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BIORTHOGONAL p–WAVELET PACKETS RELATED TO THE WALSH POLYNOMIALS

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Abstract. This paper deals with the construction of biorthogonal p-wavelet packets on $\mathbb{R}^+$ related to the Walsh polynomials and their properties are investigated by means of Walsh-Fourier transform. Three biorthogonal formulas regarding these p-wavelet packets are derived. Moreover, it is shown how to obtain several new Riesz bases of the space $L^2(\mathbb{R}^+)$ by constructing a series of subspaces of these p-wavelet packets.

1. Introduction

In the early nineties a general scheme for the construction of wavelets was defined. This scheme is based on the notion of multiresolution analysis (MRA) introduced by Mallat [15]. Immediately specialists started to implement new wavelet systems and in recent years, the concept MRA of $\mathbb{R}^n$ has been extended to many different setups, for example, Dahlke introduced multiresolution analysis and wavelets on locally compact Abelian groups [5], Lang [13,14] constructed compactly supported orthogonal wavelets on the locally compact Cantor dyadic group $\mathscr{C}$ by following the procedure of Daubechies [6] via scaling filters and these wavelets turn out to be certain lacunary Walsh series on the real line. Later on, Farkov [7] extended the results of Lang [13,14] on the wavelet analysis on the Cantor dyadic group $\mathscr{C}$ to the locally compact Abelian group G which is defined for an integer $p \geqslant 2$ and coincides with $\mathscr{C}$ when $p=2$. The construction of dyadic compactly supported wavelets for $L^2(\mathbb{R}^+)$ have been given by Protasov and Farkov in [16] where the latter author has given the general construction of all compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}^+)$ arising from scaling filters with p^n many terms in [8]. These studies were continued by Farkov and his colleagues in [9,10], where they have given some new algorithms for constructing the corresponding biorthogonal p-wavelets via biorthogonal scaling functions on the positive half-line $\mathbb{R}^+$.

It is well-known that the classical orthonormal wavelet bases have poor frequency localization. For example, if the wavelet ψ is band limited, then the measure of the supp of $(\psi_{j,k})^\wedge$ is 2^j-times that of supp $\hat{\psi}$. To overcome this disadvantage, Coifman *et al*. [4] constructed univariate orthogonal wavelet packets. The fundamental idea of wavelet packet analysis is to construct a library of orthonormal bases for $L^2(\mathbb{R})$, which can be searched in real time for the best expansion with respect to a given application.

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The standard construction is to start from a multiresolution analysis (MRA) and generate the library using the associated quadrature mirror filters (QMFs). The internal structure of the MRA and the speed of the decomposition schemes make this an efficient adaptive method for simultaneous time and frequency analysis of signals. Later on, Chui and Li [2] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be applied to the spline wavelets and so on. The introduction of biorthogonal wavelet packets attributes to Cohen and Daubechies [3]. Shen [20] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. Other notable generalizations are the orthogonal version of vector-valued wavelet packets [1] and higher dimensional wavelet packets with arbitrary dilation matrix [12].

In his recent paper, Shah [18] has constructed p-wavelet packets on the positive half-line $\mathbb{R}^+$ using the classical splitting trick of wavelets whereas Shah and Debnath in [19] have constructed the corresponding p-wavelet frame packets on $\mathbb{R}^+$ using the Walsh-Fourier transform. It is well known that the orthogonal wavelet packets have many desired properties such as compact support, good frequency localization and vanishing moments. However, there is no continuous symmetry which is a much desired property in imaging the compression and signal processing (see [6]). To achieve symmetry, several generalizations of scalar orthogonal wavelet packets have been investigated in literature. The biorthogonal wavelet packets achieve symmetry where the orthogonality is replaced by the biorthogonality (see[1,3]). As one of a series of works on positive half-line $\mathbb{R}^+$, the objective of this paper is to investigate certain properties of biorthogonal p-wavelet packets on the positive half-line $\mathbb{R}^+$ by means of the Walsh-Fourier transform and construct several new Riesz bases of space $L^2(\mathbb{R}^+)$.

In order to make the paper self-contained, we state some basic preliminaries, notation and definitions including the Walsh-Fourier transform, Walsh functions and p-MRA in Section 2. In Section 3, we examine some properties of the biorthogonal p-wavelet packets. In Section 4, we study the decomposition of space $L^2(\mathbb{R}^+)$.

2. Preliminaries and p-wavelets related to the Walsh polynomials

Let p be a fixed natural number greater than 1. As usual, let $\mathbb{R}^+ = [0,+\infty)$, $\mathbb{Z}^+ = \{0,1,2,...\}$ and $\mathbb{N} = \mathbb{Z}^+ - \{0\}$. Set $\Omega_0 = \{0,1,2,...,p-1\}$ and $\Omega = \Omega_0 - \{0\}$. Denote by $[x]$ the integer part of x. For $x \in \mathbb{R}^+$ and any positive integer j, we set

$$x_j = [p^j x](\mathrm{mod}\, p), x_{-j} = [p^{1-j}x](\mathrm{mod}\, p). \tag{2.1}$$

We consider on $\mathbb{R}^+$ the addition defined as follows: if $z = x \oplus y$, then

$$z = \sum_{j<0} \zeta_j p^{-j-1} + \sum_{j>0} \zeta_j p^{-j}$$

with $\zeta_j = x_j + y_j(\mathrm{mod} p)(j \in \mathbb{Z} \setminus \{0\})$, where $\zeta_j \in \Omega_0$ and x_j, y_j are calculated by (2.1). Moreover, we note that $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo p in $\mathbb{R}^+$.

For $x \in [0,1)$, let $r_0(x)$ be given by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0,1/p) \\ \varepsilon_p^\ell, & \text{if } x \in [\ell p^{-1}, (\ell+1)p^{-1}),\ \ell \in \Omega, \end{cases}$$

where $\varepsilon_p = \exp(2\pi i/p)$. The extension of the function r_0 to $\mathbb{R}^+$ is denoted by the equality $r_0(x+1) = r_0(x)$, $x \in \mathbb{R}^+$. Then, the generalized Walsh functions $\{w_m(x) : m \in \mathbb{Z}^+\}$ are defined by

$$w_0(x) \equiv 1, \quad w_m(x) = \prod_{j=0}^{k} \left(r_0(p^j x)\right)^{\mu_j}$$

where $m = \sum_{j=0}^{k} \mu_j p^j$, $\mu_j \in \Omega_0$, $\mu_k \neq 0$.

For $x, w \in \mathbb{R}^+$, let

$$\chi(x,w) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j w_{-j} + x_{-j} w_j)\right), \tag{2.2}$$

where x_j, w_j are given by (2.1). Note that $\chi(x, m/p^{n-1}) = \chi(x/p^{n-1}, m) = w_m(x/p^{n-1})$ for all $x \in [0, p^{n-1})$, $m \in \mathbb{Z}^+$.

The Walsh-Fourier transform of a function $f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^+} f(x) \overline{\chi(x,\xi)}\, dx, \tag{2.3}$$

where $\chi(x,\xi)$ is given by (2.2). The Walsh-Fourier operator $\mathscr{F} : L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+), \mathscr{F} f = \hat{f}$, extends uniquely to the whole space $L^2(\mathbb{R}^+)$. The properties of the Walsh-Fourier transform are quite similar to those of the classic Fourier transform (see [11, 17]). In particular, if $f \in L^2(\mathbb{R}^+)$, then $\hat{f} \in L^2(\mathbb{R}^+)$ and

$$\|\hat{f}\|_{L^2(\mathbb{R}^+)} = \|f\|_{L^2(\mathbb{R}^+)}.$$

If $x, y, \xi \in \mathbb{R}^+$ and $x \oplus y$ is p-adic irrational, then

$$\chi(x \oplus y, \xi) = \chi(x,\xi)\,\chi(y,\xi). \tag{2.4}$$

It is shown by Golubov *et al.* [11] that both the systems $\{\chi(\alpha,.)\}_{\alpha=0}^{\infty}$ and $\{\chi(.,\alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2[0,1]$.

DEFINITION 2.1. We say that a pair of functions $f(x)$, $\tilde{f}(x) \in L^2(\mathbb{R}^+)$ are biorthogonal, if their translates satisfy

$$\langle f(.), \tilde{f}(. \ominus k)\rangle = \delta_{0,k}, k \in \mathbb{Z}^+, \tag{2.5}$$

where $\delta_{0,k}$ is Kronecker symbol, i.e., $\delta_{0,k} = 1$ when $k = 0$ and $\delta_{0,k} = 0$, otherwise.

DEFINITION 2.2. Let $\mathbb{H}$ be a Hilbert space. A sequence $\{f_k\}_{k=1}^{\infty}$ of $\mathbb{H}$ is said to be a Riesz basis for $\mathbb{H}$ if there exist constants A and $B, 0 < A \leqslant B < \infty$ such that any $f \in \mathbb{H}$ can be represented as a series $f = \sum_{k=1}^{\infty} c_k f_k$ converging in $\mathbb{H}$ with

$$A\|f\|^2 \leqslant \sum_{k=1}^{\infty} |c_k|^2 \leqslant B\|f\|^2. \tag{2.6}$$

In the following subsection, we introduce the notion of p-multiresolution analysis (p-MRA) on $\mathbb{R}^+$ and some of its properties.

DEFINITION 2.3. A p-multiresolution analysis of $L^2(\mathbb{R}^+)$ is a nested sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ such that

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^+)$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(iii) $f \in V_j$ if and only if $f(p.) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(iv) there exists a function φ in V_0, called the *scaling function*, such that $\{\varphi(. \ominus k) : k \in \mathbb{Z}^+\}$ forms a Riesz basis for subspace V_0.

Since $\varphi(x) \in V_0 \subset V_1$, by Definition 2.3 and (2.6), there exists a sequence $\{a_k\}_{k\in\mathbb{Z}^+} \in l^2(\mathbb{Z}^+)$ such that

$$\varphi(x) = \sum_{k\in\mathbb{Z}^+} a_k \, \varphi(px \ominus k). \tag{2.7}$$

The Walsh-Fourier transform of (2.7) is given by

$$\hat{\varphi}(\xi) = m_0(p^{-1}\xi)\, \hat{\varphi}(p^{-1}\xi), \tag{2.8}$$

where $m_0(\xi) = \sum_{k\in\mathbb{Z}^+} a_k \overline{\chi(k,\xi)}$, is a Walsh polynomial called the *symbol* of $\varphi(x)$.

Let $W_j, j \in \mathbb{Z}$ be the direct complementary subspaces of V_j in V_{j+1}. Assume that there exist a set of $p-1$ functions $\{\psi_1, \psi_2, ..., \psi_{p-1}\}$ in $L^2(\mathbb{R}^+)$ such that their translates and dilations form a Riesz basis of W_j, i.e.,

$$W_j = \overline{\text{span}}\{\psi_\ell(p^j. \ominus k) : k \in \mathbb{Z}^+, \ell \in \Omega\}, \quad j \in \mathbb{Z}. \tag{2.9}$$

Since $\psi_\ell(x) \in W_0 \subset V_1, \ell \in \Omega$, there exists a sequence $\{a_k^\ell\}_{k\in\mathbb{Z}^+}$ in $l^2(\mathbb{Z}^+)$ such that

$$\psi_\ell(x) = \sum_{k\in\mathbb{Z}^+} a_k^\ell \, \varphi(px \ominus k), \quad \ell \in \Omega. \tag{2.10}$$

Taking the Walsh- Fourier transform for both sides of (2.10) gives

$$\hat{\psi}_\ell(\xi) = m_\ell(p^{-1}\xi)\, \hat{\varphi}(p^{-1}\xi), \tag{2.11}$$

where

$$m_\ell(\xi) = \sum_{k\in\mathbb{Z}^+} a_k^\ell \overline{\chi(k,\xi)}, \quad \ell \in \Omega.$$

If $\varphi(x), \tilde{\varphi}(x) \in L^2(\mathbb{R}^+)$ are a pair of biorthogonal scaling functions, then it follows by Definition 2.1 that

$$\langle \varphi(.), \tilde{\varphi}(. \ominus k) \rangle = \delta_{0,k}, \quad k \in \mathbb{Z}^+. \tag{2.12}$$

Moreover, we say that $\psi_\ell(x), \tilde{\psi}_\ell(x) \in L^2(\mathbb{R}^+), \ell \in \Omega,$ are pair of biorthogonl wavelets associated with a pair of biorthogonal scaling functions $\varphi(x)$ and $\tilde{\varphi}(x)$, if the family $\{\psi_\ell(.\ominus k) : k \in \mathbb{Z}^+, \ell \in \Omega\}$ is a Riesz basis of subspace W_0, and

$$\langle \varphi(.), \tilde{\psi}_\ell(.\ominus k)\rangle = 0, \quad \ell \in \Omega,\ k \in \mathbb{Z}^+, \tag{2.13}$$

$$\langle \tilde{\varphi}(.), \psi_\ell(.\ominus k)\rangle = 0, \quad \ell \in \Omega,\ k \in \mathbb{Z}^+, \tag{2.14}$$

$$\langle \psi_\ell(.), \tilde{\psi}_{\ell'}(.\ominus k)\rangle = \delta_{\ell,\ell'}\delta_{0,k}, \quad \ell,\ell' \in \Omega,\ k \in \mathbb{Z}^+. \tag{2.15}$$

Set

$$W_j^\ell = \overline{\text{span}}\left\{\psi_\ell(p^j.\ominus k) : k \in \mathbb{Z}^+\right\}, \quad \ell \in \Omega,\ j \in \mathbb{Z}. \tag{2.16}$$

By definition of W_j and formulae (2.12)–(2.15), we obtain the following proposition.

PROPOSITION 2.4. *If $\psi_\ell(x), \tilde{\psi}_\ell(x) \in L^2(\mathbb{R}^+), \ell \in \Omega$ are a pair of biorthogonal p-wavelets associated with a pair of biorthogonal scaling functions $\varphi(x), \tilde{\varphi}(x)$, then*

$$L^2(\mathbb{R}^+) = \bigoplus_{j\in\mathbb{Z}} W_j = \bigoplus_{j\in\mathbb{Z}}\bigoplus_{\ell\in\Omega} W_j^\ell \tag{2.17}$$

where $\oplus$ denotes the direct sum.

Similar to the refinement equation (2.7) and wavelet equation (2.10), in the biorthogonal setting, we have the following equations

$$\tilde{\varphi}(x) = \sum_{k\in\mathbb{Z}^+} \tilde{a}_k \tilde{\varphi}(px\ominus k). \tag{2.18}$$

$$\tilde{\psi}_\ell(x) = \sum_{k\in\mathbb{Z}^+} \tilde{a}_k^\ell \tilde{\varphi}(px\ominus k), \quad \ell \in \Omega \tag{2.19}$$

whose Walsh- Fourier transforms are respectively given by

$$\hat{\tilde{\varphi}}(\xi) = \tilde{m}_0(p^{-1}\xi)\,\hat{\tilde{\varphi}}(p^{-1}\xi), \tag{2.20}$$

$$\hat{\tilde{\psi}}_\ell(\xi) = \tilde{m}_\ell(p^{-1}\xi)\,\hat{\tilde{\varphi}}(p^{-1}\xi), \tag{2.21}$$

where $\tilde{m}_\ell(\xi) = \sum_{k\in\mathbb{Z}^+} \tilde{a}_k^\ell \overline{\chi(k,\xi)},\ \ell \in \Omega.$

For the biorthogonal scaling functions $\varphi(x), \tilde{\varphi}(x)$, we have the following properties:

LEMMA 2.5. ([8]) *Let $\varphi(x), \tilde{\varphi}(x)$ be a pair of scaling functions. Then $\varphi(x), \tilde{\varphi}(x)$ are biorthogonal if and only if*

$$\sum_{k\in\mathbb{Z}^+} \hat{\varphi}(\xi\oplus k)\overline{\hat{\tilde{\varphi}}(\xi\oplus k)} = 1 \qquad \textit{for a.e. } \xi \in \mathbb{R}^+.$$

3. The properties of biorthogonal p-wavelet packets

We begin this section with the definition of basic biorthogonal p-wavelet packets associated with the biorthogonal scaling functions φ and $\tilde{\varphi}$. Then, we examine their properties and advantages by means of the Walsh-Fourier transform.

For $n = 0,1,2,\ldots$, the *basic biorthogonal p-wavelet packets* ω_n and $\tilde{\omega}_n$ associated with a pair of biorthogonal scaling functions $\varphi(x)$ and $\tilde{\varphi}(x)$, respectively, are defined recursively by

$$\omega_n(x) = \omega_{pr+s}(x) = \sum_{k\in\mathbb{Z}^+} p\, a_k^s\, \omega_r(px \ominus k), \quad s \in \Omega_0 \tag{3.1}$$

$$\tilde{\omega}_n(x) = \tilde{\omega}_{pr+s}(x) = \sum_{k\in\mathbb{Z}^+} p\, \tilde{a}_k^s\, \tilde{\omega}_r(px \ominus k), \quad s \in \Omega_0. \tag{3.2}$$

where $r \in \mathbb{Z}^+$ is the unique element such that $n = pr+s$, $s \in \Omega_0$ holds. Note that for $r = 0$ and $s \in \Omega$, we have

$$\omega_0(x) = \varphi(x),\ \tilde{\omega}_0(x) = \tilde{\varphi}(x),\ \omega_s(x) = \psi_\ell(x) \quad \text{and} \quad \tilde{\omega}_s(x) = \tilde{\psi}_\ell(x).$$

The Walsh-Fourier transform of (3.1) and (3.2) are given by

$$\hat{\omega}_{pr+s}(\xi) = m_s(p^{-1}\xi)\hat{\omega}_r(p^{-1}\xi), \quad s \in \Omega_0, \tag{3.3}$$

and

$$\hat{\tilde{\omega}}_{pr+s}(\xi) = \tilde{m}_s(p^{-1}\xi)\hat{\tilde{\omega}}_r(p^{-1}\xi), \quad s \in \Omega_0. \tag{3.4}$$

respectively.

We are now in a position to discuss the biorthogonality properties for these wavelet packets by virtue of the Walsh-Fourier transform.

LEMMA 3.1. *Assume that $\omega_s(x), \tilde{\omega}_s(x) \in L^2(\mathbb{R}^+), s \in \Omega$ are a pair of biorthogonal p-wavelets associated with a pair of biorthogonal scaling functions $\omega_0(x)$ and $\tilde{\omega}_0(x)$. Then, we have*

$$\sum_{\ell\in\Omega_0} m_r\big(p^{-1}(\xi\oplus\ell)\big)\overline{\tilde{m}_s\big(p^{-1}(\xi\oplus\ell)\big)} = \delta_{r,s}, \quad r,s \in \Omega_0. \tag{3.5}$$

Proof. Using (2.12)–(2.16), (3.3), (3.4) and Lemma 2.5, we have

$$\begin{aligned}
\delta_{r,s} &= \sum_{k\in\mathbb{Z}^+} \omega_r(\xi\oplus k)\overline{\tilde{\omega}_s(\xi\oplus k)} \\
&= \sum_{k\in\mathbb{Z}^+} m_r\big(p^{-1}(\xi\oplus k)\big)\hat{\omega}_0\big(p^{-1}(\xi\oplus k)\big)\overline{\hat{\tilde{\omega}}_0\big(p^{-1}(\xi\oplus k)\big)}\overline{\tilde{m}_s\big(p^{-1}(\xi\oplus k)\big)} \\
&= \sum_{\ell\in\Omega_0} m_r\big(p^{-1}(\xi\oplus\ell)\big)\left\{\sum_{k\in\mathbb{Z}^+}\hat{\omega}_0\big(p^{-1}(\xi\oplus k)\oplus\ell\big)\overline{\hat{\tilde{\omega}}_0\big(p^{-1}(\xi\oplus k)\oplus\ell\big)}\right\} \\
&\quad \times\overline{\tilde{m}_s\big(p^{-1}(\xi\oplus\ell)\big)} \\
&= \sum_{\ell\in\Omega_0} m_r\big(p^{-1}(\xi\oplus\ell)\big)\overline{\tilde{m}_s\big(p^{-1}(\xi\oplus\ell)\big)}. \quad \square
\end{aligned}$$

THEOREM 3.2. *If* $\{\omega_n(x) : n \in \mathbb{Z}^+\}$ *and* $\{\tilde{\omega}_n(x) : n \in \mathbb{Z}^+\}$ *are* p*-wavelet packets with respect to a pair of biorthogonal scaling functions* $\omega_0(x)$ *and* $\tilde{\omega}_0(x)$*, respectively. Then, for* $n \in \mathbb{Z}^+$*, we have*

$$\langle \omega_n(.), \tilde{\omega}_n(. \ominus k)\rangle = \delta_{0,k}, \quad k \in \mathbb{Z}^+. \tag{3.6}$$

Proof. We prove this result by using induction on n. It follows from (2.12) and (2.15) that the claim is true for $n = 0$ and $n = 1, 2, ..., p-1$. Assume that (3.6) holds when $n < q$, where $q \in \mathbb{N}$. Then, we prove the result (3.6) for $n = q$. Let $n = pr + s$, where $r \in \mathbb{Z}^+, s \in \Omega_0$ and $r < n$. Therefore, by the inductive assumption, we have

$$\langle \omega_r(.), \tilde{\omega}_r(. \ominus k)\rangle = \delta_{0,k} \Longleftrightarrow \sum_{k \in \mathbb{Z}^+} \hat{\omega}_r(\xi \oplus k)\overline{\hat{\tilde{\omega}}_r(\xi \oplus k)} = 1, \quad \xi \in \mathbb{R}^+.$$

Using Lemmas 2.5, 3.1 and, equations (3.3) and (3.4), we obtain

$$\begin{aligned}
\langle \omega_n(.), \tilde{\omega}_n(. \ominus k)\rangle &= \langle \hat{\omega}_n(.), \hat{\tilde{\omega}}_n(. \ominus k)\rangle \\
&= \int_{\mathbb{R}^+} \hat{\omega}_{pr+s}(\xi)\,\overline{\hat{\tilde{\omega}}_{pr+s}(\xi)}\chi(k,\xi)d\xi \\
&= \int_{\mathbb{R}^+} m_s(p^{-1}\xi)\hat{\omega}_r(p^{-1}\xi)\,\overline{\tilde{m}_s(p^{-1}\xi)}\,\overline{\hat{\tilde{\omega}}_r(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \sum_{k \in \mathbb{Z}^+} \int_{p([0,1]+k)} m_s(p^{-1}\xi)\hat{\omega}_r(p^{-1}\xi)\,\overline{\tilde{m}_s(p^{-1}\xi)}\,\overline{\hat{\tilde{\omega}}_r(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{p[0,1]} m_s(p^{-1}\xi)\left\{\sum_{k \in \mathbb{Z}^+} \hat{\omega}_r\big(p^{-1}(\xi \oplus k)\big)\,\overline{\hat{\tilde{\omega}}_r\big(p^{-1}(\xi \oplus k)\big)}\right\} \\
&\quad \times \overline{\tilde{m}_s(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{p[0,1]} m_s(p^{-1}\xi)\overline{\tilde{m}_s(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]} \sum_{\ell \in \Omega_0} m_s\big(p^{-1}(\xi \oplus \ell)\big)\overline{\tilde{m}_s\big(p^{-1}(\xi \oplus \ell)\big)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]} \chi(k,\xi)d\xi = \delta_{0,k}. \quad \square
\end{aligned}$$

THEOREM 3.3. *Suppose* $\{\omega_n(x) : n \in \mathbb{Z}^+\}$ *and* $\{\tilde{\omega}_n(x) : n \in \mathbb{Z}^+\}$ *are* p*-wavelet packets with respect to a pair of biorthogonal scaling functions* $\omega_0(x)$ *and* $\tilde{\omega}_0(x)$*, respectively. Then, for* $r \in \mathbb{Z}^+$*, we have*

$$\langle \omega_{pr+s_1}(.), \tilde{\omega}_{pr+s_2}(. \ominus k)\rangle = \delta_{0,k}\delta_{s_1,s_2}, \quad s_1, s_2 \in \Omega_0,\ k \in \mathbb{Z}^+. \tag{3.7}$$

Proof. By Lemma 2.5, we have

$$\begin{aligned}
\langle \omega_{pr+s_1}, \tilde{\omega}_{pr+s_2}(. \ominus k)\rangle &= \langle \hat{\omega}_{pr+s_1}, \hat{\tilde{\omega}}_{pr+s_2}(. \ominus k)\rangle \\
&= \int_{\mathbb{R}^+} \hat{\omega}_{pr+s_1}(\xi)\,\overline{\hat{\tilde{\omega}}_{pr+s_2}(\xi)}\chi(k,\xi)d\xi \\
&= \int_{\mathbb{R}^+} m_{s_1}(p^{-1}\xi)\hat{\omega}_r(p^{-1}\xi)\,\overline{\tilde{m}_{s_2}(p^{-1}\xi)}\,\overline{\hat{\tilde{\omega}}_r(p^{-1}\xi)}\chi(k,\xi)d\xi
\end{aligned}$$

$$
\begin{aligned}
&= p \sum_{k\in\mathbb{Z}^+} \int_{([0,1]+k)} m_{s_1}(\xi)\hat{\omega}_r(\xi)\overline{\tilde{m}_{s_2}(\xi)}\,\overline{\hat{\tilde{\omega}}_r(\xi)}\chi(k,p\xi)d\xi \\
&= p\int_{[0,1]} m_{s_1}(\xi)\left\{\sum_{k\in\mathbb{Z}^+}\hat{\omega}_r(\xi\oplus k)\overline{\hat{\tilde{\omega}}_r(\xi\oplus k)}\right\}\overline{\tilde{m}_{s_2}(\xi)}\chi(k,p\xi)d\xi \\
&= \int_{p[0,1]} m_{s_1}(p^{-1}\xi)\overline{\tilde{m}_{s_2}(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]}\sum_{\ell\in\Omega_0} m_{s_1}\left(p^{-1}(\xi\oplus\ell)\right)\overline{\tilde{m}_{s_2}\left(p^{-1}(\xi\oplus\ell)\right)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]}\delta_{s_1,s_2}\chi(k,\xi)d\xi = \delta_{0,k}\delta_{s_1,s_2}. \quad \square
\end{aligned}
$$

THEOREM 3.4. *If* $\{\omega_n(x) : n\in\mathbb{Z}^+\}$ *and* $\{\tilde{\omega}_n(x) : n\in\mathbb{Z}^+\}$ *are* p*-wavelet packets with respect to a pair of biorthogonal scaling functions* $\omega_0(x)$ *and* $\tilde{\omega}_0(x)$*, respectively. Then, for* $\ell,n\in\mathbb{Z}^+$*, we have*

$$
\langle\omega_\ell(.),\tilde{\omega}_n(.\ominus k)\rangle = \delta_{\ell,n}\delta_{0,k}, \quad k\in\mathbb{Z}^+. \tag{3.8}
$$

Proof. For $\ell = n$, the result (3.8) follows by Theorem 3.2. When $\ell\neq n$, and $\ell,n\in\Omega_0$, the result (3.8) can be established from Theorem 3.3. Assuming that ℓ is not equal to n, and at least one of (ℓ,n) does not belong to Ω_0, then we can rewrite ℓ, n as $\ell = pr_1+s_1$, $n = pu_1+v_1$, where $r_1,u_1\in\mathbb{Z}^+$, $s_1,v_1\in\Omega_0$.

Case 1. If $r_1 = u_1$, then $s_1\neq v_1$. Therefore, (3.8) follows by virtue of (3.3)–(3.5) and Lemma 2.5, i.e.,

$$
\begin{aligned}
\langle\omega_\ell(.),\tilde{\omega}_n(.\ominus k)\rangle &= \langle\omega_{pr_1+s_1},\tilde{\omega}_{pu_1+v_1}(.\ominus k)\rangle \\
&= \langle\hat{\omega}_{pr_1+s_1},\hat{\tilde{\omega}}_{pu_1+v_1}(.\ominus k)\rangle \\
&= \int_{\mathbb{R}^+}\hat{\omega}_{pr_1+s_1}(\xi)\overline{\hat{\tilde{\omega}}_{pu_1+v_1}(\xi)}\chi(k,\xi)d\xi \\
&= \int_{\mathbb{R}^+} m_{s_1}(p^{-1}\xi)\hat{\omega}_{r_1}(p^{-1}\xi)\overline{\hat{\tilde{\omega}}_{u_1}(p^{-1}\xi)}\,\overline{\tilde{m}_{v_1}(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \sum_{k\in\mathbb{Z}^+}\int_{p([0,1]+k)} m_{s_1}(p^{-1}\xi)\hat{\omega}_{r_1}(p^{-1}\xi)\overline{\hat{\tilde{\omega}}_{u_1}(p^{-1}\xi)}\,\overline{\tilde{m}_{v_1}(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{p([0,1])} m_{s_1}(p^{-1}\xi)\left\{\sum_{k\in\mathbb{Z}^+}\hat{\omega}_{r_1}\left(p^{-1}(\xi\oplus k)\right)\overline{\hat{\tilde{\omega}}_{u_1}\left(p^{-1}(\xi\oplus k)\right)}\right\} \\
&\quad\times\overline{\tilde{m}_{v_1}(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]}\sum_{\ell\in\Omega_0} m_{s_1}\left(p^{-1}(\xi\oplus\ell)\right)\overline{\tilde{m}_{v_1}\left(p^{-1}(\xi\oplus\ell)\right)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]}\delta_{s_1,v_1}\chi(k,\xi)d\xi \\
&= \delta_{0,k} = 0.
\end{aligned}
$$

Case 2. If $r_1 \neq u_1$, order $r_1 = pr_2 + s_2$, $u_1 = pu_2 + v_2$, where $r_2, u_2 \in \mathbb{Z}^+$ and $s_2, v_2 \in \Omega_0$. If $r_2 = u_2$, then $s_2 \neq v_2$. Similar to Case 1, (3.8) can be established. When $r_2 \neq u_2$, we order $r_2 = pr_3 + s_3$, $u_2 = pu_3 + v_3$, where $r_3, u_3 \in \mathbb{Z}^+$ and $s_3, v_3 \in \Omega_0$. Thus, after taking finite steps (denoted by κ), we obtain $r_\kappa, u_\kappa \in \Omega_0$ and $s_\kappa, v_\kappa \in \Omega_0$. If $r_\kappa = u_\kappa$, then $s_\kappa \neq v_\kappa$. Similar to the Case 1, (3.8) follows. If $r_\kappa \neq u_\kappa$, then it gets from (2.12)–(2.15) that

$$\langle \omega_{r_\kappa}(.), \tilde{\omega}_{u_\kappa}(. \ominus k)\rangle = 0, k \in \mathbb{Z}^+ \Longleftrightarrow \sum_{k \in \mathbb{Z}^+} \hat{\omega}_{r_\kappa}(\xi \oplus k)\overline{\hat{\tilde{\omega}}_{u_\kappa}(\xi \oplus k)} = 0, \quad \xi \in \mathbb{R}^+.$$

Furthermore, we have

$$\begin{aligned}
&\langle \omega_r(.), \tilde{\omega}_u(. \ominus k)\rangle = \langle \hat{\omega}_r(.), \hat{\tilde{\omega}}_u(. \ominus k)\rangle \\
&= \langle \hat{\omega}_{pr_1+s_1}, \hat{\tilde{\omega}}_{pu_1+v_1}(. \ominus k)\rangle \\
&= \int_{\mathbb{R}^+} \hat{\omega}_{pr_1+s_1}(\xi)\, \overline{\hat{\tilde{\omega}}_{pu_1+v_1}(\xi)}\chi(k,\xi)d\xi \\
&= \int_{\mathbb{R}^+} m_{s_1}(p^{-1}\xi)m_{s_2}(p^{-2}\xi)\hat{\omega}_{r_2}(p^{-2}\xi)\,\overline{\hat{\tilde{\omega}}_{u_2}(p^{-2}\xi)}\,\overline{\tilde{m}_{v_1}(p^{-1}\xi)} \\
&\quad \times \overline{\tilde{m}_{v_2}(p^{-2}\xi)}\chi(k,\xi)d\xi \\
&\qquad \vdots \\
&= \int_{\mathbb{R}^+} \left\{\prod_{\ell=1}^{\kappa} m_{s_\ell}(p^{-\ell}\xi)\right\} \hat{\omega}_{r_\kappa}(p^{-\kappa}\xi)\,\overline{\hat{\tilde{\omega}}_{u_\kappa}(p^{-\kappa}\xi)} \left\{\prod_{\ell=1}^{\kappa} \overline{\tilde{m}_{v_\ell}(p^{-\ell}\xi)}\right\}\chi(k,\xi)d\xi \\
&= \sum_{k\in\mathbb{Z}^+} \int_{p^\kappa([0,1]+k)} \left\{\prod_{\ell=1}^{\kappa} m_{s_\ell}(p^{-\ell}\xi)\right\} \left\{\hat{\omega}_{r_\kappa}(p^{-\kappa}\xi)\,\overline{\hat{\tilde{\omega}}_{u_\kappa}(p^{-\kappa}\xi)}\right\} \\
&\quad \times \left\{\prod_{\ell=1}^{\kappa} \overline{\tilde{m}_{v_\ell}(p^{-\ell}\xi)}\right\}\chi(k,\xi)d\xi \\
&= \int_{p^\kappa[0,1]} \left\{\prod_{\ell=1}^{\kappa} m_{s_\ell}(p^{-\ell}\xi)\right\} \left\{\sum_{k\in\mathbb{Z}^+} \hat{\omega}_{r_\kappa}\big(p^{-\kappa}(\xi\oplus k)\big)\,\overline{\hat{\tilde{\omega}}_{u_\kappa}\big(p^{-\kappa}(\xi\oplus k)\big)}\right\} \\
&\quad \left\{\prod_{\ell=1}^{\kappa} \overline{\tilde{m}_{v_\ell}(p^{-\ell}\xi)}\right\}\chi(k,\xi)d\xi \\
&= \int_{p^\kappa[0,1]} \left\{\prod_{\ell=1}^{\kappa} m_{s_\ell}(p^{-\ell}\xi)\right\} .0. \left\{\prod_{\ell=1}^{\kappa} \overline{\tilde{m}_{v_\ell}(p^{-\ell}\xi)}\right\}\chi(k,\xi)d\xi \\
&= 0. \quad \square
\end{aligned}$$

4. The decomposition of $L^2(\mathbb{R}^+)$

In this section, we will decompose the subspaces $V_j, \tilde{V}_j$ and $W_j, \tilde{W}_j$ by constructing a series of subspaces of p-wavelet packets. Furthermore, we present a direct decomposition for space $L^2(\mathbb{R}^+)$.

For any $n \in \mathbb{Z}^+$, define

$$U_n = \left\{ f(x) : f(x) = \sum_{k\in\mathbb{Z}^+} b_k \omega_n(x \ominus k), \{b_k\}_{k\in\mathbb{Z}^+} \in l^2(\mathbb{Z}^+) \right\}, \tag{4.1}$$

$$\tilde{U}_n = \left\{ \tilde{f}(x) : \tilde{f}(x) = \sum_{k\in\mathbb{Z}^+} \tilde{b}_k \tilde{\omega}_n(x \ominus k), \left\{\tilde{b}_k\right\}_{k\in\mathbb{Z}^+} \in l^2(\mathbb{Z}^+) \right\}. \tag{4.2}$$

Clearly $U_0 = V_0$ and $U_s = W_0^s$, for each $s \in \Omega$. Assume that $\left\{m_s(p^{-1}(\xi \oplus k))\right\}_{s,k\in\Omega_0}$ is a unitary matrix.

LEMMA 4.1. *For $n \in \mathbb{Z}^+$, the space DU_n can be decomposed into direct sum of $U_{pn+s}, s \in \Omega_0$, i.e.,*

$$DU_n = \bigoplus_{s\in\Omega_0} U_{pn+s}, s \in \Omega_0. \tag{4.3}$$

where D is the dilation operator such that $Df(x) = f(px)$, for any $f \in L^2(\mathbb{R}^+)$.

Proof. First, we claim that

$$DU_n = \left\{ f(x) : f(x) = \sum_{s\in\Omega_0} \sum_{k\in\mathbb{Z}^+} b_k^s \omega_{pn+s}(x \ominus k), \{b_k^s\}_{k\in\mathbb{Z}^+} \in l^2(\mathbb{Z}^+) \right\}. \tag{4.4}$$

As for any $s \in \Omega_0$, by (3.1) and (4.1), $\omega_{pn+s}(x \ominus k) \in DU_n$. Assume that $f(x) \in DU_n$, then there exists a sequence $\{c_k\}_{k\in\mathbb{Z}^+} \in l^2(\mathbb{Z}^+)$ such that

$$f(x) = \sum_{k\in\mathbb{Z}^+} c_k \omega_n(px \ominus k). \tag{4.5}$$

Further, if there exists a sequence $\left\{b_k^s\right\}_{k\in\mathbb{Z}^+} \in l^2(\mathbb{Z}^+)$, $s \in \Omega_0$, as for $f(x) \in DU_n$, such that

$$f(x) = \sum_{s\in\Omega_0} \sum_{k\in\mathbb{Z}^+} b_k^s \omega_{pn+s}(x \ominus k). \tag{4.6}$$

Taking Walsh-Fourier transform on the both sides of (4.5) and (4.6), respectively and by using (3.3), we obtain

$$\hat{f}(\xi) = h(p^{-1}\xi)\hat{\omega}_n(p^{-1}\xi) = \sum_{s\in\Omega_0} g_s(\xi) m_s(p^{-1}\xi)\hat{\omega}_n(p^{-1}\xi), \tag{4.7}$$

where $h(\xi) = \sum_{k\in\mathbb{Z}^+} c_k \overline{\chi(\xi,k)}$, $g_s(\xi) = \sum_{k\in\mathbb{Z}^+} b_k^s \overline{\chi(\xi,k)}$, $\xi \in \mathbb{R}^+, s \in \Omega_0$.

The above result (4.7) follows if the following equality holds:

$$h(p^{-1}\xi) = \sum_{s\in\Omega_0} g_s(\xi) m_s(p^{-1}\xi). \tag{4.8}$$

For any $\{c_k\} \in l^2(\mathbb{Z}^+)$, we will prove that there exists a sequence $\left\{b_k^s\right\} \in l^2(\mathbb{Z}^+)$ such that (4.8) is satisfied. Moreover, (4.8) is equivalent to the following equation:

$$h\big(p^{-1}(\xi \oplus k)\big) = \sum_{s\in\Omega_0} g_s(\xi) m_s\big(p^{-1}(\xi \oplus k)\big), \ k \in \Omega_0. \tag{4.9}$$

The solvability of (4.9) for every sequence $\{c_k\} \in l^2(\mathbb{Z}^+)$ follows from the fact that the matrix $\left(m_s\left(p^{-1}(\xi \oplus k)\right)\right)_{s,k\in\Omega_0}$ is a unitary matrix (see [8]). Hence, equality (4.4) follows. Furthermore, applying Theorem 3.3, it follows that

$$\left\{\omega_{pn+s}(x \ominus k) : n \in \mathbb{Z}^+,\ s \in \Omega_0,\ k \in \mathbb{Z}^+\right\}$$

forms a Riesz basis of DU_n. $\square$

Similar to (4.3), we can establish the following result:

$$\tilde{U}_0 = \tilde{V}_0,\ \tilde{U}_s = \tilde{W}_0^s, \quad s \in \Omega,$$

and

$$D\tilde{U}_n = \bigoplus_{s\in\Omega_0} \tilde{U}_{pn+s}, \quad s \in \Omega_0. \tag{4.10}$$

For $\ell \in \mathbb{N}$, define $\tilde{\Lambda}_\ell = \sum_{j=0}^{\ell} p^j\Omega_0, \Lambda_\ell = \tilde{\Lambda}_\ell - \tilde{\Lambda}_{\ell-1}$. In what follows, we will give the direct decomposition of space $L^2(\mathbb{R}^+)$.

THEOREM 4.2. *The family of functions* $\{\omega_n(x\ominus k) : n \in \Lambda_\ell, k \in \mathbb{Z}^+\}$ *constitutes Riesz basis of* $D^\ell W_0$. *In particular,* $\{\omega_n(x\ominus k) : n \in \mathbb{Z}^+, k \in \mathbb{Z}^+\}$ *constitutes Riesz basis of* $L^2(\mathbb{R}^+)$.

Proof.. By equation (4.3), we have $DU_0 = \bigoplus_{s\in\Omega_0} U_s$, i.e., $DU_0 = U_0 \bigoplus_{s\in\Omega} U_s$. Since $U_0 = V_0$ and $W_0 = \bigoplus_{s\in\Omega} W_0^s = \bigoplus_{s\in\Omega} U_s$, then $DU_0 = V_0 \oplus W_0$. It can be inductively inferred from (4.3) that

$$D^\ell U_0 = D^{\ell-1}U_0 \bigoplus_{n\in\Lambda_\ell} U_n, \ell \in \mathbb{N}. \tag{4.11}$$

Since $V_{j+1} = V_j \oplus W_j, j \in \mathbb{Z}$, therefore, $D^\ell U_0 = D^{\ell-1}U_0 \bigoplus D^{\ell-1}W_0, \ell \in \mathbb{N}$. It follows from (4.3) and Proposition 2.4 that $D^\ell W_0 = \bigoplus_{n\in\Lambda_\ell} U_n$ and

$$L^2(\mathbb{R}^+) = V_0 \bigoplus \left(\bigoplus_{\ell\geqslant 0} D^\ell W_0\right) = U_0 \bigoplus \left(\bigoplus_{\ell\geqslant 0}\left(\bigoplus_{n\in\Lambda_\ell} U_n\right)\right) = \bigoplus_{n\in\mathbb{Z}^+} U_n. \tag{4.12}$$

In the light of Theorem 3.2, the family $\{\omega_n(x\ominus k) : k \in \mathbb{Z}^+\}$ is a Riesz basis of DW_0. Moreover, according to (4.12), the family $\{\omega_n(x\ominus k) : n \in \mathbb{Z}^+, k \in \mathbb{Z}^+\}$ forms a Riesz basis of $L^2(\mathbb{R}^+)$. $\square$

COROLLARY 4.3. *For every* $\ell \in \mathbb{N}$, *the family of functions* $\{\tilde{\omega}_n(x\ominus k) : n \in \Lambda_\ell, k \in \mathbb{Z}^+\}$ *forms a Riesz basis of* $D^\ell \tilde{W}_0$.

COROLLARY 4.4. *For every* $\ell \in \mathbb{N}$, *the family of functions*

$$\left\{\omega_n(p^j x\ominus k) : n \in \Lambda_\ell, j \in \mathbb{Z}, k \in \mathbb{Z}^+\right\}$$

forms a Riesz basis of $L^2(\mathbb{R}^+)$.

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