A CLASS OF HARMONIC STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS ASSOCIATED WITH THE SRIVASTAVA–WRIGHT GENERALIZED HYPERGEOMETRIC FUNCTION

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Abstract. Making use of Srivastava-Wright operator we introduced a new class of complex-valued harmonic functions with respect to symmetric points which are orientation preserving, univalent and starlike. We obtain coefficient conditions, extreme points, distortion bounds, convex combination.

1. Introduction

Denote by \( \mathcal{H} \) the family of functions
\[
f = h + \overline{g},
\]
which are analytic univalent and sense-preserving in the unit disc \( U = \{ z : |z| < 1 \} \). So that \( f \) is normalized by \( f(0) = f_{\bar{z}}(0) - 1 = 0 \). Thus, for \( f = h + \overline{g} \in \mathcal{H} \), we may express the analytic functions \( h \) and \( g \) in the forms
\[
h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1.
\]
where \( h \) and \( g \) are analytic in \( D \). We call \( h \) the analytic part and \( g \) the co-analytic part of \( f \). A necessary and sufficient condition for \( f \) to be locally univalent and sense-preserving in \( \mathcal{H} \) is that \( |h'(z)| > |g'(z)| \) in \( \mathcal{H} \) (see [3]).

Hence
\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k} \overline{z}^k, \quad |b_1| < 1.
\]
We denote \( \overline{\mathcal{H}} \) the subclass of \( \mathcal{H} \) consists of harmonic functions \( f = h + \overline{g} \) of the form
\[
f(z) = z - \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k} \overline{z}^k, \quad |b_1| < 1.
\]


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Let the Hadamard product (or convolution) of two power series \( \Phi(z) = z + \sum_{k=2}^{\infty} \phi_k z^k \) and \( \Psi(z) = z + \sum_{k=2}^{\infty} \psi_k z^k \) be defined by

\[
(\Phi \ast \Psi)(z) = z + \sum_{k=2}^{\infty} \phi_k \psi_k z^k = (\Psi \ast \Phi)(z).
\]

Let \( \alpha_1, A_1, \ldots, \alpha_q, A_q \) and \( \beta_1, B_1, \ldots, \beta_s, B_s \) \((q, s \in \mathbb{N})\) be positive and real parameters such that

\[
1 + \sum_{j=1}^{s} B_j - \sum_{j=1}^{q} A_j \geq 0.
\]

The Srivastava-Wright generalized hypergeometric function \([18]\) (see also \([4–5, 17, 12, 13]\) and \([14]\))

\[
q\Psi_s[(\alpha_1, A_1), \ldots, (\alpha_q, A_q); (\beta_1, B_1), \ldots, (\beta_s, B_s); z] = q\Psi_s[(\alpha_i, A_i); (\beta_i, B_i); z]
\]

is defined by

\[
q\Psi_s[(\alpha_i, A_i); (\beta_i, B_i); z] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{q} \Gamma(\alpha_i + nA_i)}{\prod_{i=1}^{s} \Gamma(\beta_i + nB_i)} \cdot \frac{z^n}{n!} \quad (z \in U).
\]

If \( A_i = 1 \) \((i = 1, \ldots, q)\) and \( B_i = 1 \) \((i = 1, \ldots, s)\), we have the relationship:

\[
\Omega q\Psi_s[(\alpha_i, 1); (\beta_i, 1); z] = qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),
\]

where \( qF_s \) is the generalized hypergeometric function (see for details \([5, 6]\)) and

\[
\Omega = \frac{\prod_{i=1}^{s} \Gamma(\beta_i)}{\prod_{i=1}^{q} \Gamma(\alpha_i)}.
\]

In \([4]\) Dziok and Riana extended the linear operator by using Srivastava-Wright generalized hypergeometric function (see also \([11]\) and \([16]\)). First we define a function

\[
q\Phi_s[(\alpha_i, A_i); (\beta_i, B_i); z] \text{ by }
\]

\[
q\Phi_s[(\alpha_i, A_i); (\beta_i, B_i); z] = \Omega z q\Psi_s[(\alpha_i, A_i); (\beta_i, B_i); z]
\]

and consider the following linear operator

\[
\theta_{q,s}[(\alpha_i, A_i); (\beta_i, B_i)] : S_H \rightarrow S_H,
\]
defined by the convolution
\[
\theta_{p,q,s} \left[ (\alpha_i, A_i)_q ; (\beta_i, B_i)_s \right] f(z) = q \Phi_s \left[ (\alpha_i, A_i)_q ; (\beta_i, B_i)_s ; z \right] * f(z).
\]
We observe that, for a function \( f(z) \) of the form (1.1), we have
\[
\theta_{q,s} \left[ (\alpha_i, A_i)_q ; (\beta_i, B_i)_s \right] f(z) = z + \sum_{k=2}^{\infty} \Omega \sigma_k (\alpha_1) a_k z^k,
\]
(1.6)
where \( \Omega \) is given by (1.5) and \( \sigma_k (\alpha_1) \) is defined by
\[
\sigma_k (\alpha_1) = \frac{\Gamma (\alpha_1 + A_1 (k - 1)) \ldots \Gamma (\alpha_1 + A_1 (k - 1))}{\Gamma (B_1 + B_1 (k - 1)) \ldots \Gamma (B_1 + B_1 (k - 1)) (k - 1)!}.
\]
(1.7)
If, for convenience, we write
\[
\theta_{q,s} [\alpha_1, A_1, B_1] f(z) = \theta_{q,s} [(\alpha_1, A_1), \ldots, (\alpha_q, A_q); (\beta_1, B_1), \ldots, (\beta_s, B_s)] f(z),
\]
then one can easily verify from the definition (1.6) that (see [10] and [16]):
\[
zA_1 \left( \theta_{q,s} [\alpha_1, A_1, B_1] f(z) \right)' = \alpha_1 \theta_{q,s} [\alpha_1 + 1, A_1, B_1] f(z) - (\alpha_1 - A_1) \theta_{q,s} [\alpha_1, A_1, B_1] f(z).
\]
(1.8)
Applying the Srivastava-Wright operator to the harmonic functions \( f = h + \overline{g} \) given by (1.1) we get
\[
\theta_{q,s} [\alpha_1, A_1, B_1] f(z) = \theta_{q,s} [\alpha_1, A_1, B_1] h(z) + \theta_{q,s} [\alpha_1, A_1, B_1] g(z)
\]
(1.9)
Motivated by Jahangiri et al. [7–9] and Ahuja and Jahangiri [1], we define a new subclass \( H S_{s^+} ([\alpha_1, A_1, B_1], \gamma) \) of \( \mathcal{H} \) that are starlike with respect symmetric points.

**Definition 1.** For \( 0 \leq \gamma < 1 \) and \( z = re^{i\theta} \in U \), we let \( \mathcal{H} S_{s^+} ([\alpha_1, A_1, B_1], \gamma) \) a subclass of \( \mathcal{H} \) of the form \( f = h + \overline{g} \) be given by (1.3) and satisfying the analytic criteria
\[
\text{Re} \left\{ \frac{2z (\theta_{q,s} [\alpha_1, A_1, B_1] (f(z))')}{z' (\theta_{q,s} [\alpha_1, A_1, B_1] f(z) - \theta_{q,s} [\alpha_1, A_1, B_1] f(-z))} \right\} > \gamma,
\]
(1.10)
where \( \theta_{q,s} [\alpha_1, A_1, B_1] f(z) \) is defined by (1.9) and \( z' = \frac{\partial}{\partial \theta} (z = re^{i\theta}) \). We also let \( \mathcal{H} S_{s^+} ([\alpha_1, A_1, B_1], \gamma) = \mathcal{H} S_{s^+} ([\alpha_1, A_1, B_1], \gamma) \cap \mathcal{H} \). The family \( \mathcal{H} S_{s^+} ([\alpha_1, A_1, B_1], \gamma) \) is of special interest because for suitable choices of \( q, s, [A_1], [B_1] \) and \( [\alpha_1] \), we note that

(i) If \( A_i = 1 \ (i = 1, \ldots, q) \) and \( B_j = 1 \ (j = 1, \ldots, s) \), we have \( \mathcal{H} S_{s^+} ([\alpha_1, 1, 1], \gamma) = \mathcal{H} S_{s^+} ([\alpha_1], \gamma) \), which was studied by Murugusundaramoorthy et al. [11];

(ii) If \( f(-z) = -f(z) \), \( A_i = 1 \ (i = 1, \ldots, q) \) and \( B_j = 1 \ (j = 1, \ldots, s) \), we have \( \mathcal{H} S_{s^+} ([\alpha_1, 1, 1], \gamma) = S_{s^+}^* (\alpha_1, \gamma) \), which was studied by Al-Kharsani and AL-Khal [2].

**Remark 1.** If the co-analytic part of \( f = h + \overline{g} \) is zero, \( \alpha_i = A_i = 1 \ (i = 1, \ldots, q) \) and \( \beta_j = B_j = 1 \ (j = 1, \ldots, s) \) then \( \mathcal{H} S_{s^+} ([\alpha_1, 1], \gamma) \) turns out to be the class \( S_{s^+}^* (\gamma) \) of
starlike functions with respect to symmetric points which was introduced by Sakaguchi [15].

In this paper, we have obtained the coefficient conditions for the classes \( \mathcal{H} S_{s^*}(\{\alpha_1,A_1,B_1\},\gamma) \) and \( \mathcal{H} S_{s^*}(\{\alpha_1,A_1,B_1\},\gamma) \). Further a representation theorem, inclusion properties and distortion bounds for the class \( \mathcal{H} S_{s^*}(\{\alpha_1,A_1,B_1\},\gamma) \) are also established.

2. Coefficient characterization

Unless otherwise mentioned, we assume throughout this paper that \( q,s \in \mathbb{N} \), \( \alpha_1 = 1 \), \( \alpha_1,A_1,...,\alpha_q,A_q, \beta_1,B_1,...,\beta_s,B_s \in \mathbb{R}^+ \), and \( 0 \leq \gamma < 1 \). We begin with a sufficient condition for functions in \( \mathcal{H} S_{s^*}(\{\alpha_1,A_1,B_1\},\gamma) \).

**Theorem 1.** Let \( f = h + \overline{g} \) be given by (1.3). Furthermore, let

\[
\sum_{k=1}^{\infty} \left[ \frac{2k - \gamma (1 - (-1)^k)}{2 (1 - \gamma)} \right] \Omega \sigma_k(\alpha_1) |a_k| + \sum_{k=1}^{\infty} \left[ \frac{2k + \gamma (1 - (-1)^k)}{2 (1 - \gamma)} \right] \Omega \sigma_k(\alpha_1) |b_k| \leq 1
\]

(2.1)

where \( \Omega \) and \( \sigma_k(\alpha_1) \) are defined by (1.5) and (1.7), respectively. Then \( f \) is sense-preserving, harmonic univalent in \( U \) and \( f \in \mathcal{H} S_{s^*}(\{\alpha_1,A_1,B_1\},\gamma) \).

**Proof.** According the condition (1.10), we only need to show that if (2.1) holds, then

\[
\text{Re} \left\{ \frac{2z (\theta_{q,s}[\alpha_1,A_1,B_1] f(z))'}{z' [\theta_{q,s}[\alpha_1,A_1,B_1] f(z) - \theta_{q,s}[\alpha_1,A_1,B_1] f(-z)]} \right\} = \text{Re} \frac{A(z)}{B(z)} > \gamma
\]

where

\[
A(z) = 2z' (\theta_{q,s}[\alpha_1,A_1,B_1] f(z))' = 2z' \left[ z + \sum_{k=2}^{\infty} k \Omega \sigma_k(\alpha_1) a_k z^k - \sum_{k=1}^{\infty} k \Omega \sigma_k(\alpha_1) b_k z^k \right]
\]

and

\[
B(z) = z' \left[ \theta_{q,s}[\alpha_1,A_1,B_1] f(z) - \theta_{q,s}[\alpha_1,A_1,B_1] f(-z) \right]
= z' \left[ z + \sum_{k=2}^{\infty} \frac{1 - (-1)^k}{2} \Omega \sigma_k(\alpha_1) a_k z^k + \sum_{k=1}^{\infty} \frac{1 - (-1)^k}{2} \Omega \sigma_k(\alpha_1) b_k z^k \right]
\]

Using the fact that \( \text{Re} \{w(z)\} > \gamma \) if and only if \( |1 - \gamma + w| > |1 + \gamma - w| \), it suffices to show that

\[
|A(z) + (1 - \gamma)B(z)| - |A(z) - (1 + \gamma)B(z)| > 0.
\]

(2.2)
Substituting for $A(z)$ and $B(z)$ in (2.2) and by using (2.1), we obtain

$$
\begin{align*}
2(2 - \gamma)z + \sum_{k=2}^{\infty} \left[ 2k + (1 - \gamma)(1 - (-1)^k) \right] \Omega \sigma_k(\alpha_1) a_k z^k \\
- \sum_{k=1}^{\infty} \left[ 2k - (1 - \gamma)(1 - (-1)^k) \right] \Omega \sigma_k(\alpha_1) b_k z^k \\
- 2\gamma z + \sum_{k=1}^{\infty} \left[ 2k - (1 + \gamma)(1 - (-1)^k) \right] \Omega \sigma_k(\alpha_1) a_k z^k \\
- \sum_{k=1}^{\infty} \left[ 2k + (1 + \gamma)(1 - (-1)^k) \right] \Omega \sigma_k(\alpha_1) b_k z^k \\
gt 4(1 - \gamma)|z| - 2 \sum_{k=2}^{\infty} \left[ 2k - \gamma(1 - (-1)^k) \right] \Omega \sigma_k(\alpha_1)|a_k||z|^k \\
- 2 \sum_{k=1}^{\infty} \left[ 2k + \gamma(1 - (-1)^k) \right] \Omega \sigma_k(\alpha_1)|b_k||z|^k \\
= 4(1 - \gamma)|z| \left[ 1 - \sum_{k=2}^{\infty} \frac{[2k - \gamma(1 - (-1)^k)]}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1)|a_k||z|^k \\
- \sum_{k=1}^{\infty} \frac{[2k + \gamma(1 - (-1)^k)]}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1)|b_k||z|^k \right]. \\
\geq 4(1 - \gamma) \left[ 1 - \sum_{k=2}^{\infty} \frac{[2k - \gamma(1 - (-1)^k)]}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1)|a_k| \\
- \sum_{k=1}^{\infty} \frac{[2k + \gamma(1 - (-1)^k)]}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1)|b_k| \right] \geq 0.
\end{align*}
$$

This last expression is non-negative by (2.1).

The harmonic univalent functions

$$
f(z) = z + \sum_{k=2}^{\infty} \frac{2(1 - \gamma)}{2k - \gamma(1 - (-1)^k)} \Omega \sigma_k(\alpha_1) X_k z^k \\
+ \sum_{k=1}^{\infty} \frac{2(1 - \gamma)}{2k - \gamma(1 - (-1)^k)} \Omega \sigma_k(\alpha_1) Y_k z^k,
$$

(2.3)

where $\sum_{k=2}^{\infty} |X_k| + \sum_{k=1}^{\infty} |Y_k| = 1$, show that the coefficient bound given by (2.1) is sharp.
The functions of the form (2.3) are in $\mathcal{H}S_{s^*}([\alpha_1, A_1, B_1], \gamma)$ because
\[
\sum_{k=2}^{\infty} \frac{2k - \gamma (1 - (-1)^k)}{2 (1 - \gamma)} \Omega \sigma_k(\alpha_1) |a_k| + \sum_{k=1}^{\infty} \frac{2k + \gamma (1 - (-1)^k)}{2 (1 - \gamma)} \Omega \sigma_k(\alpha_1) |b_k| = \sum_{k=2}^{\infty} |X_k| + \sum_{k=1}^{\infty} |Y_k| = 1.
\]

This completes the proof of Theorem 1. □

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f(z)$ are of the form (1.4).

**THEOREM 2.** Let $f = h + \overline{g}$ be given by (1.4). Then $f \in \mathcal{H}S_{s^*}([\alpha_1, A_1, B_1], \gamma)$ if and only if
\[
\sum_{k=2}^{\infty} \frac{2k - \gamma (1 - (-1)^k)}{2 (1 - \gamma)} \Omega \sigma_k(\alpha_1) |a_k| + \sum_{k=1}^{\infty} \frac{2k + \gamma (1 - (-1)^k)}{2 (1 - \gamma)} \Omega \sigma_k(\alpha_1) |b_k| \leq 1
\]
(2.4)
where $\Omega$ and $\sigma_k(\alpha_1)$ are defined by (1.5) and (1.7), respectively.

**Proof.** Since $\mathcal{H}S_{s^*}([\alpha_1, A_1, B_1], \gamma) \subset \mathcal{H}S_{s^*}([\alpha_1, A_1, B_1], \gamma)$, we only need to prove the "only if" part of the theorem. To this end, for functions $f(z)$ of the form (1.4), we notice that the condition
\[
\text{Re} \left\{ \frac{2z}{z} \left[ \theta_{q,s} [\alpha_1, A_1, B_1] f(z) \right]' \right\} > \gamma
\]
is equivalent to
\[
\text{Re} \left\{ \frac{2(1-\gamma) - \sum_{k=2}^{\infty} [2k - \gamma (1 - (-1)^k)] \Omega \sigma_k(\alpha_1) a_k r^{k-1} - \sum_{k=1}^{\infty} [2k + \gamma (1 - (-1)^k)] \Omega \sigma_k(\alpha_1) b_k r^{k-1}}{2 - \sum_{k=2}^{\infty} (1 - (-1)^k) \Omega \sigma_k(\alpha_1) a_k r^{k-1} + \sum_{k=1}^{\infty} (1 - (-1)^k) \Omega \sigma_k(\alpha_1) b_k r^{k-1}} \right\} > 0
\]
(2.5)
The above required condition (2.5) must hold for all values of $z$ in $U$. Upon choosing the values of $z$ on the positive real axis where $0 \leq z = r < 1$, we must have
\[
\text{Re} \left\{ \frac{2(1-\gamma) - \sum_{k=2}^{\infty} [2k - \gamma (1 - (-1)^k)] \Omega \sigma_k(\alpha_1) a_k r^{k-1} - \sum_{k=1}^{\infty} [2k + \gamma (1 - (-1)^k)] \Omega \sigma_k(\alpha_1) b_k r^{k-1}}{2 - \sum_{k=2}^{\infty} (1 - (-1)^k) \Omega \sigma_k(\alpha_1) a_k r^{k-1} + \sum_{k=1}^{\infty} (1 - (-1)^k) \Omega \sigma_k(\alpha_1) b_k r^{k-1}} \right\} > 0.
\]
(2.6)
If the condition (2.4) does not hold, then the numerator in (2.6) is negative for $r$ sufficiently close to 1. Hence there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.6) is negative. This contradicts the required condition for $f(z) \in \mathcal{H}S_{s^*}([\alpha_1, A_1, B_1], \gamma)$ and so the proof of Theorem 2 is completed. □
3. Extreme points and distortion theorem

Our next theorem is on the extreme points of convex hulls of $\mathcal{HS}^s([\alpha_1, A_1, B_1], \gamma)$ denoted by $\text{clco } \mathcal{HS}^s([\alpha_1, A_1, B_1], \gamma)$.

**Theorem 3.** A function $f_k(z) \in \text{clco } \mathcal{HS}^s([\alpha_1, A_1, B_1], \gamma)$ if and only if $f_k(z)$ can be expressed in the form

$$f_k(z) = \sum_{k=1}^{\infty} \left[ X_k h_k(z) + Y_k g_k(z) \right],$$

(3.1)

where $h_1(z) = z$,

$$h_k(z) = z - \frac{2(1 - \gamma)}{2k - \gamma (1 - (-1)^k)} \frac{z^k}{\Omega \sigma_k(\alpha_1)} (k \geq 2),$$

and

$$g_k(z) = z + \frac{2(1 - \gamma)}{2k + \gamma (1 - (-1)^k)} \frac{z^k}{\Omega \sigma_k(\alpha_1)} (k \geq 1),$$

$$X_k \geq 0, \quad Y_k \geq 0, \quad \sum_{k=1}^{\infty} (X_k + Y_k) = 1.$$

In particular, the extreme points of $\mathcal{HS}^s([\alpha_1, A_1, B_1], \gamma)$ are $\{h_k\}$ and $\{g_k\}$.

**Proof.** For functions $f_k(z)$ of the form (3.1), we have

$$f_k(z) = z - \sum_{k=2}^{\infty} \frac{2(1 - \gamma)}{2k - \gamma (1 - (-1)^k)} \frac{z^k}{\Omega \sigma_k(\alpha_1)} X_k + \sum_{k=1}^{\infty} \frac{2(1 - \gamma)}{2k + \gamma (1 - (-1)^k)} \frac{z^k}{\Omega \sigma_k(\alpha_1)} Y_k.$$

Then by Theorem 2

$$\sum_{k=2}^{\infty} \frac{2k - \gamma (1 - (-1)^k)}{2 (1 - \gamma)} \frac{1}{\Omega \sigma_k(\alpha_1)} |a_k| + \sum_{k=1}^{\infty} \frac{2k + \gamma (1 - (-1)^k)}{2 (1 - \gamma)} \frac{1}{\Omega \sigma_k(\alpha_1)} |b_k|$$

$$= \sum_{k=2}^{\infty} \frac{2k - \gamma (1 - (-1)^k)}{2 (1 - \gamma)} \frac{1}{\Omega \sigma_k(\alpha_1)} X_k$$

$$+ \sum_{k=1}^{\infty} \frac{2k + \gamma (1 - (-1)^k)}{2 (1 - \gamma)} \frac{1}{\Omega \sigma_k(\alpha_1)} Y_k$$

$$= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1.$$

and so $f_k \in \mathcal{HS}^s([\alpha_1, A_1, B_1], \gamma)$. 


Conversely, if \( f_k \in \text{clco } \mathcal{H}_{s^+}([\alpha_1, A_1, B_1], \gamma) \). Setting

\[
X_k = \frac{2k - \gamma (1 - (-1)^k)}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1) |a_k| \quad (k \geq 2),
\]

and

\[
Y_k = \frac{2k + \gamma (1 - (-1)^k)}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1) |b_k| \quad (k \geq 1).
\]

We obtain \( f_k(z) = \sum_{k=1}^{\infty} [X_k h_k(z) + Y_k g_k(z)] \) as required. \( \square \)

**Theorem 4.** Let the functions \( f(z) \) defined by (1.4) be in the class \( \mathcal{H}_{s^+}([\alpha_1, A_1, B_1], \gamma) \). Then for \( |z| = r < 1 \), we have

\[
(1 - |b_1|) r - \frac{1 - \gamma}{\Omega \sigma_2(\alpha_1)} \left\{ \frac{1 - \gamma}{2} - \frac{1 + \gamma}{2} |b_1| \right\} r^2 \leq |f(z)| \leq (1 + |b_1|) r + \frac{1}{\Omega \sigma_2(\alpha_1)} \left\{ \frac{1 - \gamma}{2} - \frac{1 + \gamma}{2} |b_1| \right\} r^2.
\]

The result is sharp.

**Proof.** We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Let \( f(z) \in \mathcal{H}_{s^+}([\alpha_1, A_1, B_1], \gamma) \). Taking the absolute value of \( f \) we have

\[
|f(z)| \leq (1 + |b_1|) r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^k \leq (1 + |b_1|) r + r^2 \sum_{k=2}^{\infty} (|a_k| + |b_k|)
\]

\[
\leq (1 + |b_1|) r + \frac{(1 - \gamma)}{\Omega \sigma_2(\alpha_1)} \sum_{k=2}^{\infty} \Omega \sigma_k(\alpha_1) \left( |a_k| + |b_k| \right) r^2
\]

\[
= (1 + |b_1|) r + \frac{(1 - \gamma)^2}{\Omega \sigma_2(\alpha_1)} \sum_{k=2}^{\infty} \left\{ \frac{2k - \gamma (1 - (-1)^k)}{4(1 - \gamma)} |a_k| + \frac{2k + \gamma (1 - (-1)^k)}{4(1 - \gamma)} |b_k| \right\} \Omega \sigma_k(\alpha_1)
\]

\[
= (1 + |b_1|) r + \frac{(1 - \gamma)^2}{2 \Omega \sigma_2(\alpha_1)} \sum_{k=2}^{\infty} \left\{ \frac{2k - \gamma (1 - (-1)^k)}{2(1 - \gamma)} |a_k| + \frac{2k + \gamma (1 - (-1)^k)}{2(1 - \gamma)} |b_k| \right\} \Omega \sigma_k(\alpha_1)
\]

\[
\leq (1 + |b_1|) r + \frac{(1 - \gamma)^2}{2 \Omega \sigma_2(\alpha_1)} \left( 1 - \frac{1 + \gamma}{1 - \gamma} |b_1| \right) r^2.
\]

The bounds given in Theorem 4 for functions \( f = h + \overline{g} \) of form (1.4) also hold for functions of the form (1.2) if the coefficient condition (2.1) is satisfied. The upper bound given for \( f \in \mathcal{H}_{s^+}([\alpha_1, A_1, B_1], \gamma) \) is sharp and the equality occurs for the functions

\[
f(z) = z + b_1 \overline{z} + \frac{1}{\Omega \sigma_2(\alpha_1)} \left[ \frac{1 - \gamma}{2} - \frac{1 + \gamma}{2} b_1 \right] \overline{z}^2,
\]

show that the bounds given in Theorem 4 are sharp. This completes the proof of Theorem 4. \( \square \)
4. Convolution and convex combination

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k \bar{z}^k, \quad |b_1| < 1$$  \hfill (4.1)

and

$$G(z) = z - \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k \bar{z}^k (A_k \geq 0; B_k \geq 0)$$  \hfill (4.2)

we define the convolution of two harmonic functions $f$ and $G$ as

$$(f * G)(z) = f(z) * G(z) = z - \sum_{k=2}^{\infty} a_k A_k z^k + \sum_{k=1}^{\infty} b_k B_k \bar{z}^k.$$  \hfill (4.3)

Using this definition, we show that the class $\mathcal{H}S_s^* ([\alpha_1, A_1, B_1], \gamma)$ is closed under convolution.

**Theorem 5.** For $0 \leq \mu \leq \gamma < 1$, let $f \in \mathcal{H}S_s^* ([\alpha_1, A_1, B_1], \gamma)$ and $G \in \mathcal{H}S_s^* ([\alpha_1, A_1, B_1], \mu)$. Then $f * G \in \mathcal{H}S_s^* ([\alpha_1, A_1, B_1], \gamma) \subset \mathcal{H}S_s^* ([\alpha_1, A_1, B_1], \mu)$.

**Proof.** Let the function $f(z)$ defined by (4.1) be in the class $\mathcal{H}S_s^* ([\alpha_1, A_1, B_1], \gamma)$ and let the function $G(z)$ defined by (4.2) be in the class $\mathcal{H}S_s^* ([\alpha_1, A_1, B_1], \mu)$. Then the convolution $f * G$ is given by (4.3). We wish to show that the coefficients of $f * G$ satisfy the required condition given in Theorem 2. For $G \in \mathcal{H}S_s^* ([\alpha_1, A_1, B_1], \mu)$ we note that $0 \leq A_k \leq 1$ and $0 \leq B_k \leq 1$. Now, for the convolution function $f * G$ we obtain

$$\sum_{k=2}^{\infty} \left[ 2k - \gamma \left(1 - (-1)^k\right) \right] \Omega \sigma_k (\alpha_1) |a_k| A_k + \sum_{k=1}^{\infty} \left[ 2k + \gamma \left(1 - (-1)^k\right) \right] \Omega \sigma_k (\alpha_1) |b_k| B_k \leq 2 \left(1 - \gamma\right),$$

since $0 \leq \mu \leq \gamma < 1$ and $f \in \mathcal{H}S_s^* ([\alpha_1, A_1, B_1], \gamma)$. Therefore $f * G \in \mathcal{H}S_s^* ([\alpha_1, A_1, B_1], \gamma) \subset \mathcal{H}S_s^* ([\alpha_1, A_1, B_1], \mu)$, since the above inequality bounded by $2 \left(1 - \gamma\right)$ while $2 \left(1 - \gamma\right) \leq 2 \left(1 - \mu\right)$.

Now, we show that the class $\mathcal{H}S_s^* ([\alpha_1, A_1, B_1], \gamma)$ is closed under convex combinations of its members.

**Theorem 6.** The class $\mathcal{H}S_s^* ([\alpha_1, A_1, B_1], \gamma)$ is closed under convex combination.

**Proof.** For $i = 1, 2, ..., \text{let } f_i \in \mathcal{H}S_s^* ([\alpha_1, A_1, B_1], \gamma), \text{where } f_i \text{ is given by } f_i(z) = z - \sum_{k=2}^{\infty} a_{k_i} z^k + \sum_{k=1}^{\infty} b_{k_i} \bar{z}^k, (a_{k_i} \geq 0; b_{k_i} \geq 0; z \in U).$
Then by using Theorem 2, we have

\[
\sum_{k=2}^{\infty} \left[ \frac{2k - \gamma (1 - (-1)^k)}{2 (1 - \gamma)} \right] \Omega \sigma_k (\alpha_1) |a_{k_i}| + \sum_{k=1}^{\infty} \left[ \frac{2k + \gamma (1 - (-1)^k)}{2 (1 - \gamma)} \right] \Omega \sigma_k (\alpha_1) |b_{k_i}| \leq 1. \tag{4.4}
\]

For \( \sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1 \), the convex combination of \( f_i \) may be written as

\[
\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_{k_i}| \right) z^k + \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_{k_i}| \right) z^k. \tag{4.5}
\]

Then, by using (4.4), we have

\[
\sum_{k=2}^{\infty} \left[ \frac{2k - \gamma (1 - (-1)^k)}{2 (1 - \gamma)} \right] \Omega \sigma_k (\alpha_1) \left( \sum_{i=1}^{\infty} t_i |a_{k_i}| \right) + \sum_{k=1}^{\infty} \left[ \frac{2k + \gamma (1 - (-1)^k)}{2 (1 - \gamma)} \right] \Omega \sigma_k (\alpha_1) \left( \sum_{i=1}^{\infty} t_i |b_{k_i}| \right) \leq \sum_{i=1}^{\infty} t_i = 1,
\]

this is the necessary and sufficient condition given by (2.4) and so \( \sum_{i=1}^{\infty} t_i f_i(z) \in H S_{\alpha} ([\alpha_1, A_1, B_1], \gamma) \). This completes the proof of Theorem 6. \( \square \)

**Remark 2.** Putting \( A_i = 1 \quad (i = 1, \ldots, q) \) and \( B_j = 1 \quad (j = 1, \ldots, s) \) in our results we obtain the results obtained by Murugusundaramoorthy et al. [11].

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**References**


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