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ON THE UPPER BOUND OF THE NUMBER OF REAL ZEROS OF A RANDOM ALGEBRAIC POLYNOMIAL

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Abstract. Let N_n be the number of level crossings of a random algebraic curve $f(x,w) = \sum_{r=0}^{n} a_r \xi_r(w) x^r$ where the co-efficients $\xi_r(w)$'s are identically distributed independent random variables following semi-stable distribution with characteristic function $\exp(-(C+\cos\log|t|)|t|^{\alpha})$ for $0 < \alpha \leqslant 2$ and $C > 1$. It is proved that $N_n \leqslant \mu(\log n)^2$ in the weak version outside a set of measure less than $\frac{\mu^*}{n^{3\alpha-1-\varepsilon}} + \frac{\mu^{**}}{n^{1-\varepsilon}}$ where $0 < \varepsilon < 1$, and $N_n \leqslant \mu(\log n)^3$ in the strong version according to the sense of Evans, outside a set of measure less than $\frac{\mu'}{n_0^{\alpha\log n_0 - 2-\varepsilon}} + \frac{\mu''}{n_0^{\log n_0 - 1-\varepsilon}}$ where $0 < \alpha \leqslant 2$ and $0 < \varepsilon < 1$ for all $n \geqslant n_0$.

1. Introduction

This paper is concerned with the upper estimate for the level crossings of a random algebraic curve

$$f(x,w) = \sum_{r=0}^{n} a_r \xi_r(w) x^r \tag{1}$$

where the coefficients $\xi_r(w)$'s are identically distributed independent random variables following semi-stable distribution with characteristic function

$$\exp(-(C+\cos\log|t|)|t|^{\alpha})$$

for $0 < \alpha \leqslant 2$ and $C > 1$. A characteristic function $\phi(t)$ corresponding to distribution function $F(x)$ is said to be semi-stable if for some constants $b > 1$ and $d > 1$, $\phi(t) = \phi^b(d^{-1}t)$ for every t. Shimizu [9], [10], [11] and [12] have studied a lot on the domain of partial attraction of semi-stable distribution. On the basis of his findings, we have proceeded to the proof of our theorems.

Logan and Shepp, [4] showed that a number of above curves on an average would cross the X-axis asymptotically $2\pi\log n$ time when n is large. Sambandhan, [8] also studied the level of crossings of a random hyperbolic curve in different situations. Nayak and Mohanty, [6] studied the lower bound of level crossings of a random algebraic curve, and established a strong result in the sense of Evans, [1]. Nayak and Das, [5] studied about the bounds of level crossings when independent random variable belonging to the domain of attraction of symmetric normal law have zero mean and

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$P(\xi_r \neq 0) > 0$, and proved that $\frac{\mu \log n}{\log\log n} \leqslant N_n \leqslant \mu(\log n)^2$ except for a measure tends to zero as $n \longrightarrow \infty$.

In this paper, we have studied both weak version and strong version of upper bounds of N_n when the co-efficients $\xi_r(w)$ following semi-stable distribution and obtained $N_n \leqslant \mu(\log n)^2$ in the weak version outside a set of measure less than $\frac{\mu^*}{n^{3\alpha-1-\varepsilon}} + \frac{\mu^{**}}{n^{1-\varepsilon}}$ where $0 < \varepsilon < 1$, and $N_n \leqslant \mu(\log n)^3$ in the strong version outside a set of measure less than $\frac{\mu'}{n_0^{\alpha \log n_0 - 2 - \varepsilon}} + \frac{\mu''}{n_0^{\log n_0 - 1 - \varepsilon}}$ where $0 < \alpha \leqslant 2$ and $0 < \varepsilon < 1$ for all $n \geqslant n_0$. Throughout this paper we consider μ's as positive constants assuming different values in different occurrence.

2. Main Results

THEOREM 2.1. *Let N_n be the number of real zeros of random algebraic polynomial $f(x,w) = \sum_{r=0}^{n} a_r \xi_r(w) x^r$, where ξ_r's are identically distributed independent random variables following semi-stable distribution with common characteristic function $\phi(t)$ given by*

$$\phi(t) = \begin{cases} \exp(-(C + \cos\log|t|)|t|^{\alpha}), & t \neq 0 \\ 1, & t = 0. \end{cases}$$

Let $\sum a_r$ be a series $\{a_r \in \mathbb{R} - \{0\},\ \ 0 \leqslant r \leqslant n\}$ which converges absolutely, and $k_n = max|a_r|,\ \ t_n = min|a_r|$ with $k_n^{\alpha} = O(\log n)$, and t_n is a non-zero finite. Then for all α, $0 < \alpha \leqslant 2$ and ε, $0 < \varepsilon < 1$, the probability of the event

$$\forall\, n : N_n \leqslant \mu(\log n)^2$$

is at least

$$1 - (\frac{\mu^*}{n^{3\alpha-1-\varepsilon}} + \frac{\mu^{**}}{n^{1-\varepsilon}}).$$

THEOREM 2.2. *Let N_n be the number of real zeros of random algebraic polynomial $f(x,w) = \sum_{r=0}^{n} a_r \xi_r(w) x^r$ where ξ_r's are identically distributed independent random variables following semi-stable distribution with common characteristic function $\phi(t)$ given by*

$$\phi(t) = \begin{cases} \exp(-(C + \cos\log|t|)|t|^{\alpha}), & t \neq 0 \\ 1, & t = 0. \end{cases}$$

Let $\sum a_r$ be a series $\{a_r \in \mathbb{R} - \{0\},\ \ 0 \leqslant r \leqslant n\}$ which converges absolutely, and $k_n = max|a_r|,\ \ t_n = min|a_r|$ with $k_n^{\alpha} = O(\log n)$, and t_n is a non-zero finite. Then for all α, $0 < \alpha \leqslant 2$ and ε, $0 < \varepsilon < 1$, there exists $n_0 \in \mathbb{N}$, $n_{00} \in \mathbb{N}$, $\mu' > 0$ and $\mu'' > 0$ such that for all $n_0 \geqslant n_{00}$, the probability of the event

$$\forall\, n \geqslant n_0 : N_n \leqslant \mu(\log n)^3$$

is at least

$$1-\left(\frac{\mu^{'}}{n_0^{\alpha\log n_0-2-\varepsilon}}+\frac{\mu^{''}}{n_0^{\log n_0-1-\varepsilon}}\right).$$

We need the following Lemmas for the proof of the above two Theorems.

LEMMA 2.3. *Let ξ_r be a random variable following semi-stable distribution with characteristic function given by*

$$\phi(t)=\begin{cases}\exp(-(C+\cos\log|t|)|t|^{\alpha}), & t\neq 0\\ 1, & t=0\end{cases}$$

$0<\alpha\leqslant 2$, *then* $P(\{|a\xi_r|\leqslant\varepsilon\})\leqslant\frac{\mu\varepsilon}{a}$ *where* μ *is a positive constant.*

Proof. Let $\phi(t)$ be the characteristic function of ξ_r. By inversion formula, Gnendenko and Kolmogorov, [2], P-48

$$\begin{aligned}P(\{|a\xi_r|\leqslant\varepsilon\}) &= \frac{1}{2\pi}\lim_{\lambda\longrightarrow\infty}\int_{-\lambda}^{\lambda}\frac{e^{\frac{it\varepsilon}{a}}-e^{\frac{-it\varepsilon}{a}}}{it}\phi(t)dt\\ &= \frac{1}{\pi}\frac{\varepsilon}{a}\lim_{\lambda\longrightarrow\infty}\int_{-\lambda}^{\lambda}\frac{\sin\left(\frac{t\varepsilon}{a}\right)}{\left(\frac{t\varepsilon}{a}\right)}\phi(t)dt\\ &< \frac{\varepsilon}{\pi a}\int_{-\infty}^{\infty}\phi(t)dt \qquad \left(\text{Since } \frac{\sin\left(\frac{t\varepsilon}{a}\right)}{\left(\frac{t\varepsilon}{a}\right)}\leqslant 1\right)\\ &< \frac{\mu\varepsilon}{a} \qquad \text{where} \quad \mu=\frac{2\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\pi(C-1)^{1/\alpha}}.\end{aligned}$$

LEMMA 2.4. *Let ξ_r be a random variable following semi-stable distribution with characteristic function given by*

$$\phi(t)=\begin{cases}\exp(-(C+\cos\log|t|)|t|^{\alpha}), & t\neq 0\\ 1, & t=0\end{cases}$$

$0<\alpha\leqslant 2$, *then* $P(\{|a\xi_r|>\tau\})\leqslant\mu_1\left(\frac{a}{\tau}\right)^{\alpha}$ *where* μ_1 *is a positive constant.*

Proof. Let $\phi(t)$ be the characteristic function of ξ_r. Then by Loeve, [3], P-196

$$P(\{|\xi_r|>\tau\})\leqslant 7\tau\int_0^{1/\tau}[1-\phi(t)]dt.$$

So,

$$P(\{|a\xi_r|>\tau\})=P\left(\{|\xi_r|>\frac{\tau}{a}\}\right).$$

Since the distribution is symmetric and $(C+\cos\log|t|)|t|^{\alpha}<(C+1)|t|^{\alpha}$, so

$$1-\phi(t)<(C+1)|t|^{\alpha} \quad \text{for} \quad t\in(0,a/\tau).$$

Hence,

$$P(\{|a\xi_r|>\tau\})\leqslant\mu_1\left(\frac{a}{\tau}\right)^{\alpha}.$$

LEMMA 2.5. *(Samal and Mishra, [7])*
The number of real zeros of a regular function $f(z)$ in a circle with center z_0 and radius r is at most $\log\left(\frac{\max_{|z|\leqslant 1+2/n}|f(z)|}{|f(z_0)|}\right)/\log 2$.

2.1. Proof of Theorem 2.1

Let p be a fixed number greater than $1/\log 2$ and $\gamma=[p\log n]$, we take circles with center $x_m=1-\frac{1}{2^m}$, radii $r_m=\frac{1-x_m}{2}$, for $m=1,2,\cdots,\gamma,p\log n$ and $r_0=1/n$ in the special case when $x_0=1$. The circles $C_0,C_1,\cdots,C_\gamma,C_{p\log n}$ will cover the closed segment $[1/2,1]$.

Let Γ_m be the circle concentric with C_m, and its radius is equal to $2r_m$. So, all the Γ_m's are interior to $|z|=1+2/n$. By Lemma 2.4, we have

$$P\left(\{|a_r\xi_r|>(n+1)^3\}\right)\leqslant \mu_1\frac{|a_r|^\alpha}{(n+1)^{3\alpha}}. \tag{2}$$

Considering the values $r=0,1,2,\cdots,n$ and $k_n^\alpha=max|a_r|^\alpha$ we have

$$P\left(\{|a_r\xi_r|\leqslant (n+1)^3,\ \ 0\leqslant r<n\}\right)>1-\mu_1\frac{k_n^\alpha}{(n+1)^{3\alpha-1}}. \tag{3}$$

We also have

$$\max_{|z|\leqslant 1+2/n}|f(z)|<k_n(n+1)^4e^2. \tag{4}$$

Since the characteristic function of $f(x_m)$ is

$$\exp(-\sum_{r=0}^{n}(C+\cos\log|a_rx_m^rt|)|t|^\alpha|a_r|^\alpha|x_m|^{\alpha r}),$$

so by using Lemma 2.3, we have

$$P\left(\{|f(x_m)|<\frac{1}{n}\}\right)\leqslant\frac{\mu}{nt_n}\left(\sum_{r=0}^{n}x_m^{\alpha r}\right)^{-1/\alpha} \tag{5}$$

for $m=1,2,\cdots,\gamma,p\log n$. As t_n is non-zero finite, we get similarly

$$P\left(\{|f(x_0)|<\frac{1}{n}\}\right)<\frac{\mu}{n^{1+1/\alpha}t_n}<\frac{\mu_2}{n^{1+1/\alpha}}. \tag{6}$$

Hence by using (3), (4), (5) and (6), we have from Lemma 2.5 that outside a set of measure at most

$$\frac{\mu_2}{n^{1+1/\alpha}}+\frac{\mu_1k_n^\alpha}{(n+1)^{3\alpha-1}}+\frac{\mu_3}{n}\sum_{m=1}^{p\log n}\left(\sum_{r=0}^{n}x_m^{\alpha r}\right)^{-1/\alpha} \tag{7}$$

the number of zeros of $f(x)$ in C_0 with $r_0 = 1/n$, $x_0 = 1$, and C_m, $m = 1, 2, \cdots, \gamma, p\log n$ with $x_m = 1 - \frac{1}{2^m}$, $R = 2r_m$ is at most $\dfrac{\log\left(\frac{k_n e^2 (n+1)^4}{n^{-1}}\right)}{\log 2} < \mu \log n$. So, considering all the circles $C_0, C_1, \cdots, C_\gamma, C_{p\log n}$ the total number of zeros inside all these is at most

$$\sum_{m=1}^{p\log n} N_m(r) \leqslant \sum_{m=1}^{p\log n} \mu \log n \leqslant \mu (\log n)^2, \tag{8}$$

and also in C_0, it is $\mu \log n \leqslant \mu (\log n)^2$. Let

$$\sum_{m=1}^{p\log n} \left(\sum_{r=0}^{n} x_m^{\alpha r} \right)^{-1/\alpha} = S_1 + S_2$$

where

$$S_1 = \sum_{m=1}^{\log n/\log 2} \left(\sum_{r=0}^{n} x_m^{\alpha r} \right)^{-1/\alpha} + \sum_{(\log n/\log 2)+1}^{p\log n} \left(\sum_{r=0}^{n} x_m^{\alpha r} \right)^{-1/\alpha}. \tag{9}$$

Proceeding as Samal and Mishra, [7], we get

$$S_1 \leqslant \mu'' \log n \tag{10}$$

and

$$S_2 < \frac{\mu_4 \log n}{n^{1/\alpha}}. \tag{11}$$

Using (9), (10) and (11), we get from (7) that the measure of the exceptional is at most

$$\frac{\mu_1 k_n^\alpha}{(n+1)^{3\alpha-1}} + \frac{\mu_2}{n^{1+1/\alpha}} + \frac{\mu'' \log n}{n} + \frac{\mu_5 \log n}{n^{1+\frac{1}{\alpha}}} < \frac{\mu_6}{n^{1+\frac{1}{\alpha}-\varepsilon}} + \frac{\mu_7}{n^{1-\varepsilon}} + \frac{\mu_1}{n^{3\alpha-1-\varepsilon}}. \tag{12}$$

(We have used the fact that $k_n^\alpha = O(\log n)$ and $\log n < n^\varepsilon$ for large n, $\frac{\log n}{n^\varepsilon} < \frac{1}{n^{1-\varepsilon}}$.) Now adopting the procedure of Samal and Mishra, [7], we consider the segment $(0, 1/2)$. Let us take a circle with center zero and radius $1/2$.

The circle $|z| \leqslant \frac{1}{2}$ is interior to the circle $|z| \leqslant 1$. Now applying Lemma 2.5 with $z_0 = 0$, $r = 1/2$ and $R = 1$, we get from (3) that outside a set of measure at most

$$\frac{\mu_1 k_n^\alpha}{(n+1)^{3\alpha-1}}. \tag{13}$$

We have

$$\max_{|z|\leqslant 1} |f(z)| < k_n (n+1)^4. \tag{14}$$

Again by using Lemma 2.3, we obtain

$$P\left(\{|f(0)| < \frac{1}{n}\} \right) < \frac{\mu_8}{n}. \tag{15}$$

Using (12), (13) and (14), we get from Lemma 2.5 that the number of zeros inside the circle $|z| \leqslant \frac{1}{2}$ does not exceed

$$\mu \log n < \mu (\log n)^2 \tag{16}$$

outside a set of measure at most

$$\frac{\mu_1 k_n^\alpha}{(n+1)^{3\alpha-1}} + \frac{\mu_8}{n}. \tag{17}$$

Similarly, the number of zeros of $f(x)$ inside the circle $|z-\rho| \leqslant 1-\rho$, $0<\rho<1/2$ and radius $1/2$ is at most

$$\mu (\log n)^2 \tag{18}$$

except the set of measure

$$\frac{\mu_1 k_n^\alpha}{(n+1)^{3\alpha-1}} + \frac{\mu_9}{n}. \tag{19}$$

So, from (8), (12), (16), (17), (18) and (19), we have the the number of zeros of $f(x)$ inside the interval $(0,1)$ is at most $\mu(\log n)^2$ except the set of measure at most

$$\frac{\mu_1}{n^{3\alpha-1-\varepsilon}} + \frac{\mu_6}{n^{1+\frac{1}{\alpha}-\varepsilon}} + \frac{\mu_7}{n^{1-\varepsilon}} + \frac{\mu_8}{n} + \frac{\mu_9}{n} < \frac{\mu^*}{n^{3\alpha-1-\varepsilon}} + \frac{\mu^{**}}{n^{1-\varepsilon}}. \tag{20}$$

Therefore the measure of exceptional set is at least

$$1 - \left(\frac{\mu^*}{n^{3\alpha-1-\varepsilon}} + \frac{\mu^{**}}{n^{1-\varepsilon}}\right).$$

Hence the proof.

2.2. Proof of Theorem 2.2

As proceeding in proof of Theorem 2.1, we have by using Lemma 2.3

$$P\left(\{|a_r \xi_r| \leqslant (n+1)^{\log n},\ \ 0 \leqslant r \leqslant n\}\right) > 1 - \frac{\mu_1 k_n^\alpha}{(n+1)^{\alpha \log n - 1}}. \tag{21}$$

So, we have

$$\begin{aligned} \max |f(z)| &= \max_{|z| \leqslant 1+2/n} \sum_{r=0}^{n} |a_r||\xi_r||z|^r \\ &< k_n (1+2/n)^n (n+1)^{\log n + 1}. \end{aligned}$$

Since $(1+2/n)^n \longrightarrow e^2$ as $n \longrightarrow \infty$, so

$$\max |f(z)| < k_n e^2 (n+1)^{\log n + 1} \tag{22}$$

outside a set of measure $\frac{\mu_1 k_n^\alpha}{(n+1)^{\alpha \log n - 1}}$. Also, we have by using Lemma 2.3

$$P\left(\{|f(x_m)| < \frac{1}{(n+1)^{\log n}}\}\right) \leqslant \frac{\mu_2}{(n+1)^{\log n}} \left(\sum x_m^{\alpha r}\right)^{-1/\alpha} \tag{23}$$

for $m = 1, 2, \cdots, \gamma, p\log n$, and similarly

$$P\left(\{|f(x_0)| < \frac{1}{(n+1)^{\log n}}\}\right) \leqslant \frac{\mu'}{(n+1)^{\log n + 1/\alpha}}. \tag{24}$$

So, we have from Lemma 2.5, by using (21), (22), (23) and (24) that the number of zeros of $f(x)$ in C_m, $m = 1, 2, \cdots, \gamma, p\log n$, and in C_0 with $r_0 = 1/n$, $x_0 = 1$ is at most $\mu(\log n)^2$. Hence, the total number of zeros in the circles $C_0, C_1, \cdots, C_\gamma, C_{p\log n}$ is at most

$$\sum_{m=1}^{p\log n} \mu(\log n)^2 \leqslant \mu(\log n)^3 \tag{25}$$

outside a set of measure

$$\frac{\mu_1 k_n^\alpha}{(n+1)^{\alpha\log n - 1}} + \frac{\mu'}{(n+1)^{\log n + 1/\alpha}} + \frac{\mu}{(n+1)^{\log n}} \sum_{m=1}^{p\log n} \left(\sum_{r=0}^{n} x_m^{\alpha r}\right)^{-1/\alpha}. \tag{26}$$

Proceeding as in Theorem 2.1, we have from (26) measure of exceptional set is

$$\frac{\mu_1}{n^{\alpha\log n - 1 - \varepsilon}} + \frac{\mu'}{n^{\log n + 1/\alpha}} + \frac{\mu_4}{n^{\log n - \varepsilon}} \quad (\text{Since} \quad \log n < n^\varepsilon \text{ for } \quad 0 < \varepsilon < 1)$$

which is

$$< \frac{\mu_1}{n^{\alpha\log n - 1 - \varepsilon}} + \frac{\mu_4}{n^{\log n - \varepsilon}} \tag{27}$$

where $\alpha\log n - 1 - \varepsilon > 1$ for $0 < \varepsilon < 1$ and $0 < \alpha \leqslant 2$. Now, let us consider the segment $(0, 1/2)$, and so the circle with $z_0 = 0$, $r = 1/2$ and $R = 1$. We get from (21) that outside a set of measure at most $\frac{\mu_1 k_n^\alpha}{(n+1)^{\alpha\log n - 1}}$, we have

$$\max_{|z| \leqslant 1} |f(z)| < k_n (n+1)^{\log n + 1}. \tag{28}$$

Again, by using Lemma 2.3, we obtain

$$P\left(\{|f(0)| < \frac{1}{(n+1)^{\log n}}\}\right) < \frac{\mu_5}{(n+1)^{\log n}}. \tag{29}$$

So, outside a set of measure at most $\frac{\mu_1 k_n^\alpha}{(n+1)^{\alpha\log n - 1}} + \frac{\mu_5}{(n+1)^{\log n}}$, the number of zeros in $|z| < 1/2$ is

$$< \mu(\log n)^2 < \mu(\log n)^3. \tag{30}$$

Accordingly, we have number of zeros of $f(x)$ inside the circle $|z - \rho| \leqslant 1 - \rho$ does not exceed $\mu(\log n)^3$ outside a set of measure at most

$$\frac{\mu_1 k_n^\alpha}{(n+1)^{\alpha\log n - 1}} + \frac{\mu_6}{(n+1)^{\log n}}. \tag{31}$$

So, from (25), (27), (30) and (31) the total number of zeros of $f(x)$ inside the interval $(0,1)$ is at most $\mu(\log n)^3$ except the set of measure at most

$$\frac{\mu_1}{(n+1)^{\alpha\log n-1-\varepsilon}}+\frac{\mu_2}{(n+1)^{\log n-\varepsilon}}+\frac{\mu_5}{(n+1)^{\log n}}+\frac{\mu_6}{(n+1)^{\log n}} < \frac{\mu_1}{(n+1)^{\alpha\log n-1-\varepsilon}}+\frac{\mu_2}{(n+1)^{\log n-\varepsilon}}. \quad (32)$$

Therefore, for each $n \geqslant n_0$, we have $N_n < \mu(\log n)^3$, and the measure of exceptional set is at most

$$\sum_{n=n_0}^{\infty}\left(\frac{\mu_1}{(n+1)^{\alpha\log n-1-\varepsilon}}+\frac{\mu_2}{(n+1)^{\log n-\varepsilon}}\right) < \sum_{n=n_0}^{\infty}\left(\frac{\mu_1}{(n+1)^{\alpha\log n_0-1-\varepsilon}}+\frac{\mu_2}{(n+1)^{\log n_0-\varepsilon}}\right) < \frac{\mu'}{n_0^{\alpha\log n_0-2-\varepsilon}}+\frac{\mu''}{n_0^{\log n_0-1-\varepsilon}} \quad \text{(By integral test)}$$

where $0<\alpha\leqslant 2$ and $0<\varepsilon<1$ for all $n\geqslant n_0$. Hence the measure of exceptional set is at least

$$1-\left(\frac{\mu'}{n_0^{\alpha\log n_0-2-\varepsilon}}+\frac{\mu''}{n_0^{\log n_0-1-\varepsilon}}\right).$$

Thus, the proof of Theorem 2.2 is complete.

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