CERTAIN SUBORDINATION RESULTS INVOLVING A GENERALIZED MULTIPLIER TRANSFORMATION OPERATOR

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Abstract. This paper investigates various new subordination results for certain p-valent analytic functions involving a generalized multiplier transformation operator $J_p^m(\lambda,l), m \in \mathbb{Z}$, defined recently by J. K. Prajapat [Math. Comput. Modelling, 55 (2012), 1456–1465]. Several lines of approach are followed to obtain the subordination results. We also consider some simpler and precise forms of the derived results.

1. Introduction and preliminaries

Let $\mathscr{H}(\mathbb{U})$ denote a linear space of all analytic functions defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C},\ p \in \mathbb{N}$, let

$$\mathscr{H}[a,p] = \left\{ f \in \mathscr{H}(\mathbb{U}) : f(z) = a + a_p z^p + a_{p+1} z^{p+1} + ... \right\}.$$

We denote the special class of $\mathscr{H}[0,p]$ by $\mathscr{A}_p$ whose members are of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,\ z \in \mathbb{U}. \tag{1.1}$$

Denote by $\mathscr{P}(\alpha)$, $0 \leqslant \alpha < 1$, a class of functions $p \in \mathscr{H}[1,1]$ of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + ..., \tag{1.2}$$

which satisfies the condition that $\mathrm{Re}\,(p(z)) > \alpha$.

If the functions $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, then we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write $f(z) \prec g(z)\ (z \in \mathbb{U})$, if there exists a Schwartz function $\omega(z)$ analytic in $\mathbb{U}$ such that $|\omega(z)| < 1, z \in \mathbb{U}$, and $\omega(0) = 0$ with $f(z) = g(\omega(z))$ in $\mathbb{U}$. In particular, if $f(z)$ is univalent in $\mathbb{U}$, then $f(z) \prec g(z)\ (z \in \mathbb{U})$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Recently, Prajapat [15] defined for $m \in \mathbb{Z} = \{.., -2, -1, 0, 1, 2, ..\}$, $\lambda > 0$, $l > -p$, a generalized multiplier transformation operator $J_p^m(\lambda,l) : \mathscr{A}_p \to \mathscr{A}_p$, which is defined by

$$J_p^m(\lambda,l)f(z) = \begin{cases} \dfrac{p+l}{\lambda} z^{p-\frac{p+l}{\lambda}} \displaystyle\int_0^z t^{\frac{p+l}{\lambda}-p-1} J_p^{m+1}(\lambda,l)f(t)dt & \text{for } m \in \mathbb{Z}^-, \\[2mm] \dfrac{\lambda}{p+l} z^{1+p-\frac{p+l}{\lambda}} \left(z^{\frac{p+l}{\lambda}-p} J_p^{m-1}(\lambda,l)f(z) \right)' & \text{for } m \in \mathbb{Z}^+, \\[2mm] f(z) & \text{for } m = 0. \end{cases} \tag{1.3}$$

It easily follows from the above definition of the operator that the series expansion of $J_p^m(\lambda,l)f(z)$ for $f(z)$ of the form (1.1) is given by

$$J_p^m(\lambda,l)f(z) = z^p + \sum_{n=p+1}^{\infty} \left(1 + \frac{\lambda(n-p)}{p+l}\right)^m a_n z^n, \qquad (1.4)$$

and which also satisfies the identity:

$$z\left(J_p^m(\lambda,l)f(z)\right)' = \frac{p+l}{\lambda}J_p^{m+1}(\lambda,l)f(z) + \left(p - \frac{p+l}{\lambda}\right)J_p^m(\lambda,l)f(z) \qquad (1.5)$$

and a composition property:

$$J_p^{-m}(\lambda,l)J_p^m(\lambda,l)f(z) = J_p^0(\lambda,l)f(z) = f(z). \qquad (1.6)$$

In particular, it can be verified that

$$J_p^{-1}(\lambda,l)J_p^1(\lambda,l)f(z) = f(z) = \frac{p+l}{\lambda}z^{p-\frac{p+l}{\lambda}}\int_0^z t^{\frac{p+l}{\lambda}-p-1}J_p^1(\lambda,l)f(t)dt. \qquad (1.7)$$

REMARK 1. In view of (1.4), we observe that the operator $J_p^m((p+l)/p,l) = D_p^m$ is a generalization of the familiar Sălăgean operator for any integer m, and is defined for $f \in \mathscr{A}_p$ by

$$D_p^0 f(z) = f(z), \; D_p^1 f(z) = \frac{z f'(z)}{p}, ..., D_p^m f(z) = \frac{z\left(D_p^{m-1}f(z)\right)'}{p}, m \in \mathbb{Z}^+$$

and

$$D_p^{-1}f(z) = p\int_0^z \frac{f(t)}{t}dt, ..., D_p^m f(z) = p\int_0^z \frac{D_p^{m+1}f(t)}{t}dt, m \in \mathbb{Z}^-.$$

Various special cases studied earlier of the operator $J_p^m(\lambda,l)$ are given in [15]. For reader's convenience, we reproduce here briefly some of these special cases as follows.

For $m \in \mathbb{Z}^+ \cup \{0\}$, the operator $J_p^m(\lambda,l) = I_p^m(\lambda,l)$ was studied by Cătaş [3], and for which the operator $I_p^m(1,l) = I_p(m,l)$ was earlier studied by Kumar et al. [8], and $I_p(m,0) = D_p^m$ is the generalized Sălăgean operator studied in [13], whereas, $D_1^m = D^m$ is the well-known operator Sălăgean operator [18]. Also, for $m \in \mathbb{Z}^- \cup \{0\}$, the operator $J_p^m(\lambda,l)$ was introduced and investigated by El-Ashwah and Aouf in [5] (see also [2], [20]) and the operator $J_p^m(1,1)$ by Patel and Sahoo [14]. It also includes the important and vastly used operators of Jung et al. [7].

In obtaining various subordination results, we require the following lemmas.

LEMMA 1. ([6]; see also [11], p. 71) *Let $h(z)$ be a convex (univalent) function in $\mathbb{U}$ with $h(0) = 1$, and let the function $p(z)$ of the form (1.2) be analytic in $\mathbb{U}$. If*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z) \quad (\text{Re}\,\gamma \geqslant 0, \; \gamma \neq 0, \; z \in \mathbb{U}),$$

then

$$p(z) \prec q(z) := \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1} h(t) \, dt \prec h(z) \quad (z \in \mathbb{U}),$$

where the function $q(z)$ is convex and is the best dominant.

LEMMA 2. *([21]) Let $p(z) \in \mathscr{P}(\alpha)$, $z \in \mathbb{U}$, $0 \leqslant \alpha < 1$, then*

$$\mathrm{Re}\,(p(z)) \geqslant 2\alpha - 1 + \frac{2(1-\alpha)}{1+|z|}, \quad z \in \mathbb{U}. \tag{1.8}$$

LEMMA 3. [16] *The function*

$$(1-z)^b \equiv e^{b \log(1-z)}, \quad b \neq 0,$$

is univalent in $\mathbb{U}$, if and only if either b is in the disk $|b-1| \leqslant 1$, or in the disk $|b+1| \leqslant 1$.

LEMMA 4. [12] ([11], Theorem 3.4h, p. 132) *Let $q(z)$ be univalent in $\mathbb{U}$ and let $\theta(w)$ and $\phi(w)$ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set $Q(z) = z\,q'(z)\,\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*
   (i) $Q(z)$ is starlike (univalent) in $\mathbb{U}$,
   (ii) $\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$, $z \in \mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = q(0)$, $q(\mathbb{U}) \subset \mathbb{D}$,
and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

The Gaussian hypergeometric function $_2F_1(a,b;c;z)$ (see also, for example [4]) is an analytic function in $\mathbb{U}$ and is defined for $a,b,c \in \mathbb{C}$ $(c \neq 0, -1, -2, ...)$ by

$$_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(a)_n$ is the Pochhammer symbol representing the product that

$$(a)_n = a(a+1)...(a+n-1), \ n \in \mathbb{N}; \ (a)_0 = 1.$$

We record here the following results for the function $_2F_1(a,b;c;z)$ which are quite well-known.

LEMMA 5. [1] *Let $a,b,c \in \mathbb{C}$ $(c \neq 0, -1, -2, ...)$, then the Gaussian hypergeometric function $_2F_1(a,b;c;z)$ analytic in $\mathbb{U}$ satisfies the following identities:*
   (i) $_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} \, dt$, $\mathrm{Re}\,(c) > \mathrm{Re}\,(b) > 0$.
   (ii) $_2F_1(a,b;c;z) = (1-z)^{-a} \, _2F_1\left(a, c-b; c; \frac{z}{z-1}\right).$

In this paper, we obtain some interesting new subordination results for certain p-valent analytic functions involving the operator $J_p^m(\lambda, l)$ $(m \in \mathbb{Z})$, in the open unit disk. Some of our results are essentially motivated by the works of Wang et al. [22] and Liu [9]. We adopt various methods to study and derive the different subordination results. Many of the results presented in this paper extend the results on subordination theory involving known linear operators and, for their details and references, one may refer to [15]. Further, we consider only new simpler and precise forms of some of the results.

2. Main results

We begin by establishing the following result on subordination.

THEOREM 1. *Let $h(z)$ be a convex (univalent) function in $\mathbb{U}$ with $h(0) = 1$. If a function $f \in \mathscr{A}_p$ satisfies $J_p^m(\lambda, l)f(z) J_p^{m+1}(\lambda, l)f(z) \neq 0$ $(z \in \mathbb{U})$ and for $\mu > 0$, $v > 0$, the subordinate condition that*

$$F(z) := \left(\frac{J_p^m(\lambda, l)f(z)}{z^p} \right)^\mu \left(1 - v + v \frac{J_p^{m+1}(\lambda, l)f(z)}{J_p^m(\lambda, l)f(z)} \right) \prec h(z), \quad z \in \mathbb{U}, \quad (2.1)$$

then

$$\left(\frac{J_p^m(\lambda, l)f(z)}{z^p} \right)^\mu = \frac{(p+l)\mu}{\lambda v} z^{-\frac{(p+l)\mu}{\lambda v}} \int_0^z t^{\frac{(p+l)\mu}{\lambda v} - 1} F(t)\, dt \quad (2.2)$$

$$\prec q(z) := \frac{(p+l)\mu}{\lambda v} z^{-\frac{(p+l)\mu}{\lambda v}} \int_0^z t^{\frac{(p+l)\mu}{\lambda v} - 1} h(t)\, dt$$

$$\prec h(z), \quad z \in \mathbb{U},$$

where $q(z)$ is the best dominant.

Proof. Making use of the series representation (1.4), we can express

$$\theta(z) := \left(\frac{J_p^m(\lambda, l)f(z)}{z^p} \right)^\mu = 1 + c_{p+1}z + c_{p+2}z^2 + \dots, \quad (2.3)$$

where the coefficients c_i $(i = p+1, p+2, \dots)$ depend upon the parameters λ, μ, l and m. Evidently, the function $\theta(z)$ is analytic in $\mathbb{U}$ with $\theta(0) = 1$.

Differentiating (2.3) logarithmically, we get

$$\frac{z\theta'(z)}{\theta(z)} = \mu \left(\frac{z \left(J_p^m(\lambda, l)f(z) \right)'}{J_p^m(\lambda, l)f(z)} - p \right), \quad (2.4)$$

which on using the identity (1.5) yields

$$\frac{z\theta'(z)}{\theta(z)} = \frac{(p+l)\mu}{\lambda} \left(\frac{J_p^{m+1}(\lambda, l)f(z)}{J_p^m(\lambda, l)f(z)} - 1 \right). \quad (2.5)$$

Consequently, by using (2.3) and the operator defined by (1.4) and the condition (2.1), we are lead to

$$F(z) = \theta(z) + \frac{\lambda v}{(p+l)\mu} z\theta'(z) = 1 + \sum_{n=p+1}^{\infty} \left(1 + \frac{\lambda v(n-p)}{(p+l)\mu} \right) c_n z^{n-p} \quad (2.6)$$

$$= \frac{J_p^1(\frac{\lambda v}{\mu}, l) z^p \theta(z)}{z^p} \prec h(z), \; z \in \mathbb{U},$$

which by virtue of (1.7) and Lemma 1 establishes the desired result. $\square$

The above result of Theorem 1 can also be expressed as a subordination preserving result involving the integral operator $J_p^{-1}(\frac{\lambda v}{\mu}, l)$, and is contained in the following:

COROLLARY 1. *If the subordination condition (2.1) holds (under the same parametric constraints of Theorem 1), that is, if*

$$F(z) \prec h(z), \quad z \in \mathbb{U}, \tag{2.7}$$

then

$$\frac{J_p^{-1}(\frac{\lambda v}{\mu}, l) z^p F(z)}{z^p} \prec \frac{J_p^{-1}(\frac{\lambda v}{\mu}, l) z^p h(z)}{z^p}, \quad z \in \mathbb{U}.$$

Theorem 1 also provides a sharp bound for the dominating coefficient a_{p+1} in the series expansion of the function $f \in \mathscr{A}_p$.

COROLLARY 2. *If for the function $f \in \mathscr{A}_p$ of the form (1.1), the subordination condition (2.1) holds (under the same parametric constraints of Theorem 1), that is, if (2.7) holds for $h(z) = 1 + h_1 z + h_2 z^2 + ...$, then*

$$|a_{p+1}| \leqslant \frac{|h_1|}{\left(\mu + \frac{\lambda v}{p+l} \right) \left(1 + \frac{\lambda}{p+l} \right)^m}. \tag{2.8}$$

The bound in (2.8) is sharp for the function $f(z)$ given by

$$\left(\frac{J_p^m(\lambda, l) f(z)}{z^p} \right)^{\mu} = 1 + \mu \left(1 + \frac{\lambda}{p+l} \right)^m a_{p+1} z, \quad z \in \mathbb{U}.$$

Proof. Let for the function $f \in \mathscr{A}_p$ of the form (1.1), the function $\theta(z)$ be defined by (2.3). We have then for the function $F(z)$ (defined by (2.1) with its series expansion given by (2.6)):

$$F(z) \prec h(z), \quad z \in \mathbb{U},$$

which by a well-known result of Rogosinski [17] on subordination shows that

$$\left| \left(1 + \frac{\lambda v(n-p)}{(p+l)\mu} \right) c_n \right| \leqslant |h_1|, \quad n \geqslant p+1. \tag{2.9}$$

Now comparing the coefficient of z on both the sides of (2.3) by using (1.4), we get

$$\mu \left(1 + \frac{\lambda}{p+l}\right)^m a_{p+1} = c_{p+1},$$

and the result (2.8) follows upon using the inequality (2.9) for $n = p+1$. $\square$

If we choose a bilinear transformation defined by $h(z) = \frac{1+Az}{1+Bz}$, where $-1 \leqslant A \leqslant 1$, $-1 \leqslant B \leqslant 1$, $A \neq B$, (in case $B \neq 0$; B and $B - A$ are of same sign), then from Theorem 1, we can derive the following result.

COROLLARY 3. *Let a function* $f \in \mathscr{A}_p$ *satisfy* $J_p^m(\lambda, l) f(z) J_p^{m+1}(\lambda, l) f(z) \neq 0$ $(z \in \mathbb{U})$, *and the condition that*

$$\left(\frac{J_p^m(\lambda, l) f(z)}{z^p}\right)^\mu \left(1 - v + v \frac{J_p^{m+1}(\lambda, l) f(z)}{J_p^m(\lambda, l) f(z)}\right) \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U}, \qquad (2.10)$$

where $-1 \leqslant A \leqslant 1$, $-1 \leqslant B \leqslant 1$ *with* $A \neq B$, *(in case* $B \neq 0$; B *and* $B - A$ *are of same sign),* $\mu, v > 0$, *then*

$$\frac{(p+l)\mu}{\lambda v} \int_0^1 u^{\frac{(p+l)\mu}{\lambda v} - 1} \left(\frac{1 - (|B-A| - |B|) u}{1 + |B| u}\right) du \qquad (2.11)$$

$$< \mathrm{Re} \left(\frac{J_p^m(\lambda, l) f(z)}{z^p}\right)^\mu$$

$$< \frac{(p+l)\mu}{\lambda v} \int_0^1 u^{\frac{(p+l)\mu}{\lambda v} - 1} \left(\frac{1 + (|B-A| - |B|) u}{1 - |B| u}\right) du.$$

For $|B| = 1$, *the result is sharp and the extremal function of (2.11) is given by*

$$\left(\frac{J_p^m(\lambda, l) f(z)}{z^p}\right)^\mu = \frac{(p+l)\mu}{\lambda v} \int_0^1 u^{\frac{(p+l)\mu}{\lambda v} - 1} \left(\frac{1 + Auz}{1 + Buz}\right) du \qquad (2.12)$$

as $z \to \pm 1$.

Proof. Under the given constraints on A and B, setting $h(z) = \frac{1+Az}{1+Bz}$ in Theorem 1, we get

$$\left(\frac{J_p^m(\lambda, l) f(z)}{z^p}\right)^\mu \prec \frac{(p+l)\mu}{\lambda v} \int_0^1 u^{\frac{(p+l)\mu}{\lambda v} - 1} \left(\frac{1 + Auz}{1 + Buz}\right) du \quad (z \in \mathbb{U}): \qquad (2.13)$$

and we also note that for $z \in \mathbb{U}$,

$$1 - |A|u \leqslant \operatorname{Re}\left(\frac{1+Auz}{1+Buz}\right) \leqslant 1 + |A|u, \text{ for } B = 0, \ 0 < |A| \leqslant 1, \qquad (2.14)$$

$$\frac{1-Au}{1-Bu} \leqslant \operatorname{Re}\left(\frac{1+Auz}{1+Buz}\right) \leqslant \frac{1+Au}{1+Bu}, \text{ for } -1 \leqslant B < 0 \leqslant A \leqslant 1, \qquad (2.15)$$

$$\frac{1+Au}{1+Bu} \leqslant \operatorname{Re}\left(\frac{1+Auz}{1+Buz}\right) \leqslant \frac{1-Au}{1-Bu}, \text{ for } -1 \leqslant A \leqslant 0 < B \leqslant 1. \qquad (2.16)$$

Inequalities (2.14), (2.15) and (2.16) can be expressed by a single inequality, viz.

$$\frac{1-(|B-A|-|B|)u}{1+|B|u} \leqslant \operatorname{Re}\left(\frac{1+Auz}{1+Buz}\right) \leqslant \frac{1+(|B-A|-|B|)u}{1-|B|u},$$

and the desired result (2.11) is evidently arrived at from (2.13). Sharpness can be verified from (2.12) for the case if $|B| = 1$ as $z \to \pm 1$. $\quad\square$

We next prove the following:

COROLLARY 4. *If under the same parametric constraints stated in Corollary* 3 *with* $|B| = 1$, *a function* $f \in \mathscr{A}_p$ *satisfies* $J_p^m(\lambda,l)f(z)J_p^{m+1}(\lambda,l)f(z) \neq 0$ $(z \in \mathbb{U})$ *and for* $\mu > 0$, *the condition that*

$$\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U}, \qquad (2.17)$$

then (for some $\nu > 0$)

$$\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu \left(1 - \nu + \nu\frac{J_p^{m+1}(\lambda,l)f(z)}{J_p^m(\lambda,l)f(z)}\right) \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U},$$

for $|z| < r_0(\lambda,\mu,\nu,l)$, *where*

$$r_0(\lambda,\mu,\nu,l) = \frac{-\lambda\nu + \sqrt{\lambda^2\nu^2 + (p+l)^2\mu^2}}{(p+l)\mu} \qquad (2.18)$$

which is best possible.

Proof. From (2.17), we have

$$\operatorname{Re}\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu > \inf_{z\in\mathbb{U}}\operatorname{Re}\left(\frac{1+Az}{1+Bz}\right) \geqslant \frac{1-(|B-A|-|B|)}{1+|B|}.$$

Since, under the constraints imposed on A and B in Corollary 3 with $|B| = 1$, we write $|B-A| = 1 - AB$, hence, we get

$$\operatorname{Re}\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu > \frac{1+AB}{2}.$$

Now to prove the result, we consider

$$\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu = \frac{1+AB}{2} + \frac{1-AB}{2}q(z) \quad (|B|=1, \ -1 \leqslant A \leqslant 1, \ A \neq B, \ z \in \mathbb{U}),$$
(2.19)

where $q(z)$ is analytic satisfying $q(0)=1$ with positive real part in $\mathbb{U}$. Differentiating (2.19) logarithmically and using the identity (1.5), we get

$$H := \text{Re}\left\{\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu \left(1-v+v\frac{J_p^{m+1}(\lambda,l)f(z)}{J_p^m(\lambda,l)f(z)}\right) - \frac{1+AB}{2}\right\}$$

$$= \frac{1-AB}{2}\text{Re}\left(q(z) + \frac{\lambda v}{(p+l)\mu}zq'(z)\right)$$

$$\geqslant \frac{1-AB}{2}\text{Re}\left(q(z) - \frac{\lambda v}{(p+l)\mu}\left|zq'(z)\right|\right).$$
(2.20)

On applying a well-known estimate [10]:

$$\left|zq'(z)\right| \leqslant \frac{2r}{1-r^2}\text{Re}\,(q(z)) \qquad (|z|=r<1)$$

in (2.20), we obtain that

$$H \geqslant \frac{1-AB}{2}\left(1 - \frac{2\lambda vr}{(p+l)\mu(1-r^2)}\right), \qquad \text{Re}(q(z)) > 0,$$

if $r < r_0(\lambda,\mu,v,l)$, where $r_0(\lambda,\mu,v,l)$ is given by (2.18) which is best possible. Since, for the function $f \in \mathscr{A}_p$ given by

$$\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu = \frac{1+AB}{2} + \frac{1-AB}{2}\left(\frac{1+z}{1-z}\right), \quad z \in \mathbb{U},$$

we have

$$H = \frac{1-AB}{2}\text{Re}\left(\frac{1+z}{1-z} + \frac{2\lambda vz}{(p+l)\mu(1-z)^2}\right) = 0, \ \text{ for } z = r_0(\lambda,\mu,v,l). \quad \square$$

By putting $h(z) = 1 + Az$, $-1 < A < 1$ $(A \neq 0)$ in Theorem 1, we obtain the following result which may be looked upon as providing a sufficiency condition for the starlikeness of the function $J_p^m(\lambda,l)f(z)$.

COROLLARY 5. Let $\lambda > 0$, $l > -p$ $(p \in \mathbb{N})$, $\mu, v > 0$. If a function $f \in \mathscr{A}_p$ satisfies $J_p^m(\lambda,l)f(z)J_p^{m+1}(\lambda,l)f(z) \neq 0$ $(z \in \mathbb{U})$, and for $0 < |A| \leqslant 1 - \frac{2(p+l)\mu}{p\lambda v}$, the condition that

$$\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu \left(1-v+v\frac{J_p^{m+1}(\lambda,l)f(z)}{J_p^m(\lambda,l)f(z)}\right) \prec 1+Az, \quad z \in \mathbb{U},$$
(2.21)

then

$$\left| \frac{z\left(J_p^m(\lambda,l)f(z)\right)'}{J_p^m(\lambda,l)f(z)} - p \right| < \frac{2\,(p+l)}{\lambda\,v\,(1-|A|)}.$$

Proof. Let $\theta(z)$ be defined by (2.3), then following the proof of Theorem 1, from the subordination condition (2.21), we have

$$\theta(z) + \frac{\lambda\,v}{(p+l)\,\mu}z\theta'(z) \prec 1 + Az, \quad z \in \mathbb{U}, \tag{2.22}$$

which proves that $\theta(z) \prec 1 + Az$, $z \in \mathbb{U}$, and hence, it implies that $|\theta(z)| > 1 - |A|$. Also, from (2.22), we have

$$\left| \theta(z) + \frac{\lambda\,v}{(p+l)\,\mu}z\theta'(z) \right| < 1 + |A|.$$

Thus, we get

$$\begin{aligned}
|z\theta'(z)| &= \left| \frac{(p+l)\,\mu}{\lambda\,v}\left(\theta(z) + \frac{\lambda\,v}{(p+l)\,\mu}z\theta'(z) \right) - \frac{(p+l)\,\mu}{\lambda\,v}\theta(z) \right| \\
&< \frac{(p+l)\,\mu}{\lambda\,v}\left(\frac{1+|A|}{1-|A|} + 1 \right)|\theta(z)|,
\end{aligned}$$

and hence,

$$\left| \frac{z\theta'(z)}{\theta(z)} \right| < \frac{2\,(p+l)\,\mu}{\lambda\,v\,(1-|A|)},$$

which from (2.4) yields the desired result. $\square$

Our next main result is contained in the following theorem.

THEOREM 2. *If for $i = 1,2$, $-1 \leqslant B_i < A_i \leqslant 1$, functions $f_i \in \mathscr{A}_p$ satisfies $J_p^m(\lambda,l)f_i(z)J_p^{m+1}(\lambda,l)f_i(z) \neq 0$ $(z \in \mathbb{U})$, and for $\mu, v > 0$, the condition that*

$$\left(\frac{J_p^m(\lambda,l)f_i(z)}{z^p} \right)^{\mu}\left(1 - v + v\frac{J_p^{m+1}(\lambda,l)f_i(z)}{J_p^m(\lambda,l)f_i(z)} \right) \prec \frac{1+A_iz}{1+B_iz}, \quad z \in \mathbb{U},$$

then

$$\frac{J_p^1(\frac{\lambda v}{\mu},l)z^p\,(p_1 * p_2)\,(z)}{z^p} \in \mathscr{P}\,(\gamma), \tag{2.23}$$

where $p_i(z)$ (for $i = 1,2$) is defined by

$$p_i(z) = \left(\frac{J_p^m(\lambda,l)f_i(z)}{z^p} \right)^{\mu} \tag{2.24}$$

and

$$\gamma = 1 - \frac{2\,(A_1 - B_1)\,(A_2 - B_2)}{(1-B_1)\,(1-B_2)}\left[2 - {}_2F_1\left(1,1;\frac{(p+l)\,\mu}{\lambda\,v} + 1;\frac{1}{2} \right) \right].$$

The result is sharp when $B_i = -1$, $i = 1,2$.

Proof. Let (for $i = 1, 2$)

$$F_i(z) := \left(\frac{J_p^m(\lambda, l) f_i(z)}{z^p}\right)^\mu \left(1 - v + v \frac{J_p^{m+1}(\lambda, l) f_i(z)}{J_p^m(\lambda, l) f_i(z)}\right) \prec \frac{1 + A_i z}{1 + B_i z}, \quad z \in \mathbb{U}. \quad (2.25)$$

Then, we have

$$F_i(z) \in \mathscr{P}(\alpha_i), \quad \alpha_i = \frac{1 - A_i}{1 - B_i}, \quad i = 1, 2, \quad (2.26)$$

and by Herglotz formula, we get

$$(F_1 * F_2)(z) \in \mathscr{P}(\alpha_3), \alpha_3 = 1 - 2(1 - \alpha_1)(1 - \alpha_2), \quad (2.27)$$

where the symbol $*$ appearing above (and elsewhere also) denotes the familiar convolution (or Hadamard product). Again, if $p_i(z)$ defined by (2.24) has the series expansion:

$$p_i(z) = 1 + \sum_{n=p+1}^{\infty} c_{i,n} z^{n-p},$$

then, it follows that (for $i = 1, 2$)

$$F_i(z) = p_i(z) + \frac{v\lambda}{(p+l)\mu} z p_i'(z) = 1 + \sum_{n=p+1}^{\infty} \left(1 + \frac{v\lambda(n-p)}{(p+l)\mu}\right) c_{i,n} z^{n-p}.$$

Hence, for

$$P(z) = (p_1 * p_2)(z),$$

we obtain by using (1.4) that

$$P(z) + \frac{\lambda v}{(p+l)\mu} z P'(z) = 1 + \sum_{n=p+1}^{\infty} \left(1 + \frac{\lambda v(n-p)}{(p+l)\mu}\right) c_{1,n} c_{2,n} z^{n-p}$$

$$= \frac{(p+l)\mu}{\lambda v} z^{-\frac{(p+l)\mu}{\lambda v}} \int_0^z t^{\frac{(p+l)\mu}{\lambda v} - 1} (F_1 * F_2)(t) \, dt$$

$$= \frac{J_p^1(\frac{v\lambda}{\mu}, l) z^p (p_1 * p_2)(z)}{z^p} =: G(z).$$

Using now (1.8), (2.26), (2.27) and the identities (i) and (ii) of Lemma 5, we get

$$\operatorname{Re} G(z) = \frac{(p+l)\mu}{\lambda v} \int_0^1 u^{\frac{(p+l)\mu}{\lambda v}-1} \operatorname{Re}(F_1 * F_2)(uz)\, du$$

$$\geq \frac{(p+l)\mu}{\lambda v} \int_0^1 u^{\frac{(p+l)\mu}{\lambda v}-1} \left(2\alpha_3 - 1 + \frac{2(1-\alpha_3)}{1+u|z|}\right) du$$

$$> \frac{(p+l)\mu}{\lambda v} \int_0^1 u^{\frac{(p+l)\mu}{\lambda v}-1} \left(2\alpha_3 - 1 + \frac{2(1-\alpha_3)}{1+u}\right) du$$

$$= 1 + 2(1-\alpha_3) \left[\frac{(p+l)\mu}{\lambda v} \int_0^1 \frac{u^{\frac{(p+l)\mu}{\lambda v}-1}}{1+u} du - 1\right]$$

$$= 1 - \frac{2(A_1 - B_1)(A_2 - B_2)}{(1-B_1)(1-B_2)} \left[2 - {}_2F_1\left(1,1;\frac{(p+l)\mu}{\lambda v}+1;\frac{1}{2}\right)\right].$$

This proves (2.23). Sharpness of the result (2.23) follows for the functions $f_i \in \mathscr{A}_p$ $(i=1,2)$ such that

$$\left(\frac{J_p^m(\lambda,l)f_i(z)}{z^p}\right)^\mu \left(1-v+v\frac{J_p^{m+1}(\lambda,l)f_i(z)}{J_p^m(\lambda,l)f_i(z)}\right) = \frac{1+A_iz}{1-z}, \quad i=1,2.$$

Since, for such functions, we get

$$p_i(z) = \frac{(p+l)\mu}{\lambda v} z^{-\frac{(p+l)\mu}{\lambda v}} \int_0^z t^{\frac{(p+l)\mu}{\lambda v}-1} \left(\frac{1+A_it}{1-t}\right) dt \quad (i=1,2).$$

therefore,

$$G(z) = \frac{(p+l)\mu}{\lambda v} \int_0^1 u^{\frac{(p+l)\mu}{\lambda v}-1} \left(\left(\frac{1+A_1uz}{1-uz}\right) * \left(\frac{1+A_2uz}{1-uz}\right)\right) du$$

$$= \frac{(p+l)\mu}{\lambda v} \int_0^1 u^{\frac{(p+l)\mu}{\lambda v}-1} \left(1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-uz}\right) du$$

$$= 1 - (A_1+1)(A_2+1) \left[1 - \frac{(p+l)\mu}{\lambda v} \int_0^1 \frac{u^{\frac{(p+l)\mu}{\lambda v}-1}}{1+u} du\right], \text{ as } z \to -1$$

$$= 1 - \frac{(A_1+1)(A_2+1)}{2} \left[2 - {}_2F_1\left(1,1;\frac{(p+l)\mu}{\lambda v}+1;\frac{1}{2}\right)\right].$$

This completes the proof of Theorem 2. $\quad\square$

For $\mu = 1$, Theorem 2 reduces to the following simple form.

Corollary 6. *If for* $i = 1, 2, \ -1 \leqslant B_i < A_i \leqslant 1$ *and the functions* $f_i \in \mathscr{A}_p$ *satisfy* $J_p^m(\lambda, l) f_i(z) J_p^{m+1}(\lambda, l) f_i(z) \neq 0 \ (z \in \mathbb{U})$ *and for* $\nu > 0$ *the condition that*

$$(1 - \nu) \frac{J_p^m(\lambda, l) f_i(z)}{z^p} + \nu \frac{J_p^{m+1}(\lambda, l) f_i(z)}{z^p} \prec \frac{1 + A_i z}{1 + B_i z}, \quad z \in \mathbb{U},$$

then

$$\frac{J_p^1(\lambda, l) z^p K(z)}{z^p} \in \mathscr{P}(\gamma_1),$$

where

$$K(z) = (1 - \nu) \frac{J_p^m(\lambda, l) (f_1 * f_2)(z)}{z^p} + \nu \frac{J_p^{m+1}(\lambda, l) (f_1 * f_2)(z)}{z^p}$$

and

$$\gamma_1 = 1 - \frac{2(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left[2 - {}_2F_1\left(1, 1; \frac{p+l}{\lambda \nu} + 1; \frac{1}{2}\right) \right].$$

The result is sharp when $B_i = -1$, $i = 1, 2$.

Further, on setting $\lambda = (p + l)/p$ and $\mu = 1$ in Theorem 2, in view of Remark 1, we get the following result involving generalized Sălăgean operator D_p^m for any integer m.

Corollary 7. *If for* $i = 1, 2, \ -1 \leqslant B_i < A_i \leqslant 1$, *and the functions* $f_i \in \mathscr{A}_p$ *satisfy for* $m \in \mathbb{Z}$, $D_p^m f_i(z) D_p^{m+1} f_i(z) \neq 0 \ (z \in \mathbb{U})$ *and for* $\nu > 0$, *the condition that*

$$(1 - \nu) \frac{D_p^m f_i(z)}{z^p} + \nu \frac{D_p^{m+1} f_i(z)}{z^p} \prec \frac{1 + A_i z}{1 + B_i z}, \quad z \in \mathbb{U},$$

then

$$\frac{D_p^1 z^p H(z)}{z^p} \in \mathscr{P}(\gamma_2),$$

where

$$H(z) = (1 - \nu) \frac{D_p^m (f_1 * f_2)(z)}{z^p} + \nu \frac{D_p^{m+1} (f_1 * f_2)(z)}{z^p}$$

and

$$\gamma_2 = 1 - \frac{2(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left[2 - {}_2F_1\left(1, 1; \frac{p}{\nu} + 1; \frac{1}{2}\right) \right].$$

The result is sharp when $B_i = -1$ *for* $i = 1, 2$.

Next main result is given by the following theorem.

Theorem 3. *Let* $q(z)$ *be convex univalent in* $\mathbb{U}$, *with* $q(0) = 1$, *and of positive real part in* $\mathbb{U}$. *Suppose for* $ka \geqslant 0$, $b \geqslant 0$ *(with* $|a - b - 1| \leqslant 1$*):*

$$h(z) = k(q(z))^a + z(q(z))'(q(z))^b. \tag{2.28}$$

If $f \in \mathscr{A}_p$ satisfies $J_p^m(\lambda,l)f(z)J_p^{m+1}(\lambda,l)f(z) \neq 0$ $(z \in \mathbb{U})$, and for $\mu > 0$, the condition that

$$k\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^{\mu a} + \frac{(p+l)\mu}{\lambda}\left(\frac{J_p^{m+1}(\lambda,l)f(z)}{J_p^m(\lambda,l)f(z)} - 1\right)\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^{\mu(1+b)}$$

$$\prec h(z), \ for \ z \in \mathbb{U},$$

then

$$\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^{\mu} \prec q(z), \ for \ z \in \mathbb{U} \qquad (2.29)$$

and $q(z)$ is the best dominant of (2.29).

Proof. Let $\theta(z)$ be defined by (2.3), then $\theta(0) = q(0) = 1$. Using (2.5), we put the subordinate condition in the form:

$$k(\theta(z))^a + z(\theta(z))'(\theta(z))^b \prec h(z), \ for \ z \in \mathbb{U}.$$

Let $\Theta(w) = kw^a$ and $\phi(w) = w^b$, be analytic in a domain D containing $\theta(\mathbb{U})$ and $q(\mathbb{U})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{U})$. Then, $Q(z) = zq'(z)\phi(q(z)) = zq'(z)(q(z))^b$ is starlike in $\mathbb{U}$. For $b \geqslant 0$, and for a convex (hence, starlike) function $q(z)$, we note that

$$\text{Re}\left(\frac{zQ'(z)}{Q(z)}\right) = \text{Re}\left(1 + \frac{zq''(z)}{q'(z)} + b\frac{zq'(z)}{q(z)}\right) > 0, \ for \ z \in \mathbb{U}, \qquad (2.30)$$

which ensures the condition (i) of Lemma 4. Further, from (2.28), we get

$$\frac{zh'(z)}{Q(z)} = ka(q(z))^{a-b-1} + \frac{zQ'(z)}{Q(z)},$$

and under the condition that $|a-b-1| \leqslant 1$, we have

$$\text{Re}(q(z))^{a-b-1} \geqslant (\text{Re}\,q(z))^{a-b-1} > 0,$$

and therefore, with the use of (2.30) and $ka \geqslant 0$, the condition (ii) of Lemma 4 also holds. In view of (2.29), the other conditions of Lemma 4 are also satisfied. Hence, by Lemma 4, we obtain (2.29), which establishes Theorem 3. $\quad\square$

Now, we consider some applications of Theorem 3 below.
For $-1 \leqslant B < A \leqslant 1$, $0 < \gamma \leqslant 1$, let $q(z) = \left(\frac{1+Az}{1+Bz}\right)^{\gamma}$ for $z \in \mathbb{U}$, then

$$\text{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} = \text{Re}\left\{1 + (\gamma-1)\frac{Az}{1+Az} - (\gamma+1)\frac{Bz}{1+Bz}\right\}$$

$$= -1 + (1-\gamma)\text{Re}\frac{1}{1+Az} + (1+\gamma)\text{Re}\frac{1}{1+Bz}$$

$$> -1 + \frac{1-\gamma}{1+|A|} + \frac{1+\gamma}{1+|B|} \geqslant 0,$$

and

$$\mathrm{Re}\left(\frac{1+Az}{1+Bz}\right)^{\gamma} \geqslant \left(\mathrm{Re}\left(\frac{1+Az}{1+Bz}\right)\right)^{\gamma} > \left(\frac{1-A}{1-B}\right)^{\gamma} \geqslant 0.$$

By using the above stated assertions, Theorem 3 yields the following result.

COROLLARY 8. *Let* $ka \geqslant 0$, $b \geqslant 0$ *and* $|a-b-1| \leqslant 1$. *Also, for* $-1 \leqslant B < A \leqslant 1$, $0 < \gamma \leqslant 1$, *let*

$$u(z) = k\left(\frac{1+Az}{1+Bz}\right)^{\gamma a} + \frac{\gamma(A-B)z}{(1+Az)^{1-\gamma(1+b)}(1+Bz)^{1+\gamma(1+b)}}.$$

If $f \in \mathscr{A}_p$ *satisfies* $J_p^m(\lambda,l)f(z)J_p^{m+1}(\lambda,l)f(z) \neq 0$ $(z \in \mathbb{U})$, *and for* $\theta(z)$ *defined by (2.3), the condition that*

$$k(\theta(z))^a + z(\theta(z))'(\theta(z))^b \prec u(z), \ \text{for} \ z \in \mathbb{U}, \tag{2.31}$$

then

$$\theta(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\gamma}, \ \text{for} \ z \in \mathbb{U},$$

and $\left(\frac{1+Az}{1+Bz}\right)^{\gamma}$ *is the best dominant of (2.31).*

Putting $A = 1$ and $B = -1$, in Corollary 8, we get the following simple form of the result.

COROLLARY 9. *Let* $ka \geqslant 0$, $b \geqslant 0$ *and* $|a-b-1| \leqslant 1$. *Also, for* $0 < \gamma \leqslant 1$, *let*

$$t(z) = k\left(\frac{1+z}{1-z}\right)^{\gamma a} + \frac{2\gamma z}{(1+z)^{1-\gamma(1+b)}(1-z)^{1+\gamma(1+b)}}.$$

If $f \in \mathscr{A}_p$ *satisfies* $J_p^m(\lambda,l)f(z)J_p^{m+1}(\lambda,l)f(z) \neq 0$ $(z \in \mathbb{U})$, *and for* $\theta(z)$ *defined by (2.3), the condition that*

$$k(\theta(z))^a + z(\theta(z))'(\theta(z))^b \prec t(z), \ \text{for} \ z \in \mathbb{U},$$

then

$$|\arg\theta(z)| < \frac{\pi}{2}\gamma, \ \text{for} \ 0 < \gamma \leqslant 1, \ z \in \mathbb{U}.$$

Considering now μ to be a non-zero complex number, we derive the following result.

THEOREM 4. *Let for* $0 \neq \mu \in \mathbb{C}$ *and* $0 \leqslant \beta < 1$, *either* $\left|\frac{2\mu(1-\beta)(p+l)}{\lambda} - 1\right| \leqslant 1$, *or* $\left|\frac{2\mu(1-\beta)(p+l)}{\lambda} + 1\right| \leqslant 1$. *If*

$$\mathrm{Re}\left\{\frac{J_p^{m+1}(\lambda,l)f(z)}{J_p^m(\lambda,l)f(z)}\right\} > \beta, \tag{2.32}$$

then

$$\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^{\mu} \prec \frac{1}{(1-z)^{\frac{2\mu(1-\beta)(p+l)}{\lambda}}}, \quad z \in \mathbb{U}. \tag{2.33}$$

The result is sharp.

Proof. Let $\theta(z)$ be defined by (2.3), then from (2.5) and (2.32), we obtain

$$\frac{J_p^{m+1}(\lambda,l)f(z)}{J_p^m(\lambda,l)f(z)} = 1 + \frac{\lambda}{(p+l)\mu}\frac{zp'(z)}{p(z)} \prec \frac{1+(1-2\beta)z}{1-z}, \quad z \in \mathbb{U}.$$

Suppose that

$$r(z) = \frac{1}{(1-z)^{\frac{2\mu(1-\beta)(p+l)}{\lambda}}}, \quad \vartheta(w) = 1 \quad \text{and} \quad \varphi(w) = \frac{\lambda}{(p+l)\mu w} \quad (w \neq 0),$$

then $r(z)$ is univalent by the assertion of Lemma 3, and $r(z)$, $\vartheta(w)$ and $\varphi(w)$ satisfy the conditions of Lemma 4. Also,

$$Q(z) = zr'(z)\varphi(r(z)) = \frac{2z(1-\beta)}{1-z}$$

is univalent and starlike in $\mathbb{U}$, and

$$\sigma(z) = \vartheta(r(z)) + Q(z).$$

Hence, the conditions of Lemma 4 are satisfied, and we have

$$\vartheta(\theta(z)) + z\theta'(z)\varphi(\theta(z)) \prec \vartheta(r(z)) + zr'(z)\varphi(r(z)) = \sigma(z),$$

which shows that

$$\theta(z) \prec r(z),$$

and this proves the result (2.33) of Theorem 4. Sharpness can be seen for the function $f(z)$ such that

$$\frac{J_p^{m+1}(\lambda,l)f(z)}{J_p^m(\lambda,l)f(z)} = \frac{1+(1-2\beta)z}{1-z}.$$

Since for such function and for $\theta(z)$ defined by (2.3), we have from (2.5) that

$$\frac{2\mu(1-\beta)(p+l)}{\lambda(1-z)} = \frac{\theta'(z)}{\theta(z)}$$

and hence,

$$\theta(z) = \left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^{\mu} = \frac{1}{(1-z)^{\frac{2\mu(1-\beta)(p+l)}{\lambda}}}. \quad \square$$

For μ real and positive, we obtain the following result directly from Theorem 4.

Corollary 10. *Let* $\lambda > 0$, $l > -p$, *and for* $\mu > 0$, $0 \leqslant \beta < 1$, $\frac{\mu(1-\beta)(p+l)}{\lambda} < 1$. *If*

$$\text{Re}\left\{\frac{J_p^{m+1}(\lambda,l)f(z)}{J_p^m(\lambda,l)f(z)}\right\} > \beta,$$

then

$$\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu \prec \frac{1}{(1-z)^{\frac{2\mu(1-\beta)(p+l)}{\lambda}}}, \quad z \in \mathbb{U}.$$

The result is sharp.

3. Some consequent results

Several consequent results arising from the main results can be pointed out (and derived) here in this section. But we confine ourselves to considering some worthwhile cases of derivation of Theorem 1 only. Our first consequent result shows the growth of $J_p^m(\lambda,l)f(z)$ in $\mathbb{U}$.

Corollary 11. *Under the same parametric constraints stated in Corollary 3, let the condition (2.10) holds, then (for* $|z| = r < 1$*):*

$$r^p \left(\frac{(p+l)\mu}{\lambda v}\int_0^1 u^{\frac{(p+l)\mu}{\lambda v}-1}\left(\frac{1-(|B-A|-|B|)ur}{1+|B|ur}\right)du\right)^{\frac{1}{\mu}} \tag{3.1}$$

$$\leqslant \left|J_p^m(\lambda,l)f(z)\right|$$

$$\leqslant r^p \left(\frac{(p+l)\mu}{\lambda v}\int_0^1 u^{\frac{(p+l)\mu}{\lambda v}-1}\left(\frac{1+(|B-A|-|B|)ur}{1-|B|ur}\right)du\right)^{\frac{1}{\mu}}. \tag{3.2}$$

For $|B| = 1$, *the result is sharp and the extremal function is given by (2.12) as* $z \to \pm r$.

Proof. Following the proof of Corollary 3, from (2.13), we get

$$\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu = \frac{(p+l)\mu}{\lambda v}\int_0^1 u^{\frac{(p+l)\mu}{\lambda v}-1}\left(\frac{1+Au\omega(z)}{1+Bu\omega(z)}\right)du \tag{3.3}$$

for some Schwartz function $\omega(z)$ with $|\omega(z)| \leqslant |z| = r < 1$, and as

$$\left|\frac{1+Au\omega(z)}{1+Bu\omega(z)}\right| \leqslant \begin{cases} 1+|A|ur, & \text{for } B=0, \ 0<|A|\leqslant 1, \\ \frac{1+Aur}{1+Bur}, & \text{for } -1\leqslant B<0\leqslant A\leqslant 1, \\ \frac{1-Aur}{1-Bur}, & \text{for } -1\leqslant A\leqslant 0<B\leqslant 1, \end{cases}$$

we get

$$\left| \frac{1 + Au\omega(z)}{1 + Bu\omega(z)} \right| \leqslant \frac{1 + (|B - A| - |B|)ur}{1 - |B|ur}.$$

Hence, from (3.3), we obtain the upper bound (3.2). Again, following the proof of Corollary 3, as for some Schwartz function $\omega(z)$ with $|\omega(z)| \leqslant |z| = r < 1$,

$$\mathrm{Re}\left(\frac{1 + Au\omega(z)}{1 + Bu\omega(z)} \right) \geqslant \frac{1 - (|B - A| - |B|)ur}{1 + |B|ur},$$

we obtain

$$\left| \left(\frac{J_p^m(\lambda, l)f(z)}{z^p} \right)^{\mu} \right| \geqslant \mathrm{Re}\left(\frac{J_p^m(\lambda, l)f(z)}{z^p} \right)^{\mu}$$

$$\geqslant \frac{(p+l)\mu}{\lambda v} \int_0^1 u^{\frac{(p+l)\mu}{\lambda v} - 1} \left(\frac{1 - (|B - A| - |B|)ur}{1 + |B|ur} \right) du,$$

which evidently proves the lower bound (3.1). This proves the result. From (2.12), sharpness of the result can be verified for the case if $|B| = 1$ as $z \to \pm r$. $\square$

The following corollary gives the sharp bounds for the real part of the function considered in Theorem 1.

COROLLARY 12. *Corresponding to the parametric constraints stated in Corollary* 3, *let the condition* (2.10) *hold, then (for* $-1 \leqslant B < A \leqslant 1$)

$$\mathrm{Re}\left(\frac{J_p^m(\lambda, l)f(z)}{z^p} \right)^{\mu} > \begin{cases} \frac{A}{B} - \left(\frac{A}{B} - 1 \right)(1 - B)^{-1} \, {}_2F_1\left(1, 1; \frac{(p+l)\mu}{\lambda v} + 1; \frac{B}{B-1} \right) & if\ B \neq 0, \\ 1 - \frac{A}{1 + \frac{\lambda v}{(p+l)\mu}} & if\ B = 0, \end{cases}$$

(3.4)

and (for $-1 \leqslant A < B \leqslant 1$)

$$\mathrm{Re}\left(\frac{J_p^m(\lambda, l)f(z)}{z^p} \right)^{\mu} < \begin{cases} \frac{A}{B} - \left(\frac{A}{B} - 1 \right)(1 - B)^{-1} \, {}_2F_1\left(1, 1; \frac{(p+l)\mu}{\lambda v} + 1; \frac{B}{B-1} \right) & if\ B \neq 0, \\ 1 - \frac{A}{1 + \frac{\lambda v}{(p+l)\mu}} & if\ B = 0. \end{cases}$$

(3.5)

The results are sharp and the extremal function is given by (2.12) *as* $z \to -1$.

Proof. For $-1 \leqslant B < A \leqslant 1$, we note that $B - A$ is negative, and hence, under the constraints imposed on A and B in Corollary 3 and for $-1 \leqslant B \leqslant 0$, we can express

$$\frac{1 - (|B - A| - |B|)u}{1 + |B|u} = \frac{1 - Au}{1 - Bu}.$$

Similarly, for $-1 \leqslant A < B \leqslant 1$, we note that $B - A$ is positive, and hence, under the constraints imposed on A and B in Corollary 3 and for $0 \leqslant B \leqslant 1$, we can write

$$\frac{1 + (|B - A| - |B|)u}{1 - |B|u} = \frac{1 - Au}{1 - Bu}.$$

It directly follows from (2.11) that (for $-1 \leqslant B < A \leqslant 1$)

$$\text{Re}\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu > \frac{(p+l)\mu}{\lambda\nu}\int_0^1 u^{\frac{(p+l)\mu}{\lambda\nu}-1}\left(\frac{1-Au}{1-Bu}\right)du$$

and (for $-1 \leqslant A < B \leqslant 1$)

$$\text{Re}\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu < \frac{(p+l)\mu}{\lambda\nu}\int_0^1 u^{\frac{(p+l)\mu}{\lambda\nu}-1}\left(\frac{1-Au}{1-Bu}\right)du.$$

Thus, by applying Lemma 5, we get

$$\frac{(p+l)\mu}{\lambda\nu}\int_0^1 u^{\frac{(p+l)\mu}{\lambda\nu}-1}\left(\frac{1-Au}{1-Bu}\right)du$$

$$= \begin{cases} \frac{A}{B}-\left(\frac{A}{B}-1\right)(1-B)^{-1}\ _2F_1\left(1,1;\frac{(p+l)\mu}{\lambda\nu}+1;\frac{B}{B-1}\right) & \text{if } B\neq 0, \\ 1-\frac{A}{1+\frac{\lambda\nu}{(p+l)\mu}} & \text{if } B=0, \end{cases}$$

which proves (3.4) and (3.5). For sharpness of the results, from the function given by (2.12), we get

$$\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu\left(1-\nu+\nu\frac{J_p^{m+1}(\lambda,l)f(z)}{J_p^m(\lambda,l)f(z)}\right) = \frac{1+Az}{1+Bz}, \quad z\in\mathbb{U}$$

and hence, from (2.2) of Theorem 1, we have

$$\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu = \frac{(p+l)\mu}{\lambda\nu}\int_0^1 u^{\frac{(p+l)\mu}{\lambda\nu}-1}\left(\frac{1-Au}{1-Bu}\right)du, \quad \text{as } z\to -1. \quad \square$$

A much simpler form of Corollary 12 occurs when we put $\nu = 1$, $A = 1$ and $B = -1$, and under these values of parameters, Corollary 12 gives the following result.

COROLLARY 13. *If a function $f\in\mathscr{A}_p$ satisfies $J_p^m(\lambda,l)f(z)J_p^{m+1}(\lambda,l)f(z)\neq 0$ $(z\in\mathbb{U})$ and for $\mu > 0$, the condition:*

$$\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu\frac{J_p^{m+1}(\lambda,l)f(z)}{J_p^m(\lambda,l)f(z)} \prec \frac{1+z}{1-z}, \quad z\in\mathbb{U},$$

then

$$\text{Re}\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^\mu > \ _2F_1\left(1,1;\frac{(p+l)\mu}{\lambda}+1;\frac{1}{2}\right)-1.$$

The result is sharp.

By putting $\lambda = (p+l)/p$ in Corollary 13, we get in view of the Remark 1, the following result involving generalized Sălăgean operator D_p^m for any integer m.

COROLLARY 14. *If a function $f \in \mathscr{A}_p$ satisfies $D_p^m f(z) \, D_p^{m+1} f(z) \neq 0 \ (z \in \mathbb{U})$, and for $\mu > 0$ the condition that*

$$\left(\frac{D_p^m f(z)}{z^p} \right)^{\mu} \frac{D_p^{m+1} f(z)}{D_p^m f(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{U},$$

then

$$\left(\frac{D_p^m f(z)}{z^p} \right)^{\mu} > \, _2F_1 \left(1, 1; p\mu + 1; \frac{1}{2} \right) - 1.$$

The result is sharp.

REMARK 2. If further we choose $m = 0, p = 1$ in Corollary 14, then in view of the Remark 1, we get a sharp result for a class of Bazilevic functions of type μ, see [19].

Based on the parameters A, B and ν, involved in the Corollary 3, we prove an inclusion relation in the form of following result.

COROLLARY 15. *Let a function $f \in \mathscr{A}_p$ satisfy $J_p^m(\lambda, l)f(z) J_p^{m+1}(\lambda, l)f(z) \neq 0$ $(z \in \mathbb{U})$, and suppose that for $i = 1, 2$, $-1 \leqslant A_i \leqslant 1$, $-1 \leqslant B_i \leqslant 1$ with $A_i \neq B_i$ (in case $B_i \neq 0$) B_i and $B_i - A_i$ are of same sign. If $|A_1| \geqslant |A_2|$, $|B_1| \geqslant |B_2|$, $\mu > 0$, $\nu_2 > \nu_1 > 0$, and*

$$\left(\frac{J_p^m(\lambda, l)f(z)}{z^p} \right)^{\mu} \left(1 - \nu_2 + \nu_2 \frac{J_p^{m+1}(\lambda, l)f(z)}{J_p^m(\lambda, l)f(z)} \right) \prec \frac{1 + A_2 z}{1 + B_2 z}, \quad z \in \mathbb{U}, \qquad (3.6)$$

holds, then

$$\left(\frac{J_p^m(\lambda, l)f(z)}{z^p} \right)^{\mu} \left(1 - \nu_1 + \nu_1 \frac{J_p^{m+1}(\lambda, l)f(z)}{J_p^m(\lambda, l)f(z)} \right) \prec \frac{1 + A_1 z}{1 + B_1 z}, \quad z \in \mathbb{U}.$$

Proof. Under the given constraints on A_i and B_i $(i = 1, 2)$, in view of (2.14), (2.15), (2.16), and the following inequalities:

$$1 - |A_1| \leqslant 1 - |A_2| \leqslant 1 + |A_2| \leqslant 1 + |A_1|, \quad \text{for } B_i = 0, \quad 0 < |A_i| \leqslant 1,$$

$$\frac{1 - A_1}{1 - B_1} \leqslant \frac{1 - A_2}{1 - B_2} \leqslant \frac{1 + A_2}{1 + B_2} \leqslant \frac{1 + A_1}{1 + B_1}, \quad \text{for } -1 \leqslant B_i < 0 \leqslant A_i \leqslant 1,$$

$$\frac{1 + A_1}{1 + B_1} \leqslant \frac{1 + A_2}{1 + B_2} \leqslant \frac{1 - A_2}{1 - B_2} \leqslant \frac{1 - A_1}{1 - B_1}, \quad \text{for } -1 \leqslant A_i \leqslant 0 < B_i \leqslant 1,$$

we get

$$\frac{1 + A_2 z}{1 + B_2 z} \prec \frac{1 + A_1 z}{1 + B_1 z}, \quad z \in \mathbb{U}. \qquad (3.7)$$

Thus, following Theorem 1 and the subordination condition (3.6), then in view of the inequality $0 < \frac{v_1}{v_2} < 1$, and the subordination relation (3.7) of convex functions, we have

$$\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^{\mu}\left(1-v_1+v_1\frac{J_p^{m+1}(\lambda,l)f(z)}{J_p^m(\lambda,l)f(z)}\right)$$

$$= \left(1-\frac{v_1}{v_2}\right)\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^{\mu}+\frac{v_1}{v_2}\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^{\mu}\left(1-v_2+v_2\frac{J_p^{m+1}(\lambda,l)f(z)}{J_p^m(\lambda,l)f(z)}\right)$$

$$\prec \frac{1+A_1z}{1+B_1z}, \quad z \in \mathbb{U},$$

which proves the result.   □

Our further results below include a convolution property and an integral representation:

COROLLARY 16. *If under the same parametric constraints stated in Corollary 3, a function $f \in \mathscr{A}_p$ satisfies the condition (2.10), then*

$$\frac{1}{z^p}\left[\left(z^p+\frac{1}{\delta}\sum_{n=p+1}^{\infty}\left(1+\frac{\lambda(n-p)}{p+l}\right)^m z^n\right)*f(z)\right]\neq 0 \quad (z \in \mathbb{U}, 0 < \theta < 2\pi)$$

where $\delta = 1-\left(\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}\right)^{\frac{1}{\mu}}$.

Proof. Let the condition (2.10) hold, then on applying Theorem 1 for $h(z) = \frac{1+Az}{1+Bz}$, we get

$$\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^{\mu} \neq \frac{1+Ae^{i\theta}}{1+Be^{i\theta}} \quad (z \in \mathbb{U}, 0 < \theta < 2\pi),$$

or,

$$\frac{J_p^m(\lambda,l)f(z)-z^p\left(\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}\right)^{\frac{1}{\mu}}}{z^p\left(1-\left(\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}\right)^{\frac{1}{\mu}}\right)} \neq 0 \quad (z \in \mathbb{U}, 0 < \theta < 2\pi). \tag{3.8}$$

On expressing (1.4) as the convolution of two functions of the class $\mathscr{A}_p$ by

$$J_p^m(\lambda,l)f(z) = \left(z^p+\sum_{n=p+1}^{\infty}\left(1+\frac{\lambda(n-p)}{p+l}\right)^m z^n\right)*f(z), \tag{3.9}$$

we obtain from (3.8) the desired result.   □

COROLLARY 17. *If a function $f \in \mathscr{A}_p$ satisfies the condition (2.10) under the same parametric constraints stated in Corollary 3, then for some Schwartz function $\omega(z)$:*

$$f(z) = J_p^{-m}(\lambda,l)\left(z^p\left(\frac{1+A\omega(z)}{1+B\omega(z)}\right)^{\frac{1}{\mu}}\right) \quad (z \in \mathbb{U}). \tag{3.10}$$

Proof. Let a function $f \in \mathscr{A}_p$ satisfy the condition (2.10), then on applying Theorem 1 for $h(z) = \frac{1+Az}{1+Bz}$, we get for some Schwartz function $\omega(z)$:

$$\left(\frac{J_p^m(\lambda,l)f(z)}{z^p}\right)^{\mu} = \frac{1+A\omega(z)}{1+B\omega(z)}, \quad z \in \mathbb{U},$$

or,

$$J_p^m(\lambda,l)f(z) = z^p \left(\frac{1+A\omega(z)}{1+B\omega(z)}\right)^{\frac{1}{\mu}},$$

which on using (1.6) gives the representation (3.10). $\square$

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