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ON CERTAIN SERIES INVOLVING RECIPROCALS OF BINOMIAL COEFFICIENTS

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Abstract. We evaluate the following family of series involving reciprocals of binomial coefficients in terms of elementary functions for $m = 3,4$.

$$\sum_{k=0}^{\infty} \frac{x^k}{(mk+1)\binom{mk}{k}}.$$

1. Introduction

For non-negative integers $m,$ the binomial coefficients are defined by

$$\binom{m}{n} = \begin{cases} \frac{m!}{n!(m-n)!}, & \text{if } m \geqslant n, \\ 0, & \text{if } m < n. \end{cases}$$

These numbers are a class of important combinatorial numbers and important tools in many areas of mathematics and statistics. Another area of science in which series involving reciprocal binomial coefficients appear is nuclear physics, [5]. It is known, see [9], that some of these sums appear in the calculation of massive Feynmann diagrams with several different approaches: for instance, as solutions of differential equations for Feynmann amplitudes, through a naive $\varepsilon-$expansion of hypergeometric functions within Mellin-Barnes technique, or in the framework of recently proposed algebraic approaches. Recently many authors studied series involving reciprocals of binomial coefficients via integral representations and found interesting results including explicit evaluations of their sums. These interests are largely motivated by the following series representations for $\zeta(2)$ and $\zeta(3)$:

$$\zeta(2) = 3\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}} \quad \text{and} \quad \zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}},$$

which are used by Apéry in his celebrated paper [1] to prove the irrationality of $\zeta(2)$ and $\zeta(3)$. Here ζ stands for the Riemann zeta function. For the investigations of sums

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related to the reciprocals of binomial coefficients, see for instance, [2, 3, 4, 6–17]. In [3] the author calculated explicitly the sum of the series for $n=0,1,2:$

$$\sum_{k=1}^{\infty}\frac{x^k}{k^n\binom{3k}{k}}. \tag{1}$$

Our aim in this work is to evaluate the following series for $m=3,4$ in closed form:

$$\sum_{k=0}^{\infty}\frac{x^k}{(mk+1)\binom{mk}{k}}. \tag{2}$$

For our purpose, we begin by recalling the following definition and property of the *beta function* β

$$\beta(s,t)=\int_0^1 x^s(1-x)^t dx=\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)},\ s,t>0, \tag{3}$$

where Γ is the classical gamma function.

2. Main results

The following theorems are our main results.

THEOREM 1. *For all real numbers x with $0<|x|<\frac{27}{4}$, we have*

$$\sum_{k=0}^{\infty}\frac{x^k}{(3k+1)\binom{3k}{k}}=\frac{\phi(\phi-1)}{3\phi-1}\left\{\frac{3}{2}\log\left|\frac{\phi}{\phi-1}\right|\right.$$
$$\left.+\frac{3\phi-2}{\sqrt{3\phi^2-4\phi}}\left[\tan^{-1}\frac{\phi}{\sqrt{3\phi^2-4\phi}}+\tan^{-1}\frac{2-\phi}{\sqrt{3\phi^2-4\phi}}\right]\right\}:=F(\phi), \tag{4}$$

where

$$\phi=\phi(x)=\frac{2}{3}+\frac{1}{3}\left(\frac{27-2x+3\sqrt{81-12x}}{2x}\right)^{-\frac{1}{3}}+\frac{1}{3}\left(\frac{27-2x+3\sqrt{81-12x}}{2x}\right)^{\frac{1}{3}} \tag{5}$$

is the only positive real root of $a(1-a)^2=\frac{1}{x}$.

Proof. If we use $\Gamma(k+1)=k!$ and (3) we get

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{x^k}{(3k+1)\binom{3k}{k}}&=\sum_{k=0}^{\infty}x^kB(k+1,2k+1)=\sum_{k=0}^{\infty}x^k\int_0^1 u^k(1-u)^{2k}du\\&=\int_0^1\sum_{k=0}^{\infty}[xu(1-u)^2]^k du=\frac{1}{x}\int_0^1\frac{du}{\frac{1}{x}-u(1-u)^2}.\end{aligned} \tag{6}$$

To evaluate this integral we set

$$a(1-a)^2=\frac{1}{x}, \tag{7}$$

so that we get

$$\sum_{k=0}^{\infty} \frac{1}{(a(1-a)^2)^k(3k+1)\binom{3k}{k}} = -a(1-a)^2 \int_0^1 \frac{du}{u(1-u)^2 - a(1-a)^2}. \tag{8}$$

Since

$$u(1-u)^2 - a(1-a)^2 = (u-a)(u^2+(a-2)u+(a-1)^2),$$

we can factorize

$$\frac{1}{u(1-u)^2 - a(1-a)^2} = \frac{1}{3a^2-4a+1}\left(\frac{1}{u-a} - \frac{u+2a-2}{u^2+(a-2)u+(a-1)^2}\right). \tag{9}$$

Hence, we get

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1}{(a(1-a)^2)^k(3k+1)\binom{3k}{k}} \\
&= -\frac{a(a-1)}{3a-1}\int_0^1 \left[\frac{1}{u-a} - \frac{1}{2}\frac{2u+a-2+(3a-2)}{u^2+(a-2)u+(a-1)^2}\right] du \\
&= \frac{a(a-1)}{3a-1}\left\{\log\left|\frac{a}{a-1}\right| + \frac{1}{2}\int_0^1 \frac{(2u+a-2)du}{u^2+(a-2)u+(a-1)^2}\right. \\
&\qquad \left. + \frac{1}{2}\int_0^1 \frac{(3a-2)du}{u^2+(a-2)u+(a-1)^2}\right\}.
\end{aligned} \tag{10}$$

We can easily evaluate that

$$\int_0^1 \frac{(2u+a-2)du}{u^2+(a-2)u+(a-1)^2} = \log\left|\frac{a}{a-1}\right|$$

and

$$\begin{aligned}
&\int_0^1 \frac{(3a-2)du}{u^2+(a-2)u+(a-1)^2} \\
&\qquad = \frac{2(3a-2)}{\sqrt{3a^2-4a}}\left\{\tan^{-1}\frac{a}{\sqrt{3a^2-4a}} + \tan^{-1}\frac{2-a}{\sqrt{3a^2-4a}}\right\}.
\end{aligned}$$

Using Cardano's method or any packet program like *Mathematica* or *Maple* we obtain the only positive real solution of (7) to be $a = \phi(x)$, where ϕ is as defined in (5). Replacing the values of these two integrals in (10) we complete the proof of Theorem 1.

THEOREM 2. *For all real numbers x with $0 < |x| < \frac{256}{27}$, we have*

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{x^k}{(4k+1)\binom{4k}{k}} = \frac{\varphi(\varphi^2-1)}{2(2\varphi^2+1)}\log\left|\frac{\varphi+1}{\varphi-1}\right| \\
&+ \frac{(\varphi-1)(\varphi^3+1)}{4\varphi(2\varphi^2+1)}\sqrt{\frac{\varphi}{\varphi-2}}\left[\tan^{-1}\sqrt{\frac{\varphi}{\varphi-2}} + \tan^{-1}\frac{3-\varphi}{\varphi+1}\sqrt{\frac{\varphi}{\varphi-2}}\right] \\
&+ \frac{(\varphi+1)(\varphi^3-1)}{4\varphi(2\varphi^2+1)}\sqrt{\frac{\varphi}{\varphi+2}}\left[\tan^{-1}\frac{\varphi+3}{\varphi-1}\sqrt{\frac{\varphi}{\varphi+2}} - \tan^{-1}\sqrt{\frac{\varphi}{\varphi+2}}\right] \\
&:= G(\varphi),
\end{aligned} \tag{11}$$

where

$$\varphi(x)=\Bigg[1+\frac{16\cdot(2/3)^{\frac{1}{3}}}{\left(9x^2+\sqrt{81x^4-768x^3}\right)^{\frac{1}{3}}} + \frac{2(2/3)^{\frac{2}{3}}\left(9x^2+\sqrt{81x^4-768x^3}\right)^{\frac{1}{3}}}{x}\Bigg]^{\frac{1}{2}}, \quad \text{if } -\frac{256}{27}<x<0 \tag{12}$$

and

$$\varphi(x)=\left[1+\frac{16}{\sqrt{3x}}\cos\left(\frac{1}{3}\tan^{-1}\sqrt{\frac{256}{27x}-1}\right)\right]^{\frac{1}{2}}, \text{ if } 0<x<\frac{256}{27}, \tag{13}$$

is the only positive real root of the equation $\frac{64y^2}{(y^2-1)^3}-x=0$.

Proof. If we use $\Gamma(k+1)=k!$ and (3) we get

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{x^k}{(4k+1)\binom{4k}{k}} &= \sum_{k=0}^{\infty}x^kB(k+1,3k+1)\\ &= \sum_{k=0}^{\infty}x^k\int_0^1 u^k(1-u)^{3k}du\\ &= \int_0^1\sum_{k=0}^{\infty}[xu(1-u)^3]^k du\\ &= \frac{1}{x}\int_0^1\frac{du}{\frac{1}{x}-u(1-u)^3}.\end{aligned}$$

In order to evaluate this integral we set

$$\frac{(y^2-1)^3}{64y^2}=\frac{1}{x}, \tag{14}$$

so that we find that

$$\sum_{k=0}^{\infty}\frac{\left(\frac{64y^2}{(y^2-1)^3}\right)^k}{(4k+1)\binom{4k}{k}}=\frac{(y^2-1)^3}{64y^2}\int_0^1\frac{du}{\frac{(y^2-1)^3}{64y^2}-u(1-u)^3}. \tag{15}$$

Since

$$\begin{aligned}&\frac{(y^2-1)^3}{64y^2}-u(1-u)^3\\ &\qquad=\left(u^2-\frac{y+3}{2}u+\frac{(y+1)^3}{8y}\right)\left(u^2+\frac{y-3}{2}u+\frac{(y-1)^3}{8y}\right),\end{aligned}$$

we can factorize

$$\frac{1}{\frac{(y^2-1)^3}{64y^2}-u(1-u)^3}=\frac{8y^3}{(y^2-1)^2(2y^2+1)}\times$$
$$\times\left(\frac{2u+\frac{y-3}{2}+\frac{y^3+1}{2y^2}}{u^2+\frac{y-3}{2}u+\frac{(y-1)^3}{8y}}-\frac{2u-\frac{y+3}{2}+\frac{1-y^3}{2y^2}}{u^2-\frac{y+3}{2}u+\frac{(y+1)^3}{8y}}\right).$$

Thus, by (15) we have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\left(\frac{64y^2}{(y^2-1)^3}\right)^k}{(4k+1)\binom{4k}{k}}&=\frac{y(y^2-1)}{8(2y^2+1)}\int_0^1\frac{2u+\frac{y-3}{2}}{u^2+\frac{y-3}{2}u+\frac{(y-1)^3}{8y}}\,du\\&+\frac{(y^2-1)(y^3+1)}{16y(2y^2+1)}\int_0^1\frac{du}{u^2+\frac{y-3}{2}u+\frac{(y-1)^3}{8y}}\\&-\frac{y(y^2-1)}{8(2y^2+1)}\int_0^1\frac{2u-\frac{y+3}{2}}{u^2-\frac{y+3}{2}u+\frac{(y+1)^3}{8y}}\,du\\&+\frac{(y^2-1)(y^3-1)}{16y(2y^2+1)}\int_0^1\frac{du}{u^2-\frac{y+3}{2}u+\frac{(y+1)^3}{8y}}.\end{aligned}\tag{16}$$

Simply calculations give

$$\int_0^1\frac{2u-\frac{y+3}{2}}{u^2-\frac{y+3}{2}u+\frac{(y+1)^3}{8y}}\,du=2\log\left|\frac{y-1}{y+1}\right|,\tag{17}$$

$$\int_0^1\frac{2u+\frac{y-3}{2}}{u^2+\frac{y-3}{2}u+\frac{(y-1)^3}{8y}}\,du=2\log\left|\frac{y+1}{y-1}\right|,\tag{18}$$

$$\begin{aligned}&\int_0^1\frac{du}{u^2-\frac{y+3}{2}u+\frac{(y+1)^3}{8y}}\\&=\frac{4}{y-1}\sqrt{\frac{y}{y+2}}\left[\tan^{-1}\frac{y+3}{y-1}\sqrt{\frac{y}{y+2}}-\tan^{-1}\sqrt{\frac{y}{y+2}}\right],\end{aligned}\tag{19}$$

and

$$\begin{aligned}&\int_0^1\frac{du}{u^2+\frac{y-3}{2}u+\frac{(y-1)^3}{8y}}\\&=\frac{4}{y+1}\sqrt{\frac{y}{y-2}}\left[\tan^{-1}\sqrt{\frac{y}{y-2}}+\tan^{-1}\frac{3-y}{y+1}\sqrt{\frac{y}{y-2}}\right].\end{aligned}\tag{20}$$

Replacing the values of these integrals obtained in (17), (18), (19) and (20) in (16), we get

$$\sum_{k=0}^{\infty}\frac{\left(\frac{64y^2}{(y^2-1)^3}\right)^k}{(4k+1)\binom{4k}{k}}=\frac{y(y^2-1)}{2(2y^2+1)}\log\left|\frac{y+1}{y-1}\right|+\frac{(y-1)(y^3+1)}{4y(2y^2+1)}\sqrt{\frac{y}{y-2}}\times$$
$$\times\left[\tan^{-1}\sqrt{\frac{y}{y-2}}+\tan^{-1}\frac{3-y}{y+1}\sqrt{\frac{y}{y-2}}\right]$$
$$+\frac{(y+1)(y^3-1)}{4y(2y^2+1)}\sqrt{\frac{y}{y+2}}\left[\tan^{-1}\frac{y+3}{y-1}\sqrt{\frac{y}{y+2}}-\tan^{-1}\sqrt{\frac{y}{y+2}}\right]. \tag{21}$$

If we set again

$$\frac{64y^2}{(y^2-1)^3}=x$$

here, we get the only positive real solution to be $y=\varphi(x)$, where φ is as defined in (12) and (13). This completes the proof of Theorem 2.

3. Examples

In this section F and G are as defined in (4) and (11), respectively.

EXAMPLE 1. Putting $x=1$ in (4)

$$\sum_{k=0}^{\infty}\frac{1}{(3k+1)\binom{3k}{k}}=F(a),$$

where

$$a=\frac{2}{3}+\frac{1}{3}\left(\frac{25+3\sqrt{69}}{2}\right)^{-\frac{1}{3}}+\frac{1}{3}\left(\frac{25+3\sqrt{69}}{2}\right)^{\frac{1}{3}}.$$

EXAMPLE 2. Putting $x=-1$ in (4)

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+1)\binom{3k}{k}}=F(b),$$

where

$$b=\frac{2}{3}-\frac{1}{3}\left(\frac{29+3\sqrt{93}}{2}\right)^{-\frac{1}{3}}-\frac{1}{3}\left(\frac{29+3\sqrt{93}}{2}\right)^{\frac{1}{3}}.$$

EXAMPLE 3. Putting $x=6$ in (4), we get

$$\sum_{k=0}^{\infty}\frac{6^k}{(3k+1)\binom{3k}{k}}=F(c),$$

where

$$c=\frac{1}{3}\left(2^{1/6}+2^{-1/6}\right)^2.$$

EXAMPLE 4. Putting $x=1$ in (11), we get

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)\binom{4k}{k}} = G(d),$$

where

$$d = \left[1 + \frac{16}{\sqrt{3}} \cos\left(\frac{1}{3} \tan^{-1} \sqrt{\frac{229}{27}}\right)\right]^{\frac{1}{2}}.$$

EXAMPLE 5. Putting $x=-1$ in (11), we get

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)\binom{4k}{k}} = G(e),$$

where

$$e = \left[1 + \frac{16(2/3)^{1/3}}{\left(9+\sqrt{849}\right)^{\frac{1}{3}}} - 2(2/3)^{\frac{2}{3}} \left(9+\sqrt{849}\right)^{\frac{1}{3}}\right]^{\frac{1}{2}}.$$

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