g \in \mathscr{S}^*(1/2)$ such that

$$-\frac{z^2 f'(z) + \lambda z^3 f''(z)}{g(z)g(-z)} \in h(\mathbb{U}), \quad (z \in \mathbb{U}; 0 \leqslant \lambda \leqslant 1). \tag{5}$$

REMARK 1. There are many choices of function h which would provide interesting subclasses of analytic functions. For example, if we let

$$h(z) = \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}; -1 \leqslant B < A \leqslant 1),$$

then it is easily verified that h is a convex function in $\mathbb{U}$ and satisfies the hypothesis of Definition 5. If $f \in \mathscr{K}_s^{\lambda}(h)$, then

$$-\frac{z^2 f'(z) + \lambda z^3 f''(z)}{g(z)g(-z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

where $0 \leqslant \lambda \leqslant 1$, $-1 \leqslant B < A \leqslant 1$ and $g \in \mathscr{S}^*(1/2)$.

That is, $f \in \mathscr{K}_s(\lambda, A, B)$, a class recently studied by Wang and Chen [5].

REMARK 2. Further, for $\lambda = 0$ and

$$h(z) = \frac{1+(1-2\gamma)z}{1-z} \quad (0 \leqslant \gamma < 1; z \in \mathbb{U}),$$

in (5), we get the class $\mathscr{K}_s(\gamma)$ given in Definition 2 and for $\gamma = 0$, we obtain a class recently studied by Xu et al. [8].

In this work, by using the definition of principle of subordination, we obtain convolution properties, coefficient bounds, covering theorem, inclusion theorem and distortion theorem for functions in the function class $\mathscr{K}_s^{\lambda}(h)$. Our results unify and extend the corresponding results obtained by Xu et al. [8], Wang et al. [6], Gao and Zhou [1] and Kowalczyk and Les-Bomba [2].

2. Main Results

To prove our main results, we shall require the following:

LEMMA 1. (see [4]) *Let the function $h(z)$ given by*

$$h(z) = \sum_{n=1}^{\infty} h_n z^n$$

be convex in $\mathbb{U}$. *Suppose also that the function* $f(z)$ *given by*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

is holomorphic in $\mathbb{U}$. *If* $f(z) \prec h(z)$ $(z \in \mathbb{U})$, *then*

$$|a_n| \leqslant |h_1| \quad (n \in \mathbb{N}).$$

LEMMA 2. (see [1]) *Let* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathscr{S}^*(1/2)$. *Then*

$$G(z) = -\frac{g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in \mathscr{S}^* \quad (z \in \mathbb{U})$$

is an odd starlike function and

$$|B_{2n-1}| = |2b_{2n-1} - 2b_2 b_{2n-2} + ... + (-1)^n 2b_{n-1} b_{n+1} + (-1)^{n+1} b_n^2| \leqslant 1,$$
$$(n \in \mathbb{N}^* := \mathbb{N} \backslash \{1\} = \{2, 3, 4....\}).$$

LEMMA 3. *Let* $\gamma = 0$ *and* $f \in \mathscr{K}$. *Then*

$$H(z) = \frac{1+\gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt \in \mathscr{K}.$$

The above Lemma is a special case of Theorem 4 obtained by Wu [7].

We now state and prove that the main results of our present investigation. The result contained in following theorem is obvious

THEOREM 1. *An analytic function* $f \in \mathscr{K}_s^\lambda(h)$ *if and only if there exists a function* $g \in \mathscr{S}^*(1/2)$ *such that*

$$-\frac{z^2 f'(z) + \lambda z^3 f''(z)}{g(z)g(-z)} \prec h(z) \quad (0 \leqslant \lambda \leqslant 1; z \in \mathbb{U}). \tag{6}$$

THEOREM 2. *Let* $0 \leqslant \lambda . \leqslant 1$. *Then*

$$\mathscr{K}_s^\lambda(h) \subset \mathscr{K} \subset \mathscr{S}^*. \tag{7}$$

Proof. Suppose that $F(z) := (1-\lambda) f(z) + \lambda z f'(z)$ and $G(z) = -\frac{g(z)g(-z)}{z}$ with $f \in \mathscr{K}_s^\lambda(h)$.
Then (6) can be written as

$$\frac{zF'(z)}{G(z)} \prec h(z) \quad (z \in \mathbb{U}).$$

By Lemma 2, we know that $G \in \mathscr{S}^*$. Thus we have

$$F(z) := (1-\lambda)f(z) + \lambda z f'(z) \in \mathscr{K}.$$

Now we consider two cases :
Case I: When $\lambda = 0$, it is obvious that $f = F \in \mathscr{K}$.
Case II: When $\lambda \in (0,1]$. By definition of $F(z)$, we have

$$f(z) = \frac{1}{\lambda} z^{1-\left(\frac{1}{\lambda}\right)} \int_0^z t^{\left(\frac{1}{\lambda}\right)-2} F(t)dt \tag{8}$$

Since $\gamma = \left(\frac{1}{\lambda}\right) - 1 \geqslant 0$, therefore by Lemma 3 that $f \in \mathscr{K}$. Hence we have $\mathscr{K}_s^{\lambda}(h) \subset \mathscr{K} \subset \mathscr{S}^*$. □

THEOREM 3. *Let the function $f(z)$ be given by (1) and $f \in \mathscr{K}_s^{\lambda}(h)$. Then*

$$1 + \sum_{k=2}^{\infty} \left(\frac{k + \lambda k(k-1)}{1-h(x)} \right) a_k z^k - \sum_{k=2}^{\infty} \left(\frac{h(x) B_{2k-1}}{1-h(x)} \right) z^{2k-2} \neq 0 \tag{9}$$

and $|x| = 1$.

Proof. Suppose that

$$-\frac{z^2 f'(z) + \lambda z^3 f''(z)}{g(z)g(-z)} \neq h(x) \quad (|x| = 1; z \in U)$$

or

$$f'(z) + \lambda z f''(z) \neq h(x) \left(-\frac{g(z)g(-z)}{z^2} \right)$$

Using Lemma 2 and simplifying, we get the desired result (9). □

On taking

$$h(x) = h(e^{i\theta}) = \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} \quad (0 < \theta < 2\pi; -1 \leqslant B < A \leqslant 1) \tag{10}$$

in Theorem 3, we obtain

COROLLARY 1. *Let the function $f(z)$ defined by (1) and $f \in \mathscr{K}_s^{\lambda}[A,B]$. Then*

$$(1+|B|) \sum_{k=2}^{\infty} \{k + \lambda k(k-1)\} |a_k| + (1+|A|) \sum_{k=2}^{\infty} |B_{2k-1}| \leqslant A - B. \tag{11}$$

Proof. Using (10) in (9), we get

$$1 + \sum_{k=2}^{\infty} \frac{\{k + \lambda k(k-1)\}(B + e^{-i\theta})}{B - A} a_k z^{k-1} - \sum_{k=2}^{\infty} \frac{(A + e^{-i\theta})}{B - A} B_{2k-1} z^{2k-2} \neq 0.$$

Now

$$\left|1+\sum_{k=2}^{\infty}\frac{\{k+\lambda k(k-1)\}(B+e^{-i\theta})}{B-A}a_k z^{k-1}-\sum_{k=2}^{\infty}\frac{(A+e^{-i\theta})}{B-A}B_{2k-1}z^{2k-2}\right|$$
$$\geqslant 1-\left|\sum_{k=2}^{\infty}\frac{\{k+\lambda k(k-1)\}(B+e^{-i\theta})}{B-A}a_k z^{k-1}-\sum_{k=2}^{\infty}\frac{(A+e^{-i\theta})}{B-A}B_{2k-1}z^{2k-2}\right|$$
$$\geqslant 1-\left|\sum_{k=2}^{\infty}\frac{\{k+\lambda k(k-1)\}(B+e^{-i\theta})}{B-A}a_k z^{k-1}\right|-\left|\sum_{k=2}^{\infty}\frac{(A+e^{-i\theta})}{B-A}B_{2k-1}z^{2k-2}\right|$$
$$\geqslant 1-\sum_{k=2}^{\infty}\frac{\{k+\lambda k(k-1)\}(|B|+1)}{A-B}|a_k|-\sum_{k=2}^{\infty}\frac{(|A|+1)}{A-B}|B_{2k-1}|\geqslant 0.$$

From it result (11) follows at once. □

On substituting $A=1-2\gamma\,(0\leqslant\gamma<1/2)$ and $B=-1$ in Corollary 1, we get

COROLLARY 2. *Let $f(z)$ be defined by (1) and $f\in\mathscr{K}_s(\gamma)$. Then*

$$\sum_{k=2}^{\infty}\{k+\lambda k(k-1)\}|a_k|+(1-\gamma)\sum_{k=2}^{\infty}|B_{2k-1}|\leqslant 1-\gamma. \tag{12}$$

REMARK 3. On putting $A=\beta$ and $B=-\alpha\beta$ and $\lambda=0$ in Corollary, we get a known result obtained by Wang et al. [6].

THEOREM 4. *Let the function $f(z)$ be defined by (1) and $f\in\mathscr{K}_s^{\lambda}(h)$. Then*

$$|a_{2k}|\leqslant\frac{k|h'(0)|}{2\{1+(2k-1)\lambda\}}\quad(k\in\mathbb{N}) \tag{13}$$

and

$$|a_{2k+1}|\leqslant\frac{k|h'(0)|+1}{(2k+1)\{1+2k\lambda\}}\quad(k\in\mathbb{N}). \tag{14}$$

.

Proof. According to Definition 1, there exists $g(z)\in\mathscr{S}^*(1/2)$ such that inclusion relationship (7) holds true.

Furthermore, by using Lemma 2, we know that the function $G(z)$ is defined by

$$G(z)=-\frac{g(z)g(-z)}{z}=z+\sum_{k=2}^{\infty}B_{2k-1}z^{2k-1},$$

is an odd starlike function and that

$$|B_{2k-1}|\leqslant 1. \tag{15}$$

We thus find from (5) that

$$\frac{zf'(z)+\lambda z^2 f''(z)}{G(z)}\in h(\mathbb{U}).$$

Next by setting

$$p(z)=\frac{zf'(z)+\lambda z^2 f''(z)}{G(z)} \quad (z\in\mathbb{U}). \tag{16}$$

We deduce that

$$p(0)=h(0)=1 \text{ and } p(z)\in h(\mathbb{U}).$$

Thus we have

$$p(z)\prec h(z) \ \ (z\in\mathbb{U}).$$

According to Lemma 1, we find that

$$|p_k|=\left|\frac{p_k(0)}{k!}\right|\leqslant |h'(0)| \quad (k\in\mathbb{N}). \tag{17}$$

On other hand, we readily find from (16) that

$$zf'(z)+\lambda z^2 f''(z)=p(z)G(z) \quad (z\in\mathbb{U}) \tag{18}$$

Further, by setting

$$p(z)=1+p_1 z+p_2 z^2+...(z\in\mathbb{U}) \tag{19}$$

and using (15), (18) and (19), we deduce that

$$2k[1+(2k-1)\lambda]|a_{2k}|=p_{2k-1}+p_{2k-3}B_3+p_{2k-5}B_5+...+p_2B_{2k-1} \ \ (k\in\mathbb{N})$$

and

$$(2k+1)[1+(2k+1)\lambda]|a_{2k+1}|=p_{2k}+p_{2k-2}B_3+p_{2k-4}B_5+...+p_2B_{2k-1}+B_{2k+1} \ \ (k\in\mathbb{N}).$$

Again using Lemma 2, and (17) we get

$$2k[1+(2k-1)\lambda]|a_{2k}|\leqslant k|h'(0)|$$

and

$$(2k+1)[1+2k\lambda]|a_{2k+1}|\leqslant k|h'(0)|+1$$

which gives desired result. □

REMARK 4. The known result due to Wang and Chen [5] and Xu et al. [8] are special cases of Theorem 4.

THEOREM 5. *Let the function $f(z)$ is defined by (1) and $f\in\mathscr{K}_s^{\lambda}(h)$. Then the unit disk $\mathbb{U}$ is mapped by f on a domain that contain the disk $|w|<r_1$ where*

$$r_1=\frac{2(1+\lambda)}{|h'(0)|+4(1-\lambda)}. \tag{20}$$

Proof. Suppose that $f\in\mathscr{K}_s^{\lambda}(h)$ and let w_0 be any complex number such that $f(z)\neq w_0$ for $z\in\mathbb{U}$. Then $w_0\neq 0$ and

$$\frac{w_0 f(z)}{w_0-f(z)}=z+\left(a_2+\frac{1}{w_0}\right)z^2+...$$

is univalent by Theorem 2. This leads to

$$\left|a_2+\frac{1}{w_0}\right| \leqslant 2. \tag{21}$$

On other hand, form Theorem 4, we have

$$|a_2| \leqslant \frac{|h'(0)|}{2(1+\lambda)}. \tag{22}$$

Combining (21) and (22), we deduce that

$$|w_0| \geqslant \frac{1}{|a_2|+2} \geqslant \frac{2(1+\lambda)}{|h'(0)|+4(1-\lambda)} = r_1$$

which completes the proof of Theorem 5. □

THEOREM 6. *Let the function $f(z)$ be defined by (1) and $f \in \mathcal{K}_s^{\lambda}(h)$. Then*
(i) If $\lambda = 0$, then for $|z| = r < 1$, we have

$$\int_0^r \frac{\min\{h(\tau),h(-\tau)\}}{1+\tau^2}d\tau \leqslant |f(z)| \leqslant \int_0^r \frac{\max\{h(\tau),h(-\tau)\}}{1-\tau^2}d\tau. \tag{23}$$

(ii) If $0 < \lambda \leqslant 1$, then for $|z| = r < 1$, we have

$$\frac{1}{\lambda}r^{1-(\frac{1}{\lambda})}\int_0^r\int_0^s \frac{\min\{h(\tau),h(-\tau)\}}{1+\tau^2}s^{\frac{2}{\lambda}-1}dsd\tau \leqslant |f(z)|$$

$$\leqslant \frac{1}{\lambda}r^{1-(\frac{1}{\lambda})}\int_0^r\int_0^s \frac{\max\{h(\tau),h(-\tau)\}}{1-\tau^2}s^{\frac{2}{\lambda}-1}dsd\tau. \tag{24}$$

Proof. Since $f \in \mathcal{K}_s^{\lambda}(h)$, therefore there exists a function $g \in \mathcal{S}^*(1/2)$ such that (4) holds. It follows from Lemma 2, that the function given by

$$G(z) = -\frac{g(z)g(-z)}{z} \quad (z \in \mathbb{U})$$

is an odd starlike function. It is well known that

$$\frac{r}{1+r^2} \leqslant |G(z)| \leqslant \frac{r}{1-r^2} \quad (|z| = r; 0 \leqslant r < 1). \tag{25}$$

Again we have

$$\frac{zf'(z)+\lambda z^2 f''(z)}{G(z)} \prec h(z) \quad (z \in \mathbb{U}).$$

Also using (4), we have

$$\min\{h(r),h(-r)\} \leqslant \left|\frac{zf'(z)+\lambda z^2 f''(z)}{G(z)}\right| \leqslant \min\{h(r),h(-r)\} \quad (|z|=r;0\leqslant r<1). \tag{26}$$

From (25) and (26), we get

$$\frac{\min\{h(r),h(-r)\}}{1+r^2} \leqslant \left|zf'(z)+\lambda z^2 f''(z)\right| \leqslant \frac{\max\{h(r),h(-r)\}}{1-r^2} \quad (|z|=r;0\leqslant r<1).$$

Let

$$F(z) = (1-\lambda)f(z)+\lambda zf'(z).$$

Then

$$F'(z) = zf'(z)+\lambda z^2 f''(z)$$

and inequality (27) can be written as

$$\frac{\min\{h(r),h(-r)\}}{1+r^2} \leqslant \left|F'(z)\right| \leqslant \frac{\max\{h(r),h(-r)\}}{1-r^2} \quad (|z|=r;0\leqslant r<1). \tag{28}$$

Let $z=re^{i\theta}$ $(0<r<1)$. If ξ denote the closed line - segment in the complex ζ-plane form $\zeta=0$ to $\zeta=z$, we have

$$F(z) = \int_{\xi} F'(\zeta)d\zeta = \int_0^r F'(\tau e^{i\theta})e^{i\theta}d\tau \quad (|z|=r;0\leqslant r<1).$$

Thus using the upper estimate in (28), we have

$$\begin{aligned}|F(z)| &= \left|\int_{\xi} F'(\zeta)d\zeta\right| \leqslant \int_0^r \left|F'(\tau e^{i\theta})\right| d\tau \\ &\Rightarrow |(1-\lambda)f(z)+\lambda zf'(z)| \leqslant \int_0^r \frac{\max\{h(\tau),h(-\tau)\}}{1-\tau^2} \quad (|z|=r;0\leqslant r<1).\end{aligned}$$

In order to prove lower bound in (23), it is sufficient to show that it holds true for z_0 nearest to zero where

$$|z_0| = r \ \ (0\leqslant r<1).$$

Moreover, we have

$$|F(z)| \geqslant |F(z_0)| \quad (|z|=r;0\leqslant r<1).$$

Since $f(z)$ is close-to-convex function in the open unit disk $\mathbb{U}$, it is univalent in $\mathbb{U}$. We deduce that the original image of the closed-line segment ξ_0 in the complex ζ-plane from $\zeta=0$ and $\zeta=f(z_0)$ along the piece of arc Γ in the disk $\mathbb{U}_r$ given by $\mathbb{U}_r=\{z: z\in\mathbb{C}\ and\ |z|=r;0\leqslant r<1\}$.
Hence we have

$$|F(z_0)| = \int_{F(\Gamma)} |dw| = \int_{\Gamma} |F'(z)||dz| \geqslant \int_0^r \frac{\min\{h(\tau),h(-\tau)\}}{1+\tau^2}d\tau \quad (|z|=r;0\leqslant r<1).$$

Thus

$$\int_0^r \frac{\min\{h(\tau),h(-\tau)\}}{1+\tau^2}d\tau \leqslant |(1-\lambda)f(z)+\lambda zf'(z)| \leqslant \int_0^r \frac{\max\{h(\tau),h(-\tau)\}}{1-\tau^2}d\tau. \quad (29)$$

To complete the proof, we consider following to cases :
(i) When $\lambda = 0$. From (29), we easily get (23).
(ii) When $0 < \lambda \leqslant 1$. Then by using (8) together with (29), we easily arrive at (24). □

REMARK 5. The results obtained by Xu et al. ([8], Th. 3), and Kowalcyk and Les-Bomba ([2], Th. 6) and Wang et al. ([6], Th. 5.1) are special cases of our Theorem 6.

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