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A NOTE ON RELATIVE TYPE OF ENTIRE FUNCTIONS REPRESENTED BY VECTOR VALUED DIRICHLET SERIES

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Abstract. We introduce in this paper a new parameter $T_g(f)$, the relative growth type of entire function $f(s)$ represented by a vector valued Dirichlet series with respect to another entire VVDS $g(s)$ when their relative order is one. We establish a few lemmas, and show that under certain conditions type and relative type of VVDS are equal. Several basic results have also been obtained.

1. Introduction

Let $f(s)$ be an entire function defined by an everywhere absolutely convergent vector valued Dirichlet series (VVDS)

$$f(s)=\sum_{n=1}^{\infty}a_n e^{s\lambda_n}\quad s=\sigma+it,\qquad (\sigma,\,t \text{ are real variables}) \tag{1}$$

where $a_n's$ belong to a Banach space $(E,\|.\|)$, $\lambda_n's$ are non-negative real numbers such that $0\leqslant\lambda_1<\lambda_2<\cdots\cdots<\lambda_n\to\infty$ as $n\to\infty$, and satisfy the conditions

$$\limsup_{n\to\infty}\frac{\log n}{\lambda_n}=D<\infty \tag{2}$$

and

$$\limsup_{n\to\infty}\frac{\log\|a_n\|}{\lambda_n}=-\infty. \tag{3}$$

B. L. Srivastava [4] defined the growth parameters such as order, type, lower order and lower type of entire functions represented by VVDS as defined above. He also obtained the results for coefficient characterization of order and type. Bernal [1] introduced the concept of relative order of entire functions represented by a power series. Lahiri and Banerjee [2] extended these results for entire functions reprsented by Dirichlet series.In this paper,we have introduced relative type of two entire VVDS to measure the growth rate when their relative order is one.We have given the alternate definition of relative order also which is different from that given in [2] and is more suited for Dirichlet series.

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For an entire function $f(s)$ defined by (1), we define its maximum modulus as

$$M(\sigma)=M(\sigma,f)=\sup_{-\infty<t<\infty}\|f(\sigma+it)\|.$$

Then, its order ρ is given by

$$\rho=\limsup_{\sigma\to\infty}\frac{\log\log M(\sigma)}{\sigma},\quad (0\leqslant\rho\leqslant\infty). \tag{4}$$

If $0<\rho<\infty$, the type of f is defined as

$$T=\limsup_{\sigma\to\infty}\frac{\log M(\sigma)}{e^{\rho\sigma}},\quad (0\leqslant T\leqslant\infty)$$

and the lower type τ of $f(s)$ is defined as

$$\tau=\liminf_{\sigma\to\infty}\frac{\log M(\sigma)}{e^{\sigma\rho}}.$$

The function $f(s)$ is said to be of perfectly regular growth if $0<\tau=T<\infty$.

We also define the maximum term $m(\sigma)$ and the rank $N(\sigma)$ of the maximum term as

$$m(\sigma)=\max\{||a_n||e^{\sigma\lambda_n};n\in N^+\},$$
$$N(\sigma)=\max\{n;||a_n||e^{\sigma\lambda_n}=m(\sigma),n\in N^+\}.$$

Then $m(\sigma)$ and $N(\sigma)$ are indefinitely increasing functions of σ and $m(\sigma)\leqslant M(\sigma)$.

Let f and g be two entire VVDS of the form (1), $F(\sigma)$ and $G(\sigma)$ denote their respective maximum moduli.

DEFINITION 1. The relative order of $f(s)$ with respect to $g(s)$, denoted by $\rho_g(f)$ is defined as

$$\rho_g(f)=\inf\{\mu>0:F(\sigma)<G(\sigma\mu)\text{ for all }\sigma>\sigma_0(\mu)\}$$

i.e. $\rho_g(f)=\limsup_{\sigma\to\infty}\frac{G^{-1}F(\sigma)}{\sigma}$.

If we choose $g(s)=\exp(\exp(s))$ as the comparison function, the above definition coincides with the classical definition of Ritt order as in (4).

When $\rho_g(f)=1$ i.e., $\rho(f)=\rho(g)=\rho$ then to discuss their relative growth we need to have further refinement of the growth parameter. Hence we introduce the relative type of $f(s)$ with respect to $g(s)$.

DEFINITION 2. The relative type of $f(s)$ with respect to $g(s)$, denoted by $T_g(f)$ when $\rho_g(f)=1$ is defined as

$$T_g(f)=\inf\left\{\mu>0:F(\sigma)<G\left[\frac{1}{\rho}\log(\mu e^{\sigma\rho})\right]\text{ for all }\sigma>\sigma_0(\mu)\right\}$$

i.e.

$$T_g(f) = \limsup_{\sigma\to\infty} \frac{\exp[\rho G^{-1}F(\sigma)]}{e^{\sigma\rho}}.$$

DEFINITION 3. The relative lower type of $f(s)$ with respect to $g(s)$, denoted by $\tau_g(f)$ when $\rho_g(f) = 1$ is defined as

$$\tau_g(f) = \liminf_{\sigma\to\infty} \frac{\exp[\rho G^{-1}F(\sigma)]}{e^{\sigma\rho}}$$

and if $T_g(f) = \tau_g(f)$ then f is said to be of regular type with respect to g.

DEFINITION 4. A non-constant entire function $g(s)$ defined by a VVDS is said to have property (A), if for any $\delta > 1$ and $\sigma > \sigma_0(\delta)$,

$$[G(\sigma)]^2 \leqslant G(\sigma\delta).$$

Property (A) has been closely studied by Bernal [1].

2. Auxiliary results

In this section we present some results which will be used in the sequel. We prove

LEMMA 1. *Let $g(s)$ be an entire function represented by VVDS given in* (1) *and let $\alpha > 1$, $0 < \beta < \alpha$, be given. Then,*

$$G(\alpha\sigma) > e^{\beta\sigma}G(\sigma) \quad \textit{for all large } \sigma.$$

Proof. Let $g(s) = \sum_{n=1}^{\infty} b_n e^{s\lambda_n}$. Then following [5, Lemma 1, p. 2653], we have

$$G(\sigma) \leqslant O(1)\, m(\sigma + D + \varepsilon)$$

where $\varepsilon > 0$ is arbitrary and the bounded constant in $O(1)$ does not contain σ or λ_n. Let us choose $\alpha_0 > 1$, $\beta < \alpha_0 < \alpha$. Then

$$e^{\beta\sigma}G(\sigma) \leqslant O(1)e^{\beta\sigma}m(\sigma + D + \varepsilon) = O(1)\,||b_{N(\sigma')}||\; e^{\beta\sigma + \sigma'\lambda_{N(\sigma'')}}$$

where $\sigma' = \sigma + D + \varepsilon$ and $N(\sigma')$ is the rank of the maximum term $m(\sigma')$. Hence for suitably large σ,

$$e^{\beta\sigma}G(\sigma) \leqslant ||b_{N(\sigma')}||\; e^{\beta\sigma + \alpha_0\sigma\lambda_{N(\sigma'')}}.$$

Also $\beta + \alpha_0\lambda_n < \alpha\lambda_n$ for all sufficiently large n. Hence we get

$$e^{\beta\sigma}G(\sigma) \leqslant ||b_{N(\sigma')}||\; e^{\beta\sigma + \alpha_0\sigma\lambda_{N(\sigma'')}} \leqslant m(\alpha\sigma) \leqslant M(\alpha\sigma) = G(\alpha\sigma).$$

Thus the proof of Lemma 1 is complete. □

Next we prove

LEMMA 2. *Let $g(s)$ satisfy the property (A); then for any positive integer n and for all $\delta > 1$, we have*

$$[G(\sigma)]^n < G(\delta\sigma) \quad \textit{for all large } \sigma.$$

Proof. Let n be any positive integer. Then there exists an integer m such that $2^m > n$. Now we can write

$$\begin{aligned} G(\delta\sigma) &= G(\sigma\delta^{1/2}\delta^{1/2}) \geqslant \left[G(\sigma\delta^{1/2})\right]^2 \\ &= \left[G(\sigma\delta^{1/4}\delta^{1/4})\right]^2 \\ &\geqslant \left[G(\sigma\delta^{1/4})\right]^4 \geqslant \cdots\cdots \geqslant \left[G(\sigma\delta^{1/2^m})\right]^{2^m}. \end{aligned}$$

Since $\delta^{1/2^k} > 1$, $k \geqslant 1$ and $2^m > n$, we have $G(\delta\sigma) \geqslant [G(\sigma)]^n$. This proves Lemma 2. □

Next we have

LEMMA 3. *Let $f(s)$ be VVDS defined by (1), $k > 1$, $0 < \mu < \lambda$ and n is any positive integer. Then*

$$\lim_{\sigma\to\infty} \frac{F(k\sigma)}{F(\sigma)} = \lim_{\sigma\to\infty} \frac{F(\lambda\sigma)}{F(\mu\sigma)} = \infty \tag{5}$$

and

$$\lim_{\sigma\to\infty} \frac{F(k\sigma)}{\sigma^n F(\sigma)} = \lim_{\sigma\to\infty} \frac{F(\lambda\sigma)}{\sigma^n F(\mu\sigma)} = \infty. \tag{6}$$

Proof. Since $k > 1$, choosing $0 < \beta < k$, we have from Lemma 1,

$$\lim_{\sigma\to\infty} \frac{F(k\sigma)}{F(\sigma)} \geqslant \lim_{\sigma\to\infty} e^{\beta\sigma} = \infty.$$

The second part of (5) follows on taking $k = \lambda/\mu$.
Again for $k > 1$, we choose β, $0 < \beta < k$. Then from Lemma 1,

$$\lim_{\sigma\to\infty} \frac{F(k\sigma)}{\sigma^n F(\sigma)} \geqslant \lim_{\sigma\to\infty} \frac{e^{\beta\sigma}}{\sigma^n} = \infty.$$

This proves (6) and the proof of Lemma 3 is complete. □

LEMMA 4. *Let $f(s)$ be an entire VVDS and n is any positive integer. Then $\rho_g(f) = \rho_g(f^n)$.*

Proof. By Lemma 2, we get $[F(\sigma)]^n < F(\delta\sigma)$ where $\delta > 1$ is arbitrary. By definition of relative order, we have

$$\rho_g(f^n) = \lim_{\sigma\to\infty} \sup \frac{G^{-1}[F(\sigma)]^n}{\sigma} \leqslant \lim_{\sigma\to\infty} \sup \frac{G^{-1}[F(\delta\sigma)]}{\sigma}.$$

Now letting $\delta \to 1^{+}$, we obtain $\rho_g(f) \geqslant \rho_g(f^n)$. The inequality $\rho_g(f^n) \geqslant \rho_g(f)$ is evident. From the above two inequalities we have $\rho_g(f) = \rho_g(f^n)$ which proves Lemma 4. □

Using the definition of relative type, we easily get the following results

LEMMA 5.

1. *Let* f_1, f_2 *and* g *be entire VVDS such that* $F_1(\sigma) \leqslant F_2(\sigma)$ *for all large* σ *and* $\rho_g(f_1) = \rho_g(f_2) = 1$. *Then* $T_g(f_1) \leqslant T_g(f_2)$.
2. *Let* f, g_1 *and* g_2 *be entire VVDS such that* $G_1(\sigma) \leqslant G_2(\sigma)$ *for all large* σ *and* $\rho_{g_1}(f) = \rho_{g_2}(f) = 1$. *Then* $T_{g_2}(f) \leqslant T_{g_1}(f)$.

LEMMA 6. *Let* f, g *and* h *be three vector valued entire Dirichlet series such that* $\rho_g(f) = \rho_g(h) = 1$. *Then*

$$\frac{\tau_g(f)}{T_g(h)} \leqslant \liminf_{\sigma\to\infty} \frac{\exp\left[\rho G^{-1}F(\sigma)\right]}{\exp\left[\rho G^{-1}H(\sigma)\right]} \leqslant \frac{\tau_g(f)}{\tau_g(h)} \leqslant \limsup_{\sigma\to\infty} \frac{\exp\left[\rho G^{-1}F(\sigma)\right]}{\exp\left[\rho G^{-1}H(\sigma)\right]} \leqslant \frac{T_g(f)}{\tau_g(h)}.$$

3. Main results

In this section we present the main results of the paper. First we prove

THEOREM 1. *If* f_1, f_2 *and* g *are entire VVDS defined by* (1) *such that* $\rho_g(f_1) = \rho_g(f_2) = 1$ *and* g *satisfies property* (A) *then*

1. $T_g(f_1 \pm f_2) \leqslant \max\left\{T_g(f_1), T_g(f_2)\right\}$. *The equality holds when* $T_g(f_1) \neq T_g(f_2)$.
2. $T_g(f_1 \cdot f_2) \leqslant T_g(f_1) + T_g(f_2)$.

Proof. (1) We may suppose that both $T_g(f_1)$ and $T_g(f_2)$ are finite since otherwise the result is trivial.

Case 1. Let $f = f_1 + f_2$, $T = T_g(f)$, $T_i = T_g(f_i)$; $i = 1,2$ and $T_1 \leqslant T_2$. For arbitrary $\varepsilon > 0$ and for all large σ, we have

$$F_1(\sigma) < G\left[\frac{1}{\rho}\log\left[(T_1+\varepsilon)e^{\sigma\rho}\right]\right] \leqslant G\left[\frac{1}{\rho}\log\left[(T_2+\varepsilon)e^{\sigma\rho}\right]\right] \tag{7}$$

and

$$F_2(\sigma) < G\left[\frac{1}{\rho}\log\left[(T_2+\varepsilon)e^{\sigma\rho}\right]\right]. \tag{8}$$

So for all large σ, $F(\sigma) \leqslant F_1(\sigma) + F_2(\sigma)$. Using inequalities (7) and (8), we get

$$F(\sigma) < 2G\left[\frac{1}{\rho}\log\left[(T_2+\varepsilon)e^{\sigma\rho}\right]\right] < \left\{G\left[\frac{1}{\rho}\log\left[(T_2+\varepsilon)e^{\sigma\rho}\right]\right]\right\}^2.$$

Since G satisfies property (A), therefore $F(\sigma) < G\left[\frac{\delta}{\rho}\log\left[(T_2+\varepsilon)e^{\sigma\rho}\right]\right]$, where $\delta > 1$.

Hence $\exp\left[\frac{\rho}{\delta}G^{-1}F(\sigma) - \sigma\rho\right] < [T_2+\varepsilon]$ for all large σ. Since $\delta > 1$ and $\varepsilon > 0$ are arbitrary, proceeding to limits as $\sigma \to \infty$ we have $T \leqslant T_2$.

Case 2. Let $T_1 < T_2$ and suppose that $T_1 < \mu < \lambda < T_2$. Then for all large σ, we have

$$F_1(\sigma) < G\left[\frac{1}{\rho}\log\left(\mu e^{\sigma\rho}\right)\right]$$

and there exists an increasing sequence $\{\sigma_n\} \to \infty$ such that

$$F_2(\sigma_n) > G\left[\frac{1}{\rho}\log\left(\lambda e^{\sigma_n\rho}\right)\right]. \tag{9}$$

By (1) of Lemma 3, we have

$$\lim_{\sigma\to\infty}\frac{G\left[\frac{1}{\rho}\log\left(\lambda e^{\sigma\rho}\right)\right]}{G\left[\frac{1}{\rho}\log\left(\mu e^{\sigma\rho}\right)\right]} = \infty.$$

Then for all large σ and arbitrary $\varepsilon > 0$, we obtain

$$\frac{G\left[\frac{1}{\rho}\log\left[\lambda e^{\sigma\rho}\right]\right]}{G\left[\frac{1}{\rho}\log\left[\mu e^{\sigma\rho}\right]\right]} > \frac{2}{\varepsilon}. \tag{10}$$

Now for all large σ, we have $F(\sigma) \geqslant F_2(\sigma) - F_1(\sigma)$. Hence using (9) and (10), we obtain

$$\begin{aligned} F(\sigma) &> G\left[\frac{1}{\rho}\log\left[\lambda e^{\sigma\rho}\right]\right] - \frac{\varepsilon}{2}G\left[\frac{1}{\rho}\log\left[\lambda e^{\sigma\rho}\right]\right] \\ &= \left(1-\frac{\varepsilon}{2}\right)G\left[\frac{1}{\rho}\log\left[\lambda e^{\sigma\rho}\right]\right] \\ &> G\left[\frac{1}{\rho}\log[(1-\varepsilon)\lambda e^{\sigma\rho}]\right] \end{aligned}$$

for all large σ.

Proceeding to limits as $\sigma \to \infty$, we obtain $T_g(f) > \mu$ where $\mu = (1-\varepsilon)\lambda$.

Considering case (1) and the above obtained results, we have

$$T_g(f_1 \pm f_2) \leqslant \max\left\{T_g(f_1), T_g(f_2)\right\}$$

which proves part (1).

(2) Let us put $T(f_i) = T_i$ and $T(g) = T'$. Using the definition of type, we have $G(\sigma) = \exp(T'e^{\rho\sigma})$ for all large σ.

Therefore

$$G^{-1}(\sigma) = \left[\frac{1}{\rho}\log\left(\frac{1}{T'}\log\sigma\right)\right] \text{ for all large } \sigma \to \infty. \tag{11}$$

Now for all large σ, we have

$$F(\sigma) \leqslant F_1(\sigma) \cdot F_2(\sigma)$$

By definition of relative type

$$T_g(f) = \limsup_{\sigma \to \infty} \frac{\exp[\rho G^{-1} F(\sigma)]}{e^{\sigma\rho}} \leqslant \limsup_{\sigma \to \infty} \frac{\exp\left[\rho G^{-1}(F_1(\sigma) \cdot F_2(\sigma))\right]}{e^{\sigma\rho}}.$$

Using the relation (11), we have

$$T_g(f) \leqslant \limsup_{\sigma \to \infty} \frac{\log(F_1(\sigma) \cdot F_2(\sigma))}{T' e^{\sigma\rho}}.$$

By the definition of type,

$$F_1(\sigma) \leqslant \exp(T_1 e^{\sigma\rho}) \tag{12}$$

and

$$F_2(\sigma) \leqslant \exp(T_2 e^{\sigma\rho}). \tag{13}$$

So, from inequalities (12) and (13), we have

$$T_g(f) \leqslant \limsup_{\sigma \to \infty} \frac{\log(\exp(T_1 e^{\sigma\rho}) \exp(T_2 e^{\sigma\rho}))}{T' e^{\sigma\rho}}.$$

Hence, $T_g(f) \leqslant \frac{T_1}{T'} + \frac{T_2}{T'}$ and therefore $T_g(f) \leqslant T_g(f_1) + T_g(f_2)$. This completes the proof of Theorem 1. □

Next we prove

THEOREM 2. *Let f and g be entire VVDS such that $\rho_g(f) = 1$, P be a vector valued Dirichlet polynomial and n be a positive integer. Then*

1. $T_g(f) = T_g(Pf) = T_{Pg}(f)$.
2. $T_g(f) = T_g(f') = T_{g'}(f)$.
3. $T_g(f^n) = nT_g(f)$.

Proof. (1) There exists $\alpha \in (0,1)$ and let m be a positive integer such that $\alpha \leqslant |P(s)| \leqslant e^{m\sigma}$.

Let $h = Pf$, therefore $\alpha F(\sigma) < H(\sigma) < e^{m\sigma} F(\sigma)$. Since $\beta > 1$, using Lemma 1 we have

$$F(\alpha\sigma) < \alpha F(\sigma) < H(\sigma) < F(\beta\sigma).$$

Hence

$$\frac{\exp\left[\rho G^{-1}(F(\alpha\sigma))\right]}{e^{\alpha\sigma\rho}} \cdot \frac{e^{\alpha\sigma\rho}}{e^{\sigma\rho}} < \frac{\exp\left[\rho G^{-1}(H(\sigma))\right]}{e^{\sigma\rho}} < \frac{\exp\left[\rho G^{-1}(F(\beta\sigma))\right]}{e^{\beta\sigma\rho}} \cdot \frac{e^{\beta\sigma\rho}}{e^{\sigma\rho}}.$$

Proceeding to limits as $\sigma \to \infty$, since $\alpha < 1$ and $\beta > 1$ are arbitrary, we have

$$T_g(f) \leqslant T_g(h) \leqslant T_g(f).$$

Hence $T_g(f) = T_g(Pf)$. Similarly, if we put $h = Pg$, then by above result,

$$\liminf_{\sigma\to\infty} \frac{\exp[\rho F^{-1}G(\sigma)]}{e^{\sigma\rho}} = \liminf_{\sigma\to\infty} \frac{\exp[\rho F^{-1}H(\sigma)]}{e^{\sigma\rho}}. \tag{14}$$

By definition of relative type, we have

$$\begin{aligned} T_h(f) &= \limsup_{\sigma\to\infty} \frac{\exp[\rho H^{-1}F(\sigma)]}{e^{\sigma\rho}} = \limsup_{\sigma\to\infty} \frac{\exp\left[\rho H^{-1}(\sigma)\right]}{\exp\left[\rho F^{-1}(\sigma)\right]} \\ &= \limsup_{\sigma\to\infty} \frac{\exp[\rho\sigma]}{\exp\left[\rho F^{-1}(H(\sigma))\right]} \\ &= \limsup_{\sigma\to\infty} \frac{\exp[\rho\sigma]}{\exp\left[\rho F^{-1}(G(\sigma))\right]}, \end{aligned}$$

from (14) above,

$$= \limsup_{\sigma\to\infty} \frac{\exp\left[\rho G^{-1}(\sigma)\right]}{\exp\left[\rho F^{-1}(\sigma)\right]} = \limsup_{\sigma\to\infty} \frac{\exp[\rho G^{-1}(F(\sigma))]}{e^{\sigma\rho}}.$$

Hence we get $T_g(f) = T_h(f)$. This proves part (1).

(2) From [3, p. 139], we have for any entire Dirichlet function $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$, $M(\sigma, f) - \varepsilon < (\sigma - \sigma_0)M'(\sigma, f) + |f(s_0)|$ for all large σ and $M'(\sigma, f) - \varepsilon \leqslant \frac{1}{\delta}M(\sigma + \delta, f)$ where $\varepsilon, \delta > 0$ and $M'(\sigma, f)$ denotes the maximum modulus of the derivative $f'(s)$. We can similarly show that for entire VVDS given by (1),

$$F(\sigma) < \sigma F'(\sigma) + O(1), \quad F'(\sigma) - \varepsilon \leqslant \frac{1}{\delta} F(\sigma + \delta) \tag{15}$$

Using the above inequalities and the definition of relative order, we have $\rho_g(f) = \rho_g(f')$ i.e., $\rho(f) = \rho(f') = \rho(g) = \rho$. Let $k > 1$, then by Lemma 3(ii), $F(k\sigma) > \sigma^n F(\sigma)$ for all large σ. Now replacing σ by $q\sigma$ such that $0 < q < 1$ and $qk = 1$, we get

$$F(qk\sigma) > (q\sigma)^n F(q\sigma).$$

Hence

$$F(\sigma) > (q\sigma)^n F(q\sigma) > \sigma F(q\sigma) \quad \text{for all large} \quad \sigma. \tag{16}$$

From inequalities (15) and (16), we obtain $F(q\sigma) < \frac{F(\sigma)}{\sigma} < F'(\sigma)$ for all large σ. Therefore $T_g(f') = \limsup_{\sigma\to\infty} \frac{\exp[\rho G^{-1}(F'(\sigma))]}{e^{\sigma\rho}} > \limsup_{\sigma\to\infty} \frac{\exp[\rho G^{-1}(F(q\sigma))]}{e^{\sigma\rho}}$.

Letting $q \to 1^-$, we obtain $T_g(f') \geqslant T_g(f)$.

For the reverse inequality, by (i) of Lemma 3, we have $F(r\sigma) > \frac{1}{\delta}F(\sigma)$, where $\delta > 0$ and $r > 1$.

Therefore $F'(\sigma) \leqslant \frac{1}{\delta}F(\sigma) < F(r\sigma)$.

By the definition of relative type we have $T_g(f') = \lim_{\sigma\to\infty} \sup \frac{\exp[\rho G^{-1}(F'(\sigma))]}{e^{\sigma\rho}} \leqslant \lim_{\sigma\to\infty} \sup \frac{\exp[\rho G^{-1}(F(r\sigma))]}{e^{r\sigma\rho}} \cdot \frac{e^{r\sigma\rho}}{e^{\sigma\rho}}$ where $r > 1$.

Letting $r \to 1^+$, we obtain $T_g(f') \leqslant T_g(f)$ which leads to $T_g(f') = T_g(f)$.

Similarly, from the above proved equality $T_f(g) = T_f(g')$. Therefore

$$\lim_{\sigma\to\infty} \sup \frac{\exp[\rho F^{-1}(G(\sigma))]}{e^{\sigma\rho}} = \lim_{\sigma\to\infty} \sup \frac{\exp[\rho F^{-1}(G'(\sigma))]}{e^{\sigma\rho}}.$$

By the definition of relative type,

$$\begin{aligned} T_g(f) &= \lim_{\sigma\to\infty} \sup \frac{\exp[\rho G^{-1}(F(\sigma))]}{e^{\sigma\rho}} \\ &= \lim_{\sigma\to\infty} \sup \frac{\exp[\rho G^{-1}(\sigma)]}{\exp[\rho F^{-1}(\sigma)]}, \quad \text{replacing } F(\sigma) \text{ by } \sigma \\ &= \lim_{\sigma\to\infty} \sup \frac{e^{\sigma\rho}}{\exp[\rho F^{-1}G(\sigma)]}, \quad \text{replacing } \sigma \text{ by } G(\sigma) \\ &= \lim_{\sigma\to\infty} \sup \frac{e^{\sigma\rho}}{\exp[\rho F^{-1}G(\sigma)]} = \lim_{\sigma\to\infty} \sup \frac{\exp[\rho G^{-1}(\sigma)]}{\exp[\rho F^{-1}(\sigma)]} \\ &= \lim_{\sigma\to\infty} \sup \frac{\exp[\rho G^{-1}(F(\sigma))]}{e^{\sigma\rho}}. \end{aligned}$$

Hence $T_g(f) = T_{g'}(f)$ which proves the result.

(3) Since $\rho_g(f) = 1$, by Lemma 4 we have $\rho(f) = \rho(g) = \rho(f^n) = \rho$. Let $[F(\sigma)]^n$ be the maximum modulus of f^n. Then,

$$\begin{aligned} T_g(f^n) &= \lim_{\sigma\to\infty} \sup \frac{\exp\left[\rho G^{-1}[F(\sigma)]^n\right]}{e^{\sigma\rho}} \\ &= \lim_{\sigma\to\infty} \sup \frac{\exp\left[\rho G^{-1}[F(\sigma)]^n\right]}{\log F(\sigma)} \cdot \frac{\log F(\sigma)}{e^{\sigma\rho}} \\ &= \lim_{\sigma\to\infty} \sup \frac{\exp\left[\rho G^{-1}[F(\sigma)]^n\right]}{\log F(\sigma)} T(f) \\ &= \lim_{\sigma\to\infty} \sup \frac{\exp[\rho\sigma]}{\log [G(\sigma)]^{1/n}} T(f). \end{aligned}$$

Hence $T_g(f^n) = nT_g(f)$. This completes the proof of Theorem 2. □

THEOREM 3. *If f is of perfectly regular growth and regular relative type with respect to g, $\rho_g(f) = 1$ and $T_g(f) = T(f) = T$ then g is of perfectly regular growth and of type one. Conversely, if g is of perfectly regular growth and of type one then $T_g(f) = T(f)$ for all entire VVDS f of regular relative type with respect to g.*

Proof. Given that $T_g(f) = T(f) = T$ and $\rho(f) = \rho(g) = \rho(\text{say})$. Let $\varepsilon > 0$ be given. By definition of type, since f is of regular type, we have

$$\exp[(T-\varepsilon)e^{\sigma\rho}] < F(\sigma) < \exp[(T+\varepsilon)e^{\sigma\rho}], \quad \sigma > \sigma_0. \tag{17}$$

Similarly, since f is of regular relative type with respect to g, we have

$$G\left[\frac{1}{\rho}\log[(T-\varepsilon)e^{\sigma\rho}]\right] < F(\sigma) < G\left[\frac{1}{\rho}\log[(T+\varepsilon)e^{\sigma\rho}]\right], \quad \sigma > \sigma_1 \tag{18}$$

Combining (17) and (18), we have for $\sigma > \sigma_2 = \max(\sigma_0, \sigma_1)$,

$$G\left[\frac{1}{\rho}\log[(T-\varepsilon)e^{\sigma\rho}]\right] < \exp[(T+\varepsilon)\,e^{\sigma\rho}],$$

$$\exp[(T-\varepsilon)e^{\sigma\rho}] < G\left[\frac{1}{\rho}\log[(T+\varepsilon)e^{\sigma\rho}]\right]$$

or $\exp\left[\frac{(T-\varepsilon)}{(T+\varepsilon)}e^{\sigma\rho}\right] < G(\sigma) < \exp\left[\frac{(T+\varepsilon)}{(T-\varepsilon)}e^{\sigma\rho}\right]$.

Since $\varepsilon > 0$ is arbitrary, on proceeding to limits, it follows that g is of perfectly regular growth and of type one. Conversely, if $g(s)$ is of perfectly regular growth and of type 1 then for arbitrary $\varepsilon > 0$ and $\sigma > \sigma'(\varepsilon)$,

$$\exp[(1-\varepsilon)e^{\sigma\rho}] < G(\sigma) < \exp[(1+\varepsilon)e^{\sigma\rho}]. \tag{19}$$

From the definition of relative type, for $\sigma > \sigma''(\varepsilon)$,

$$G\left[\frac{1}{\rho}\log[(T_g(f)-\varepsilon)e^{\sigma\rho}]\right] < F(\sigma) < G\left[\frac{1}{\rho}\log[(T_g(f)+\varepsilon)e^{\sigma\rho}]\right]. \tag{20}$$

Using inequalities (19) and (20), we obtain for

$$\sigma > \max(\sigma', \sigma''), \quad G\left[\frac{1}{\rho}\log[(T_g(f)-\varepsilon)e^{\sigma\rho}]\right] > \exp[(1-\varepsilon)\,(T_g(f)-\varepsilon)\,e^{\sigma\rho}]$$

and

$$G\left[\frac{1}{\rho}\log[(T_g(f)+\varepsilon)e^{\sigma\rho}]\right] < \exp[(1+\varepsilon)\,(T_g(f)+\varepsilon)\,e^{\sigma\rho}].$$

Hence

$$\exp[(1-\varepsilon)\,(T_g(f)-\varepsilon)\,e^{\sigma\rho}] < F(\sigma) < \exp[(1+\varepsilon)\,(T_g(f)+\varepsilon)\,e^{\sigma\rho}]$$

or

$$\exp\left[\left\{T_g(f)-\varepsilon\,(1+T_g(f)-\varepsilon)\right\}e^{\sigma\rho}\right] < F(\sigma) < \exp\left[\left\{T_g(f)+\varepsilon\,(1+T_g(f)+\varepsilon)\right\}e^{\sigma\rho}\right]$$

Since $\varepsilon > 0$ is arbitrary, on proceeding to limits as $\sigma \to \infty$, we obtain $T(f) = T_g(f)$ which completes the proof of Theorem 3. $\square$

4. Asymptotic behaviour of two functions

DEFINITION. Two entire VVDS f_1 and f_2 are said to be asymptotically equivalent if

$$\lim_{\sigma\to\infty}\frac{F_1(\sigma)}{F_2(\sigma)}=1.$$

In this case we write $f_1 \sim f_2$ then clearly $f_2 \sim f_1$.

Now we prove

THEOREM 4. *Let f_1, f_2, f, g_1, g_2 and g be entire VVDS. Then*

1. *If $\rho_g(f_1)=\rho_g(f_2)=1$ and $f_1\sim f_2$ then $T_g(f_1)=T_g(f_2)$.*
2. *If $\rho_{g_1}(f)=\rho_{g_2}(f)=1$ and $g_1\sim g_2$ then $T_{g_1}(f)=T_{g_2}(f)$.*

Proof. (1) Consider $\rho(f_1)=\rho(f_2)=\rho(g)=\rho$. Given that $f_1\sim f_2$, therefore there exists $\varepsilon>0$ such that $F_1(\sigma)<(1+\varepsilon)F_2(\sigma)$ for all large σ.

Using Lemma 1 and assuming that $(1+\varepsilon)<\beta$, we have $F_1(\sigma)<F_2(\beta\sigma)$.

By definition of relative type

$$T_g(f_1)=\limsup_{\sigma\to\infty}\frac{\exp\left[\rho G^{-1}\left(F_1(\sigma)\right)\right]}{e^{\sigma\rho}}\leqslant\limsup_{\sigma\to\infty}\frac{\exp\left[\rho G^{-1}\left(F_2(\beta\sigma)\right)\right]}{e^{\sigma\rho}}.$$

As $\beta\to 1$ i.e., $\varepsilon\to 0$, we obtain $T_g(f_1)\leqslant T_g(f_2)$. The reverse inequality follows similarly. Hence $T_g(f_1)=T_g(f_2)$.

(2) Let $\rho(f)=\rho(g_1)=\rho(g_2)=\rho$. Given that $g_1\sim g_2$, therefore there exists $\varepsilon>0$ such that for all large σ

$$G_1(\sigma)<(1+\varepsilon)G_2(\sigma).$$

Consider $\alpha>1$.Then by Lemma 1, we have $G_1(\sigma)<G_2(\alpha\sigma)$ and thus

$$\sigma<G_1^{-1}(G_2(\alpha\sigma)).$$

Now put $G_2(\alpha\sigma)=t$, then $\frac{1}{\alpha}G_2^{-1}(t)<G_1^{-1}(t)$ for all large value of σ. Hence by definition of relative type,

$$T_{g_2}(f)=\limsup_{\sigma\to\infty}\frac{\exp\left[\rho G_2^{-1}\left(F(\sigma)\right)\right]}{e^{\sigma\rho}}\leqslant\limsup_{\sigma\to\infty}\frac{\exp\left[\rho\alpha G_1^{-1}\left(F(\sigma)\right)\right]}{e^{\sigma\rho}}.$$

Now letting $\alpha\to 1^+$, we have $T_{g_2}(f)\leqslant T_{g_1}(f)$. The reverse inequality $T_{g_2}(f)\geqslant T_{g_1}(f)$ holds similarly. Hence $T_{g_1}(f)=T_{g_2}(f)$. This completes the proof of Theorem 4. □

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