Journal of
Classical
Analysis
Volume 2, Number 2 (2013), 107–127
doi:10.7153/jca-02-09

FOURIER EXPANSIONS FOR A LOGARITHMIC FUNDAMENTAL SOLUTION OF THE POLYHARMONIC EQUATION

HOWARD S. COHL

Abstract. In even-dimensional Euclidean space for integer powers of the Laplacian greater than or equal to the dimension divided by two, a fundamental solution for the polyharmonic equation has logarithmic behavior. We give two approaches for developing an azimuthal Fourier expansion of this logarithmic fundamental solution. The first approach is algebraic and relies upon the construction of two-parameter polynomials which we call logarithmic polynomials. The second approach depends on the computation of parameter derivatives of Fourier series expressions for a power-law fundamental solution of the polyharmonic equation. We conclude by comparing the two approaches and giving the azimuthal Fourier series for a logarithmic fundamental solution of the polyharmonic equation in rotationally-invariant coordinate systems.

1. Introduction

Solutions of the polyharmonic equation (powers of the Laplacian operator) are ubiquitous in many areas of computational, pure, applied mathematics, physics and engineering. We concern ourselves, in this paper, with a fundamental solution of the polyharmonic equation (Laplace, biharmonic, etc.), which by convolution yields a solution to the inhomogeneous polyharmonic equation. Solutions to inhomogeneous polyharmonic equations are useful in many physical applications including those areas related to Poisson's equation such as Newtonian gravity, electrostatics, magnetostatics, quantum direct and exchange interactions (cf. Section 1 in [3]), etc. Furthermore, applications of higher-powers of the Laplacian include such varied areas as minimal surfaces [12], Continuum Mechanics [8], Mesh deformation [6], Elasticity [9], Stokes Flow [7], Geometric Design [20], Cubature formulae [17], mean value theorems (cf. Pizzetti's formula) [13], and Hartree-Fock calculations of nuclei [21].

Closed-form expressions for the Fourier expansions of a logarithmic fundamental solution for the polyharmonic equation are extremely useful when solving inhomogeneous polyharmonic problems in even-dimensional Euclidean space, especially when a degree of rotational symmetry is involved. A fundamental solution of the polyharmonic equation on d-dimensional Euclidean space $\mathbf{R}^d$ has two arguments and therefore maps from a $2d$-dimensional space to the reals. Solutions to the inhomogeneous polyharmonic equation can be obtained by convolution of a fundamental solution with an integrable source distribution. Eigenfunction decompositions of a fundamental solution

Mathematics subject classification (2010): 35A08, 31B30, 31C12, 33C05, 42A16.
Keywords and phrases: Fundamental solutions, polyharmonic equation, Fourier series, polynomials, associated Legendre functions.

© ELEMENT, Zagreb
Paper JCA-02-09

reduces the dimension of the resulting convolution integral to obtain Dirichlet boundary values, replacing it instead by a sum or an integral over some discrete or continuous parameter space. By taking advantage of rotational or nearly rotational symmetry of the source distribution, one reduces the dimensionality of the resulting convolution integral and obtains a rapidly convergent Fourier cosine expansion. In the case of an axisymmetric (constant angular dependence) source distribution, the entire contribution to the boundary values are determined by a single term in the azimuthal Fourier series. These kinds of expansions have been previously shown to be extremely effective in solving inhomogeneous problems (see for instance the discussion in Cohl & Tohline (1999) [4]). In general, these methods can be applied to any of the applications described in the previous paragraph.

It is a well-known fact (see for instance Schwartz (1950) ([16], p. 45); Gel'fand & Shilov (1964) ([5], p. 202)) that a fundamental solution of the polyharmonic equation on $\mathbf{R}^d$ is given by combinations of power-law and logarithmic functions of the global distance between two points. In a recent paper (Cohl & Dominici (2010) [3]), we derived an identity over the complex numbers which determined the Fourier coefficients of a power-law fundamental solution of the polyharmonic equation. The present work is concerned with computing the Fourier coefficients of a logarithmic fundamental solution of the polyharmonic equation. One obtains a logarithmic fundamental solution for the polyharmonic equation only on even-dimensional Euclidean space and only when the power of the Laplacian is greater than or equal to the dimension divided by two. The most familiar example of a logarithmic fundamental solution of the polyharmonic equation occurs in two-dimensions, for a single-power of the Laplacian, i.e., Laplace's equation.

We present two different approaches for obtaining Fourier series of a logarithmic fundamental solution for the polyharmonic equation. The first approach is algebraic and involves the generation of a certain set of naturally arising two-index polynomials which we refer to as logarithmic polynomials. The second approach starts with the main result from [3] and determines the Fourier series expansion for a logarithmic fundamental solution of the polyharmonic equation through parameter differentiation.

This paper is organized as follows. In Section 2 we introduce the problem. In Section 3 we describe our algebraic approach to computing a Fourier series for a logarithmic fundamental solution of the polyharmonic equation. In Section 4 we give our limit derivative approach for computing the Fourier series for a logarithmic fundamental solution of the polyharmonic equation. In Section 5 we give some comparisons between the two approaches. In Section 6 we obtain azimuthal Fourier expansions for a logarithmic fundamental solution of the polyharmonic equation in rotationally-invariant coordinate systems which parametrize points on d-dimensional Euclidean space. In Appendix A we give some necessary formulae relating to differentiation of associated Legendre functions of the first kind with respect to the degree. In Appendix B we present and derive some properties of the logarithmic polynomials.

Throughout this paper we rely on the following definitions. The set of natural numbers is given by $\mathbf{N} := \{1,2,3,\ldots\}$, the set $\mathbf{N}_0 := \{0,1,2,\ldots\} = \mathbf{N} \cup \{0\}$, the set of integers is given by $\mathbf{Z} := \{0,\pm 1,\pm 2,\ldots\}$. The sets $\mathbf{Q}$ and $\mathbf{R}$ represents the rational and real numbers respectively. For $a_1, a_2, a_3, \ldots \in \mathbf{C}$, if $i, j \in \mathbf{Z}$ and $j < i$ then $\sum_{n=i}^{j} a_n = 0$ and $\prod_{n=i}^{j} a_n = 1$, where $\mathbf{C}$ represents the complex numbers.

2. Fundamental solution of the polyharmonic equation and the non-logarithmic Fourier series

If $\Phi : \mathbf{R}^d \to \mathbf{R}$ satisfies the polyharmonic equation given by

$$(-\Delta)^k \Phi(\mathbf{x}) = 0, \tag{1}$$

where $\mathbf{x} \in \mathbf{R}^d$, $\Delta : C^p(\mathbf{R}^d) \to C^{p-2}(\mathbf{R}^d)$ for $p \geqslant 2$, is the Laplacian operator defined by $\Delta := \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2}$, $k \in \mathbf{N}$ and $\Phi \in C^{2k}(\mathbf{R}^d)$, then Φ is called polyharmonic. We use the nonnegative Laplacian $-\Delta \geqslant 0$. The inhomogeneous polyharmonic equation is given by

$$(-\Delta)^k \Phi(\mathbf{x}) = \rho(\mathbf{x}), \tag{2}$$

where we take ρ to be an integrable function so that a solution to (2) exists. A fundamental solution for the polyharmonic equation on $\mathbf{R}^d$ is a function $\mathfrak{g}_k^d : (\mathbf{R}^d \times \mathbf{R}^d) \setminus \{(\mathbf{x},\mathbf{x}) : \mathbf{x} \in \mathbf{R}^d\} \to \mathbf{R}$ which satisfies, in the sense of distributions, the equation

$$(-\Delta)^k \mathfrak{g}_k^d(\mathbf{x},\mathbf{x}') = c\delta(\mathbf{x}-\mathbf{x}'), \tag{3}$$

for some $c \in \mathbf{R}, c \neq 0$, where δ is the Dirac delta distribution and $\mathbf{x}' \in \mathbf{R}^d$. When $c = 1$, we call a fundamental solution of the polyharmonic equation normalized, and denote it by $\mathscr{G}_k^d : (\mathbf{R}^d \times \mathbf{R}^d) \setminus \{(\mathbf{x},\mathbf{x}) : \mathbf{x} \in \mathbf{R}^d\} \to \mathbf{R}$. The Euclidean inner product $(\cdot,\cdot) : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ defined by $(\mathbf{x},\mathbf{x}') := x_1 x_1' + \ldots + x_d x_d'$, induces a norm (the Euclidean norm) $\|\cdot\| : \mathbf{R}^d \to [0,\infty)$, on the finite-dimensional vector space $\mathbf{R}^d$, given by $\|\mathbf{x}\| := \sqrt{(\mathbf{x},\mathbf{x})}$.

In the rest of this paper, we will use the gamma function $\Gamma : \mathbf{C} \setminus -\mathbf{N}_0 \to \mathbf{C}$, which is a natural generalization of the factorial function (see for instance Chapter 5 in Olver *et al.* (2010) [14]). A fundamental solution of the polyharmonic equation is given as follows.

THEOREM 1. *Let* $d, k \in \mathbf{N}$. *Define* $\mathscr{G}_k^d : (\mathbf{R}^d \times \mathbf{R}^d) \setminus \{(\mathbf{x},\mathbf{x}) : \mathbf{x} \in \mathbf{R}^d\} \to \mathbf{R}$ *such that*

$$\mathscr{G}_k^d(\mathbf{x},\mathbf{x}') := \begin{cases} \dfrac{(-1)^{k+d/2+1}\, \|\mathbf{x}-\mathbf{x}'\|^{2k-d}}{(k-1)!\,(k-d/2)!\,2^{2k-1}\pi^{d/2}} \left(\log\|\mathbf{x}-\mathbf{x}'\| - \beta_{k-d/2,d}\right) & \text{if } d \text{ even, } k \geqslant d/2, \\[2ex] \dfrac{\Gamma(d/2-k)\|\mathbf{x}-\mathbf{x}'\|^{2k-d}}{(k-1)!\,2^{2k}\pi^{d/2}} & \text{otherwise,} \end{cases}$$

where $\beta_{p,d} \in \mathbf{Q}$ *is defined as* $\beta_{p,d} := \frac{1}{2}\left[H_p + H_{d/2+p-1} - H_{d/2-1}\right]$, *with* H_j *being the* j*th harmonic number*

$$H_j := \sum_{i=1}^{j} \frac{1}{i}.$$

Proof. See Boyling (1996) [2, (2.1)] and [17, Section II.2]. In regard to a logarithmic fundamental solution of the polyharmonic equation, note that in [17] the term proportional to $\|\mathbf{x}-\mathbf{x}'\|^{2k-d}$ is missing. This term is in the kernel of the polyharmonic operator $(-\Delta)^k$, so any constant multiple of this term $\beta_{p,d}$ may be added to a fundamental solution of the polyharmonic equation. Our choice for this constant is given so that

$$-\Delta\mathscr{G}_k^d = \mathscr{G}_{k-1}^d, \tag{4}$$

is satisfied for all $k > d/2$, and that for $k = d/2$, the constant vanishes. Boyling's fundamental solution satisfies (4) for all $k > d/2$, but is missing the term proportional to $H_{d/2-1}$, and therefore only vanishes when $k = d/2$ for $d = 2$. Sobolev does not include this constant term, so for him $\mathscr{G}_k^d$ is purely logarithmic for all $k \geqslant d/2$, $d \geqslant 2$ even. However (4) is not strictly satisfied for $k > d/2$. $\square$

In this paper we restrict our attention to separable rotationally-invariant coordinate systems for the polyharmonic equation on $\mathbf{R}^d$ which are given by

$$\left.\begin{array}{l} x_1 = R(\xi_1,\ldots,\xi_{d-1})\cos\phi \\ x_2 = R(\xi_1,\ldots,\xi_{d-1})\sin\phi \\ x_3 = x_3(\xi_1,\ldots,\xi_{d-1}) \\ \quad\vdots \\ x_d = x_d(\xi_1,\ldots,\xi_{d-1}) \end{array}\right\}. \tag{5}$$

These coordinate systems are described by d-coordinates: an angle $\phi \in \mathbf{R}$ plus $(d-1)$-curvilinear coordinates $(\xi_1,\ldots,\xi_{d-1})$. Rotationally-invariant coordinate systems parametrize points on the $(d-1)$-dimensional half-hyperplane given by $\phi = \mathrm{const}$ and $R \geqslant 0$ using the curvilinear coordinates $(\xi_1,\ldots,\xi_{d-1})$. A separable rotationally-invariant coordinate system transforms the polyharmonic equation into a set of d-uncoupled ordinary differential equations with separation constants $m \in \mathbf{Z}$ and k_j for $1 \leqslant j \leqslant d-2$. For a separable rotationally-invariant coordinate system, this uncoupling is accomplished, in general, by assuming a solution to (1) of the form

$$\Phi(\mathbf{x}) = e^{im\phi}\,\mathscr{R}(\xi_1,\ldots,\xi_{d-1})\prod_{i=1}^{d-1} A_i(\xi_i,m,k_1,\ldots,k_{d-2}),$$

where the properties of the functions $\mathscr{R}$ and A_i, for $1 \leqslant i \leqslant d-1$, and the constants k_j for $1 \leqslant j \leqslant d-2$, depend on the specific separable rotationally-invariant coordinate system in question. Note that separable coordinate systems are divided into two distinct classes, those which are simply separable ($\mathscr{R} = \mathrm{const}$), and those which are $\mathscr{R}$-separable. For a general description of the theory of separation of variables see Miller (1977) [11].

The Euclidean distance between two points $\mathbf{x}, \mathbf{x}' \in \mathbf{R}^d$, expressed in a rotationally-invariant coordinate system, is given by

$$\|\mathbf{x}-\mathbf{x}'\| = \sqrt{2RR'}\left[\chi - \cos(\phi-\phi')\right]^{1/2},$$

where the toroidal parameter $\chi > 1$, is given by

$$\chi := \frac{R^2 + R'^2 + \sum_{i=3}^{d} (x_i - x_i')^2}{2RR'}, \tag{6}$$

where $R, R' \in (0, \infty)$ are defined in (5). The hypersurfaces given by $\chi > 1$ equals constant are independent of coordinate system and represent hyper-tori of revolution.

From Theorem 1 we see that, apart from multiplicative constants, the algebraic expression $\mathfrak{l}_k^d : (\mathbf{R}^d \times \mathbf{R}^d) \setminus \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbf{R}^d\} \to \mathbf{R}$ of an unnormalized fundamental solution for the polyharmonic equation on Euclidean space $\mathbf{R}^d$ for d even, $k \geqslant d/2$, is given by

$$\mathfrak{l}_k^d(\mathbf{x}, \mathbf{x}') := \|\mathbf{x} - \mathbf{x}'\|^{2k-d} \left(\log \|\mathbf{x} - \mathbf{x}'\| - \beta_{k-d/2,d} \right). \tag{7}$$

By expressing $\mathfrak{l}_k^d$ in a rotationally-invariant coordinate system (5) we obtain

$$\begin{aligned} \mathfrak{l}_k^d(\mathbf{x}, \mathbf{x}') = {} & \left(2RR'\right)^p \left[\frac{1}{2} \log\left(2RR'\right) - \beta_{p,d} \right] \left[\chi - \cos(\phi - \phi')\right]^p \\ & + \frac{1}{2} \left(2RR'\right)^p \left[\chi - \cos(\phi - \phi')\right]^p \log\left[\chi - \cos(\phi - \phi')\right], \end{aligned} \tag{8}$$

where $p = k - d/2 \in \mathbf{N}_0$. For the polyharmonic equation on even-dimensional Euclidean space $\mathbf{R}^d$ with $1 \leqslant k \leqslant d/2 - 1$, apart from multiplicative constants, the algebraic expression for an unnormalized fundamental solution of the polyharmonic equation $\mathfrak{h}_k^d : (\mathbf{R}^d \times \mathbf{R}^d) \setminus \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbf{R}^d\} \to \mathbf{R}$ is given by

$$\mathfrak{h}_k^d(\mathbf{x}, \mathbf{x}') := \|\mathbf{x} - \mathbf{x}'\|^{2k-d}.$$

By expressing $\mathfrak{h}_k^d$ in a rotationally-invariant coordinate system we obtain

$$\mathfrak{h}_k^d(\mathbf{x}, \mathbf{x}') = \left(2RR'\right)^{-q} \left[\chi - \cos(\phi - \phi')\right]^{-q}, \tag{9}$$

where $q = d/2 - k$.

For computation of Fourier expansions about the azimuthal separation angle $(\phi - \phi')$ of $\mathfrak{l}_k^d$ and $\mathfrak{h}_k^d$, all that is required is to compute the Fourier cosine series for the following three functions $f_\chi, h_\chi : \mathbf{R} \to (0, \infty)$ and $g_\chi : \mathbf{R} \to \mathbf{R}$ defined as

$$\begin{aligned} f_\chi(\psi) &:= (\chi - \cos\psi)^p, \\ g_\chi(\psi) &:= (\chi - \cos\psi)^p \log(\chi - \cos\psi), \qquad \text{and} \\ h_\chi(\psi) &:= (\chi - \cos\psi)^{-q}, \end{aligned}$$

where $p \in \mathbf{N}_0$, $q \in \mathbf{N}$ and $\chi > 1$ is a fixed parameter.

The Fourier series of f_χ is given in [3] (cf. (4.4) therein), namely

$$(z - \cos\psi)^p = (z^2 - 1)^{p/2} \sum_{n=0}^{p} \varepsilon_n \cos(n\psi) \frac{(-p)_n (p-n)!}{(p+n)!} P_p^n\left(\frac{z}{\sqrt{z^2 - 1}}\right), \tag{10}$$

where the Neumann factor $\varepsilon_n := 2 - \delta_{n,0}$ commonly occurs in Fourier series, $\delta_{n,0}$ is the Kronecker delta, and

$$(z)_n := \prod_{i=1}^{n}(z+i-1),$$

for $z \in \mathbf{C}$ and $n \in \mathbf{N}_0$, is the Pochhammer symbol (rising factorial). We have used Whipple's formula in (10) (see for instance, (8.2.7) in Abramowitz & Stegun (1972) [1]) to convert the associated Legendre function of the second kind $Q_\nu^\mu : (1,\infty) \to \mathbf{C}$ appearing in [3] to the associated Legendre function of the first kind $P_\nu^\mu : (1,\infty) \to \mathbf{R}$. The associated Legendre function of the first kind can be defined (and analytically continued) using the Gauss hypergeometric function ((14.3.6) and Section 14.21(i) in [14]) defined as

$$P_\nu^\mu(z) := \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1\left(-\nu,\nu+1;1-\mu;\frac{1-z}{2}\right)$$

where $|1-z|<2$.

The Fourier series of h_χ is given in [3] (where the Whipple formula (8.2.7) in Abramowitz & Stegun (1972) [1] has been used and cf. (4.5) therein), namely

$$\frac{1}{(z-\cos\psi)^q} = \frac{(z^2-1)^{-q/2}}{(q-1)!}\sum_{n=0}^{\infty}\varepsilon_n\cos(n\psi)(n+q-1)!P_{q-1}^{-n}\left(\frac{z}{\sqrt{z^2-1}}\right), \qquad (11)$$

where $q \in \mathbf{N}$. Since the Fourier series of h_χ is computed in [3], we understand how to compute Fourier expansions of $\mathfrak{h}_k^d$ (9) in separable rotationally-invariant coordinate systems. In order to compute Fourier expansion of $\mathfrak{l}_k^d$ (8) in separable rotationally-invariant coordinate systems, all that remains is to determine the Fourier series of g_χ. This is the goal of the next two sections.

3. Algebraic approach to the logarithmic Fourier series

Since $\chi>1$, one may make the substitution $\chi=\cosh\eta$ to evaluate the Fourier series of g_χ, which is given in the form of $(\cosh\eta-\cos\psi)^p\log(\cosh\eta-\cos\psi)$, where $p\in\mathbf{N}_0$. For $p=0$ the result is well-known (see for instance Magnus, Oberhettinger & Soni (1966) [10], p. 259)

$$\log(\cosh\eta-\cos\psi) = \eta - \log 2 - 2\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos(n\psi), \qquad (12)$$

which as we will see, should be compared with (11) for $q=1$, namely

$$\frac{1}{\cosh\eta-\cos\psi} = \frac{1}{\sinh\eta}\sum_{n=0}^{\infty}\varepsilon_n\cos(n\psi)e^{-n\eta}. \qquad (13)$$

Note that for $\eta>0$ we may write e^η and therefore η as a function of $\cosh\eta$ since $\sinh\eta=\sqrt{\cosh^2\eta-1}$, $e^\eta=\cosh\eta+\sqrt{\cosh^2\eta-1}$, and therefore $\eta=\log(\cosh\eta+\sqrt{\cosh^2\eta-1})$.

We now examine the $p=1$ case for g_χ. If we multiply both sides of (12) by $(\cosh\eta-\cos\psi)$ and take advantage of the formula

$$\cos(n\psi)\cos\psi=\frac{1}{2}\Big\{\cos[(n+1)\psi]+\cos[(n-1)\psi]\Big\}, \tag{14}$$

then we have

$$\begin{aligned}(\cosh\eta-\cos\psi)\log(\cosh\eta-\cos\psi)&=(\eta-\log 2)\cosh\eta\\ &-(\eta-\log 2)\cos\psi-2\cosh\eta\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos(n\psi)\\ &+\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos[(n+1)\psi]+\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos[(n-1)\psi].\end{aligned} \tag{15}$$

Collecting the contributions to the Fourier cosine series, we obtain

$$\begin{aligned}(\cosh\eta-\cos\psi)\log(\cosh\eta-\cos\psi)&=(1+\eta-\log 2)\cosh\eta\\ &-\sinh\eta+\cos\psi\left(\log 2-1-\eta-\frac{1}{2}e^{-2\eta}\right)\\ &+2\sum_{n=2}^{\infty}\frac{e^{-n\eta}\cos n\psi}{n(n^2-1)}(\cosh\eta+n\sinh\eta).\end{aligned} \tag{16}$$

If we compare (16) with (11) for $q=2$, namely

$$\frac{1}{(\cosh\eta-\cos\psi)^2}=\frac{1}{\sinh^3\eta}\sum_{n=0}^{\infty}\varepsilon_n\cos(n\psi)e^{-n\eta}(\cosh\eta+n\sinh\eta), \tag{17}$$

we notice that the factor $(\cosh\eta+n\sinh\eta)$ appears in both series.

For $p=2$ in g_χ, we use (14) and similarly have

$$\begin{aligned}(\cosh\eta-\cos\psi)^2\log(\cosh\eta-\cos\psi)&=(\eta-\log 2)\cosh^2\eta\\ &-2(\eta-\log 2)\cosh\eta\cos\psi\\ &+(\eta-\log 2)\cos^2\psi-(2\cosh^2\eta+1)\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos(n\psi)\\ &+2\cosh\eta\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos[(n+1)\psi]+2\cosh\eta\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos[(n-1)\psi]\\ &-\frac{1}{2}\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos[(n+2)\psi]-\frac{1}{2}\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos[(n-2)\psi].\end{aligned}$$

If we collect the contributions of the Fourier cosine series, we obtain

$$\begin{aligned}&(\cosh\eta-\cos\psi)^2\log(\cosh\eta-\cos\psi)\\ &=(\eta-\log 2)\left(\cosh^2\eta+\frac{1}{2}\right)+2\cosh\eta\, e^{-\eta}-\frac{1}{4}e^{-2\eta}\end{aligned}$$

$$+\left[-2(\eta-\log 2)\cosh\eta-\left(2\cosh^2\eta+\frac{3}{2}\right)e^{-\eta}+\cosh\eta\, e^{-2\eta}-\frac{1}{6}e^{-3\eta}\right]\cos\psi$$
$$+\left[\frac{1}{2}(\eta-\log 2)+2\cosh\eta\, e^{-\eta}-\frac{1}{2}(2\cosh^2\eta+1)e^{-2\eta}\right.$$
$$\left.+\frac{2}{3}\cosh\eta\, e^{-3\eta}-\frac{1}{8}e^{-4\eta}\right]\cos 2\psi$$
$$-4\sum_{n=3}^{\infty}\frac{e^{-n\eta}\cos(n\psi)}{n(n^2-1)(n^2-4)}\left[(n^2-1)\sinh^2\eta+3n\sinh\eta\cosh\eta+3\cosh^2\eta\right]. \quad (18)$$

By comparing (18) with (11) for $q=3$, namely

$$\frac{1}{(\cosh\eta-\cos\psi)^3}=\frac{1}{2\sinh^5\eta}\sum_{n=0}^{\infty}\varepsilon_n\cos(n\psi)e^{-n\eta}$$
$$\times\left[(n^2-1)\sinh^2\eta+3n\sinh\eta\cosh\eta+3\cosh^2\eta\right], \quad (19)$$

then we notice that the factor $\big((n^2-1)\sinh^2\eta+3n\sinh\eta\cosh\eta+3\cosh^2\eta\big)$ appears in both series. We will demonstrate in Section 5, why the identification mentioned in (17) and (19) occurs.

The algebraic approach for determining the Fourier series of g_χ described above is generalized by the following theorem.

THEOREM 2. *Let* $p\in\mathbf{N}_0$, $\eta\in(0,\infty)$, $\psi\in\mathbf{R}$, P_n^p *the associated Legendre function of the first kind (with* $P_n:=P_n^0$ *being the Legendre polynomial), and* $R_p^k(\cosh\eta)$ *the logarithmic polynomials. Then*

$$(\cosh\eta-\cos\psi)^p\log(\cosh\eta-\cos\psi)=\sum_{n=0}^{\infty}\cos(n\psi)\mathfrak{Q}_{n,p}(\cosh\eta), \quad (20)$$

where the function $\mathfrak{Q}_{n,p}:(1,\infty)\to\mathbf{R}$ *is defined such that if* $n=0$,

$$\mathfrak{Q}_{0,p}(\cosh\eta):=(\eta-\log 2)\sinh^p\eta\, P_p(\coth\eta)+2\sum_{k=1}^{p}\frac{(-1)^{k+1}e^{-k\eta}}{k}R_p^k(\cosh\eta),$$

if $1\leqslant n\leqslant p-1$,

$$\mathfrak{Q}_{n,p}(\cosh\eta):=2(\eta-\log 2)\sinh^p\eta\,\frac{(-p)_n(p-n)!}{(p+n)!}P_p^n(\coth\eta)$$
$$+e^{-n\eta}\sum_{k=0}^{n-1}\varepsilon_k(-1)^{k+1}\left[\frac{e^{-k\eta}}{n+k}+\frac{e^{k\eta}}{n-k}\right]R_p^k(\cosh\eta)$$
$$+2e^{-n\eta}\sum_{k=n}^{p}(-1)^{k+1}\frac{e^{-k\eta}}{n+k}R_p^k(\cosh\eta)$$

$$+2e^{n\eta}\sum_{k=n+1}^{p}(-1)^k\frac{e^{-k\eta}}{n-k}R_p^k(\cosh\eta),$$

if $1\leqslant n=p$,

$$\mathfrak{Q}_{p,p}(\cosh\eta):=2(-1)^p(\eta-\log 2)\sinh^p\eta\,\frac{p!}{(2p)!}P_p^p(\coth\eta)$$
$$+e^{-p\eta}\sum_{k=0}^{p-1}\varepsilon_k(-1)^{k+1}\left[\frac{e^{-k\eta}}{p+k}+\frac{e^{k\eta}}{p-k}\right]R_p^k(\cosh\eta)$$
$$+(-1)^{p+1}\frac{e^{-2p\eta}}{p}R_p^p(\cosh\eta),$$

and if $n\geqslant p+1$,

$$\mathfrak{Q}_{n,p}(\cosh\eta):=\frac{e^{-n\eta}(n+p)!}{(n-p-1)!(n^2-p^2)\cdots(n^2-1)n}$$
$$\times\sum_{k=0}^{p}\varepsilon_k(-1)^{k+1}\left[\frac{e^{-k\eta}}{n+k}+\frac{e^{k\eta}}{n-k}\right]R_p^k(\cosh\eta).$$

Proof. By starting with (12) and repeatedly multiplying by factors of $(\cosh\eta-\cos\psi)$, we see that the general Fourier series of g_χ can be given in terms of a sequence of polynomials $R_p^k:(1,\infty)\to\mathbf{R}$, with $p\in\mathbf{N}_0$ and $k\in\mathbf{Z}$, as

$$(\cosh\eta-\cos\psi)^p\log(\cosh\eta-\cos\psi)=(\eta-\log 2)(\cosh\eta-\cos\psi)^p$$
$$+2\sum_{k=-p}^{p}(-1)^{k+1}R_p^k(\cosh\eta)\sum_{n=1}^{\infty}\frac{e^{-n\eta}}{n}\cos[(n+k)\psi]. \tag{21}$$

We will refer to $R_p^k(\cosh\eta)$ as logarithmic polynomials in $\cosh\eta$ (in our notation p and k are both indices). See Appendix B for a description of some of the properties of the logarithmic polynomials.

The double sum in (21) is simplified by making the replacement $n+k\mapsto n$. It then follows that the resulting double sum naturally breaks into two disjoint regions, one triangular

$$\mathscr{A}:=\{(k,n):-p\leqslant k\leqslant p-1,\ k+1\leqslant n\leqslant p\},$$

with $p(2p+1)$ terms and the other infinite rectangular

$$\mathscr{B}:=\{(k,n):-p\leqslant k\leqslant p,\ p+1<n<\infty\}.$$

By rearranging the order of the k and n summations in (21), we derive

$$(\cosh\eta-\cos\psi)^p\log(\cosh\eta-\cos\psi)=(\eta-\log 2)(\cosh\eta-\cos\psi)^p$$
$$+\sum_{n=0}^{p}\cos(n\psi)e^{-n\eta}\,\mathfrak{r}_{n,p}^{-p,n-1}(\cosh\eta)$$

$$+\sum_{n=1}^{p-1}\cos(n\psi)e^{n\eta}\,\mathfrak{r}_{-n,p}^{-p,-n-1}(\cosh\eta)$$
$$+\sum_{n=p+1}^{\infty}\frac{\cos(n\psi)e^{-n\eta}}{n(n^2-1)\cdots(n^2-p^2)}\mathfrak{R}_{n,p}(\cosh\eta), \tag{22}$$

where $\mathfrak{r}_{n,p}^{k_1,k_2},\mathfrak{R}_{n,p}:(1,\infty)\to\mathbf{R}$ are defined as

$$\mathfrak{r}_{n,p}^{k_1,k_2}(\cosh\eta):=2\sum_{k=k_1}^{k_2}\frac{(-1)^{k+1}e^{k\eta}R_p^k(\cosh\eta)}{n-k},$$

and

$$\mathfrak{R}_{n,p}(\cosh\eta):=\frac{(n+p)!}{(n-p-1)!}\mathfrak{r}_{n,p}^{-p,p}(\cosh\eta),$$

$n\geqslant p+1$, respectively. We can also write the Fourier series directly in terms of the logarithmic polynomials $R_p^k(\cosh\eta)$ as follows

$$(\cosh\eta-\cos\psi)^p\log(\cosh\eta-\cos\psi)=(\eta-\log 2)(\cosh\eta-\cos\psi)^p$$
$$+2\sum_{n=0}^{p}\cos(n\psi)e^{-n\eta}\sum_{k=-p}^{n-1}\frac{(-1)^{k+1}e^{k\eta}R_p^k(\cosh\eta)}{n-k}$$
$$-2\sum_{n=1}^{p-1}\cos(n\psi)e^{n\eta}\sum_{k=-p}^{-n-1}\frac{(-1)^{k+1}e^{k\eta}R_p^k(\cosh\eta)}{n+k}$$
$$+2\sum_{n=p+1}^{\infty}\cos(n\psi)e^{-n\eta}\sum_{k=-p}^{p}\frac{(-1)^{k+1}e^{k\eta}R_p^k(\cosh\eta)}{n-k}. \tag{23}$$

Note that by using (10), then we can express $(\cosh\eta-\cos\psi)^p$ as a Fourier series, namely

$$(\cosh\eta-\cos\psi)^p=\sinh^p\eta\sum_{n=0}^{p}\varepsilon_n\cos(n\psi)\frac{(-p)_n(p-n)!}{(p+n)!}P_p^n(\coth\eta), \tag{24}$$

where $p\in\mathbf{N}_0$.

If we define the function $\mathfrak{P}_{n,p}:(1,\infty)\to\mathbf{R}$ such that

$$\mathfrak{P}_{n,p}(\cosh\eta):=\begin{cases}\mathfrak{r}_{0,p}^{-p,-1}(\cosh\eta) & \text{if } n=0,\\ D_{n,p}(\cosh\eta)+E_{n,p}(\cosh\eta) & \text{if } 1\leqslant n\leqslant p-1,\\ e^{-p\eta}\mathfrak{r}_{p,p}^{-p,p-1}(\cosh\eta) & \text{if } 1\leqslant n=p,\\ \dfrac{e^{-n\eta}}{(n^2-p^2)\cdots(n^2-1)n}\mathfrak{R}_{n,p}(\cosh\eta) & \text{if } n\geqslant p+1,\end{cases} \tag{25}$$

where $D_{n,p},E_{n,p}:(1,\infty)\to\mathbf{R}$ are defined as

$$D_{n,p}(\cosh\eta)=\begin{cases}0 & \text{if } p=0,1,\\ e^{n\eta}\mathfrak{r}_{-n,p}^{-p,-n-1}(\cosh\eta) & \text{if } p\geqslant 2,\end{cases}$$

and

$$E_{n,p}(\cosh\eta) = \begin{cases} 0 & \text{if } p = 0, \\ e^{-n\eta}\mathfrak{r}_{n,p}^{-p,n-1}(\cosh\eta) & \text{if } p \geqslant 1, \end{cases}$$

respectively, then we can write (22) as

$$(\cosh\eta - \cos\psi)^p \log(\cosh\eta - \cos\psi) = (\eta - \log 2)(\cosh\eta - \cos\psi)^p + \sum_{n=0}^{\infty} \cos(n\psi)\mathfrak{P}_{n,p}(\cosh\eta).$$

Note that if $p = 1$ then $E_{n,p}(\cosh\eta)$ gives the $n = 0$ contribution to (25). In fact, if we use (24), then we can express g_χ as

$$(\cosh\eta - \cos\psi)^p \log(\cosh\eta - \cos\psi) = \sum_{n=0}^{\infty} \cos(n\psi)\mathfrak{P}_{n,p}(\cosh\eta) + (\eta - \log 2)\sinh^p\eta \sum_{n=0}^{p} \varepsilon_n \cos(n\psi) \frac{(-p)_n(p-n)!}{(p+n)!} P_p^n(\coth\eta).$$

Furthermore, if we define $\mathfrak{Q}_{n,p}$ as

$$\mathfrak{Q}_{n,p}(\cosh\eta) := \mathfrak{P}_{n,p}(\cosh\eta) + \frac{\varepsilon_n(-p)_n(p-n)!}{(p+n)!}(\eta - \log 2)\sinh^p\eta P_p^n(\coth\eta),$$

we have completed our proof. □

4. Limit derivative approach to the logarithmic Fourier series

We now use a second approach to compute the Fourier series for a logarithmic fundamental solution of the polyharmonic equation (7). We would like to match our results to the computations in Section 3, which clearly demonstrate different behaviors for the two regimes, $0 \leqslant n \leqslant p$ and $n \geqslant p+1$.

THEOREM 3. *Let* $p \in \mathbf{N}_0$, $\eta \in (0,\infty)$, $\psi \in \mathbf{R}$, P_n^p *the associated Legendre function of the first kind. Then*

$$\begin{aligned}(\cosh\eta - \cos\psi)^p &\log(\cosh\eta - \cos\psi) \\ &= (\eta - \log 2)(\cosh\eta - \cos\psi)^p + p!\sinh^p\eta \sum_{n=0}^{p} \frac{(-1)^n \varepsilon_n \cos(n\psi)}{(p+n)!} \\ &\times [2\psi(2p+1) - \psi(p+1+n) - \psi(p+1-n)] P_p^n(\coth\eta) \\ &+ (-1)^p p!\sinh^p\eta \sum_{n=0}^{p-1} \frac{\varepsilon_n \cos(n\psi)}{(p+n)!} \\ &\times \sum_{k=0}^{p-n-1} \frac{(-1)^k(2n+2k+1)\left[1 + \frac{k!(p+n)!}{(2n+k)!(p-n)!}\right]}{(p-n-k)(p+n+k+1)} P_{n+k}^n(\coth\eta)\end{aligned}$$

$$+2(-1)^p p!\sinh^p\eta\sum_{n=1}^{p}\frac{(-1)^n\cos(n\psi)}{(p-n)!}\sum_{k=0}^{n-1}\frac{(-1)^k(2k+1)}{(p-k)(p+k+1)}P_k^{-n}(\coth\eta)$$
$$+2(-1)^{p+1}p!\sinh^p\eta\sum_{n=p+1}^{\infty}\cos(n\psi)(n-p-1)!P_p^{-n}(\coth\eta). \qquad (26)$$

Proof. By applying the identity

$$(\cosh\eta-\cos\psi)^p\log(\cosh\eta-\cos\psi)=\lim_{\nu\to 0}\frac{\partial}{\partial\nu}(\cosh\eta-\cos\psi)^{\nu+p}, \qquad (27)$$

where $p\in\mathbf{N}_0$, to

$$(\cosh\eta-\cos\psi)^\nu=\sinh^\nu\eta\sum_{n=0}^{\infty}\frac{(-1)^n\varepsilon_n\cos(n\psi)}{(\nu+1)_n}P_\nu^n(\coth\eta), \qquad (28)$$

where $\nu\in\mathbf{C}\setminus-\mathbf{N}$ (cf. (3.11b) in [3]), one can compute the Fourier cosine series of $(\cosh\eta-\cos\psi)^p\log(\cosh\eta-\cos\psi)$, provided availability of the necessary parameter derivatives.

Applying (27) to (28), we obtain

$$(\cosh\eta-\cos\psi)^p\log(\cosh\eta-\cos\psi)$$
$$=\left[\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\sinh^{\nu+p}\eta\right]\sum_{n=0}^{\infty}\frac{(-1)^n\varepsilon_n\cos(n\psi)}{(p+1)_n}P_p^n(\coth\eta)$$
$$+\sinh^p\eta\sum_{n=0}^{\infty}(-1)^n\varepsilon_n\cos(n\psi)\left[\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\frac{1}{(\nu+p+1)_n}\right]P_p^n(\coth\eta)$$
$$+\sinh^p\eta\sum_{n=0}^{\infty}\frac{(-1)^n\varepsilon_n\cos(n\psi)}{(p+1)_n}\left[\lim_{\nu\to 0}\frac{\partial}{\partial\nu}P_{\nu+p}^n(\coth\eta)\right].$$

Note that for $p,n\in\mathbf{N}_0$, the associated Legendre function of the first kind P_p^n vanishes if $n\geqslant p+1$. This is easily seen using the Rodrigues-type formula (cf. (14.7.11) in [14])

$$P_p^n(z)=(z^2-1)^{n/2}\frac{d^nP_p(z)}{dz^n},$$

and the fact that $P_p(z)$ (the Legendre polynomial) is a polynomial in z of degree p. The derivatives are given as follows:

$$\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\sinh^{\nu+p}\eta=\sinh^p\eta\log\sinh\eta, \qquad (29)$$

$$\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\frac{1}{(\nu+p+1)_n}=\frac{p!}{(p+n)!}[\psi(p+1)-\psi(p+1+n)], \qquad (30)$$

and

$$\lim_{\nu\to 0}\frac{\partial}{\partial\nu}P_{\nu+p}^n(\coth\eta)=\left[\frac{\partial}{\partial\nu}P_\nu^n(\coth\eta)\right]_{\nu=p}, \qquad (31)$$

where $\psi : \mathbf{C} \setminus -\mathbf{N}_0 \to \mathbf{C}$ is the digamma function defined in terms of the derivative of the gamma function

$$\frac{d}{dz}\Gamma(z) =: \psi(z)\Gamma(z)$$

(see for instance (5.2.2) in [14]). The degree-derivative of the associated Legendre function of the first kind in (31) is determined using (34) and (35). By collecting terms and using (29), (30), and (31), we have completed our proof. □

5. Comparison of the two approaches

The limit derivative approach presented in Section 4 might be considered, of the two methods, preferred for computing the azimuthal Fourier series for a logarithmic fundamental solution of the polyharmonic equation. This is because it produces azimuthal Fourier coefficients in terms of the well-known special functions, associated Legendre functions. On the other hand, the algebraic approach presented in Section 3 produces results in terms of the two-parameter logarithmic polynomials $R_p^k(x)$. As far as the author is aware, these polynomials are previously unencountered in the literature. By comparison of the two approaches we see how the logarithmic polynomials $R_p^k(x)$ (perhaps a new type of special function) are related to the associated Legendre functions. In this section we make this comparison concrete. We should also mention that the following comparison equations resolve to become quite complicated as p increases, and they have been checked for $0 \leqslant p \leqslant 10$ using Mathematica with the assistance of an algorithm generated using (39) and (40) from Appendix B.

By equating the Fourier coefficients of g_χ using the two approaches we can obtain the following corollary summation formulae which are satisfied by the logarithmic polynomials.

COROLLARY 1. *Let* $n = 0,\ p \geqslant 1, \eta \in (0,\infty)$. *Then*

$$\sum_{k=1}^{p} \frac{(-1)^{k+1} e^{-k\eta} R_p^k(\cosh\eta)}{k} = \sinh^p \eta \,[\psi(2p+1) - \psi(p+1)] P_p(\coth\eta) + (-1)^p \sinh^p \eta \sum_{k=0}^{p-1} \frac{(-1)^k (2k+1)}{(p-k)(p+k+1)} P_k(\coth\eta).$$

COROLLARY 2. *Let* $1 \leqslant n \leqslant p-1,\ p \geqslant 2,\ \eta \in (0,\infty)$. *Then*

$$\sum_{k=-p}^{n-1} \frac{(-1)^{k+1} e^{k\eta} R_p^k(\cosh\eta)}{n-k} + e^{2n\eta} \sum_{k=-p}^{-n-1} \frac{(-1)^k e^{k\eta} R_p^k(\cosh\eta)}{n+k}$$

$$= p! e^{n\eta} \sinh^p \eta \left\{ \frac{(-1)^n}{(p+n)!} [2\psi(2p+1) - \psi(p+1+n) - \psi(p+1-n)] P_p^n(\coth\eta) \right.$$

$$+ \frac{(-1)^p}{(p+n)!} \sum_{k=0}^{p-n-1} \frac{(-1)^k (2n+2k+1) \left[1 + \frac{k!(p+n)!}{(2n+k)!(p-n)!}\right]}{(p-n-k)(p+n+k+1)} P_{n+k}^n(\coth\eta)$$

$$+\frac{(-1)^{p+n}}{(p-n)!}\sum_{k=0}^{n-1}\frac{(-1)^k(2k+1)}{(p-k)(p+k+1)}P_k^{-n}(\coth\eta)\Bigg\}.$$

COROLLARY 3. *Let* $n=p,\ p\geqslant 1,\ \eta\in(0,\infty)$. *Then*

$$\sum_{k=-p}^{p-1}\frac{(-1)^{k+1}e^{k\eta}R_p^k(\cosh\eta)}{p-k}=p!e^{p\eta}\sinh^p\eta\Bigg\{\frac{(-1)^p}{(2p)!}[\psi(2p+1)-\psi(1)]P_p^p(\coth\eta)$$
$$+\sum_{k=0}^{p-1}\frac{(-1)^k(2k+1)}{(p-k)(p+k+1)}P_k^{-p}(\coth\eta)\Bigg\}.$$

COROLLARY 4. *Let* $n\geqslant p+1,\ p\geqslant 0,\ \eta\in(0,\infty)$. *Then*

$$\sum_{k=-p}^{p}\frac{(-1)^{k+1}e^{k\eta}R_p^k(\cosh\eta)}{n-k}=(-1)^{p+1}p!(n-p-1)!e^{n\eta}\sinh^p\eta P_p^{-n}(\coth\eta).\quad(32)$$

Note that these corollaries may all be written in a form where the sums are over non-negative indices k as in Theorem 2 using (37). We leave this exercise to the reader.

We now have closed-form expressions for the finite terms involving logarithmic polynomials in (20), fully in terms of the associated Legendre function of the first kind. We also have a proof of the correspondence for the function $\mathfrak{R}_{n,q}$ discussed in Section 3. Through (32), the function $\mathfrak{R}_{n,p}$ (cf. (16), (17), (18), (19), and (22)) is directly related to the associated Legendre function of the first kind, namely

$$\mathfrak{R}_{n,p}(\cosh\eta)=2(-1)^{p+1}p!(p+n)!e^{n\eta}\sinh^p\eta P_p^{-n}(\coth\eta).$$

Therefore through (11) we have

$$\frac{1}{(\cosh\eta-\cos\psi)^q}=\frac{(-1)^q}{2[(q-1)!]^2\sinh^{2q-1}\eta}\sum_{n=0}^{\infty}\varepsilon_n\cos(n\psi)e^{-n\eta}\mathfrak{R}_{n,q-1}(\cosh\eta).$$

This demonstrates the correspondences which were mentioned near (13), (17), and (19) for (22) and (26).

6. Fourier expansion for a logarithmic fundamental solution of the polyharmonic equation

Now that we have computed the Fourier series for g_χ, namely (22) (cf. (20)) and (26), we are in a position to compute the azimuthal Fourier series for a logarithmic fundamental solution of the polyharmonic equation (7). For instance, using (8), (24) and (26), we have the following corollary.

COROLLARY 5. *Let $\mathbf{x},\mathbf{x}' \in \mathbf{R}^d$, $d \geqslant 2$ even, with points parametrized using a rotationally-invariant coordinate system (5). Then the azimuthal Fourier cosine series of a logarithmic fundamental solution of the polyharmonic equation (8) may be given by*

$$\begin{aligned}
\mathfrak{l}_k^d(\mathbf{x},\mathbf{x}') &= \frac{1}{2}\left(2RR'\right)^p\left[\log\left(RR'\right)+\eta-\beta_{p,d}\right](\chi^2-1)^{p/2} \\
&\times\sum_{n=0}^{p}\varepsilon_n\cos[n(\phi-\phi')]\frac{(-p)_n(p-n)!}{(p+n)!}P_p^n\left(\frac{\chi}{\sqrt{\chi^2-1}}\right) \\
&+\frac{1}{2}(2RR')^p p!(\chi^2-1)^{p/2}\sum_{n=0}^{p}\frac{(-1)^n\varepsilon_n\cos[n(\phi-\phi')]}{(p+n)!} \\
&\times[2\psi(2p+1)-\psi(p+1+n)-\psi(p+1-n)]P_p^n\left(\frac{\chi}{\sqrt{\chi^2-1}}\right) \\
&+\frac{1}{2}(2RR')^p p!(\chi^2-1)^{p/2}\sum_{n=0}^{p-1}\frac{\varepsilon_n\cos[n(\phi-\phi')]}{(p+n)!} \\
&\times\sum_{k=0}^{p-n-1}\frac{(-1)^k(2n+2k+1)\left[1+\frac{k!(p+n)!}{(2n+k)!(p-n)!}\right]}{(p-n-k)(p+n+k+1)}P_{n+k}^n\left(\frac{\chi}{\sqrt{\chi^2-1}}\right) \\
&+(2RR')^p(-1)^p p!(\chi^2-1)^{p/2}\sum_{n=1}^{p}\frac{(-1)^n\cos[n(\phi-\phi')]}{(p-n)!} \\
&\times\sum_{k=0}^{n-1}\frac{(-1)^k(2k+1)}{(p-k)(p+k+1)}P_k^{-n}\left(\frac{\chi}{\sqrt{\chi^2-1}}\right) \\
&+(2RR')^p(-1)^{p+1}p!(\chi^2-1)^{p/2} \\
&\times\sum_{n=p+1}^{\infty}\cos[n(\phi-\phi')](n-p-1)!P_p^{-n}\left(\frac{\chi}{\sqrt{\chi^2-1}}\right).
\end{aligned}$$

Alternatively, using (8), (10), and (23) one has

COROLLARY 6. *Under the same conditions as Corollary 5, one alternatively has*

$$\begin{aligned}
\mathfrak{l}_k^d(\mathbf{x},\mathbf{x}') &= \frac{1}{2}\left(2RR'\right)^p\left[\log\left(RR'\right)+\eta-\beta_{p,d}\right](\chi^2-1)^{p/2} \\
&\times\sum_{n=0}^{p}\varepsilon_n\cos[n(\phi-\phi')]\frac{(-p)_n(p-n)!}{(p+n)!}P_p^n\left(\frac{\chi}{\sqrt{\chi^2-1}}\right) \\
&+(2RR')^p\sum_{n=0}^{p}\cos[n(\phi-\phi')]\left(\chi-\sqrt{\chi^2-1}\right)^n
\end{aligned}$$

$$\times \sum_{k=-p}^{n-1} \frac{(-1)^{k+1}\left(\chi+\sqrt{\chi^2-1}\right)^k R_p^k(\chi)}{n-k}$$

$$-(2RR')^p \sum_{n=1}^{p-1} \cos[n(\phi-\phi')] \left(\chi+\sqrt{\chi^2-1}\right)^n$$

$$\times \sum_{k=-p}^{-n-1} \frac{(-1)^{k+1}\left(\chi+\sqrt{\chi^2-1}\right)^k R_p^k(\chi)}{n+k}$$

$$+(2RR')^p \sum_{n=p+1}^{\infty} \cos[n(\phi-\phi')] \left(\chi-\sqrt{\chi^2-1}\right)^n$$

$$\times \sum_{k=-p}^{p} \frac{(-1)^{k+1}\left(\chi+\sqrt{\chi^2-1}\right)^k R_p^k(\chi)}{n-k}.$$

Using the above two corollaries, one can, for instance, obtain the axisymmetric component of a logarithmic fundamental solution of the polyharmonic equation, namely

$$\mathfrak{l}_k^d(\mathbf{x},\mathbf{x}')\Big|_{n=0} = \frac{1}{2}(2RR')^p(\chi^2-1)^{p/2}$$
$$\times \left[\log(RR')+\eta-\beta_{p,d}+2\psi(2p+1)-2\psi(p+1)\right] P_p\left(\frac{\chi}{\sqrt{\chi^2-1}}\right)$$
$$+(2RR')^p(\chi^2-1)^{p/2}\sum_{k=0}^{p-1}\frac{(-1)^k(2k+1)}{(p-k)(p+k+1)}P_k\left(\frac{\chi}{\sqrt{\chi^2-1}}\right),$$

or

$$\mathfrak{l}_k^d(\mathbf{x},\mathbf{x}')\Big|_{n=0} = \frac{1}{2}(2RR')^p(\chi^2-1)^{p/2}\left[\log(RR')+\eta-\beta_{p,d}\right] P_p\left(\frac{\chi}{\sqrt{\chi^2-1}}\right)$$
$$+(2RR')^p\sum_{k=1}^{p}\frac{(-1)^{k+1}(\chi-\sqrt{\chi^2-1})^k R_p^k(\chi)}{k}.$$

The above expressions for the axisymmetric component of a logarithmic fundamental solution of the polyharmonic equation is one type of expression sought after in Tsai, Chen & Hsu (2009) [19].

A. Derivatives with respect to the degree of certain integer-order associated Legendre functions of the first kind

The derivative with respect to its degree for the associated Legendre function of the first kind evaluated at degree zero, is given in Section 4.4.3 of Magnus, Oberhettinger

& Soni (1966) [10] as

$$\left[\frac{\partial}{\partial\nu}P_\nu(z)\right]_{\nu=0} = \frac{z-1}{2}\,{}_2F_1\left(1,1;2;\frac{1-z}{2}\right). \tag{33}$$

An important generalization of this formula has recently been derived in Szmytkowski (2011) [18]. The degree-derivative of the associated Legendre function of the first kind for $p,m\in\mathbf{N}_0$ and $0\leqslant m\leqslant p$ (cf. (5.12) in [18]) is given by

$$\begin{aligned}\left[\frac{\partial}{\partial\nu}P_\nu^m(z)\right]_{\nu=p} &= P_p^m(z)\log\frac{z+1}{2}\\ &+[2\psi(2p+1)-\psi(p+1)-\psi(p-m+1)]P_p^m(z)\\ &+(-1)^{p+m}\sum_{k=0}^{p-m-1}(-1)^k\frac{(2k+2m+1)\left[1+\frac{k!(p+m)!}{(k+2m)!(p-m)!}\right]}{(p-m-k)(p+m+k+1)}P_{k+m}^m(z)\\ &+(-1)^p\frac{(p+m)!}{(p-m)!}\sum_{k=0}^{m-1}(-1)^k\frac{2k+1}{(p-k)(p+k+1)}P_k^{-m}(z),\end{aligned} \tag{34}$$

and for $m\geqslant p+1$ (cf. (5.16) in [18]) there is

$$\left[\frac{\partial}{\partial\nu}P_\nu^m(z)\right]_{\nu=p} = (-1)^{p+m+1}(p+m)!(m-p-1)!P_p^{-m}(z). \tag{35}$$

Some special cases of (34) include for $m=0$

$$\begin{aligned}\left[\frac{\partial}{\partial\nu}P_\nu(z)\right]_{\nu=p} &= P_p(z)\log\frac{z+1}{2}+2\left[\psi(2p+1)-\psi(p+1)\right]P_p(z)\\ &+2(-1)^p\sum_{k=0}^{p-1}(-1)^k\frac{2k+1}{(p-k)(p+k+1)}P_k(z),\end{aligned}$$

and for $m=p$

$$\begin{aligned}\left[\frac{\partial}{\partial\nu}P_\nu^p(z)\right]_{\nu=p} &= P_p^p(z)\log\frac{z+1}{2}+[2\psi(2p+1)-\psi(p+1)+\gamma]P_p^p(z)\\ &+(-1)^p(2p)!\sum_{k=0}^{p-1}(-1)^k\frac{2k+1}{(p-k)(p+k+1)}P_k^{-p}(z),\end{aligned}$$

where $\gamma=-\psi(1)$ is Euler's constant (see (5.2.3) in [14]). Of course we also have for $m=p=0$

$$\left[\frac{\partial}{\partial\nu}P_\nu(z)\right]_{\nu=0} = \log\frac{z+1}{2},$$

which exactly matches (33).

B. The logarithmic polynomials

The logarithmic polynomials $R_p^k(x)$ are nonvanishing only for $-p \leqslant k \leqslant p$ and by construction, they satisfy the recurrence relation

$$R_p^k(x) = \frac{1}{2}R_{p-1}^{k-1}(x) + xR_{p-1}^k(x) + \frac{1}{2}R_{p-1}^{k+1}(x). \tag{36}$$

From (12) we have that $R_0^0(x) = 1$. This gives us the starting point for the recursion. It is evident by construction that these polynomials are even in the index k, i.e.,

$$R_p^k(x) = R_p^{-k}(x). \tag{37}$$

Some of the first few logarithmic polynomials are given by

$$\begin{gathered} R_0^0(x) = 1, \\ R_1^0(x) = x,\ R_1^{\pm 1}(x) = \tfrac{1}{2} \\ R_2^0(x) = \tfrac{1}{2} + x^2,\ R_2^{\pm 1}(x) = x,\ R_2^{\pm 2}(x) = \tfrac{1}{4}, \\ R_3^0(x) = \tfrac{3}{2}x + x^3,\ R_3^{\pm 1}(x) = \tfrac{3}{8} + \tfrac{3}{2}x^2,\ R_3^{\pm 2}(x) = \tfrac{3}{4}x,\ R_3^{\pm 3}(x) = \tfrac{1}{8}. \end{gathered}$$

We can find a generating function for the logarithmic polynomials as follows. Let

$$F(x,y,z) = \sum_{p=0}^{\infty}\sum_{k=-\infty}^{\infty} R_p^k(x) y^k z^p$$

be a generating function for the logarithmic polynomials $R_p^k(x)$. If we define the function

$$S_p(x,y) = \sum_{k=-\infty}^{\infty} R_p^k(x) y^k,$$

then using the recurrence relation for $R_p^k(x)$ (36) we can show

$$S_p(x,y) = \left(x + \frac{1}{2}\left(y + \frac{1}{y}\right)\right) S_{p-1}(x,y).$$

Combining this result along with the fact that $R_0^0(x) = 1$, we have

$$S_p(x,y) = \left(x + \frac{1}{2}\left(y + \frac{1}{y}\right)\right)^p.$$

We have therefore derived for the logarithmic polynomials $R_p^k(x)$, the bilateral generating functions

$$\left(x + \frac{1}{2}\left(y + \frac{1}{y}\right)\right)^p = \sum_{k=-\infty}^{\infty} R_p^k(x) y^k, \tag{38}$$

$$\left\{1 - z\left(x + \frac{1}{2}\left(y + \frac{1}{y}\right)\right)\right\}^{-1} = \sum_{p=0}^{\infty}\sum_{k=-\infty}^{\infty} R_p^k(x) y^k z^p.$$

PROPOSITION 1. *The derivative of the logarithmic polynomials $R_p^k(x)$ is given by*

$$\frac{d}{dx}R_p^{\pm k}(x) = pR_{p-1}^{\pm k}(x).$$

Proof. Differentiating both sides of (38) with respect to x and comparing with the original bilateral series completes the proof. □

COROLLARY 7. *The logarithmic polynomials $R_p^k(x)$ are in the Appell sequence.*

Proof. By Proposition 1 and [15, Theorem 2.5.6]. □

An algorithm for generating the logarithmic polynomials can be obtained by solving the set of difference equations

$$\left.\begin{aligned} a_0(p) &= \tfrac{1}{2}a_0(p-1) \\ a_1(p) &= \tfrac{1}{2}a_1(p-1) + xa_0(p-1) \\ a_2(p) &= \tfrac{1}{2}a_2(p-1) + xa_1(p-1) + \tfrac{1}{2}a_0(p-1) \\ &\vdots \\ a_n(p) &= \tfrac{1}{2}a_n(p-1) + xa_{n-1}(p-1) + \tfrac{1}{2}a_{n-2}(p-1) \end{aligned}\right\}, \tag{39}$$

subject to the boundary conditions

$$\left.\begin{aligned} a_0(0) &= 1 \\ a_1(1) &= xa_0(0) \\ a_2(2) &= xa_1(1) + a_0(1) \\ &\vdots \\ a_n(n) &= xa_{n-1}(n-1) + a_{n-2}(n-1) \end{aligned}\right\}, \tag{40}$$

where $R_p^k = a_{p-|k|}(p)$ is given along diagonals for a fixed $p-|k|$. For instance, one can obtain

$$R_p^{\pm p}(x) = \frac{1}{2^p},$$

$$R_p^{\pm(p-1)}(x) = \frac{p}{2^{p-1}}x,$$

$$R_p^{\pm(p-2)}(x) = \frac{p}{2^p} + \frac{p(p-1)}{2^{p-1}}x^2,$$

$$R_p^{\pm(p-3)}(x) = \frac{p(p-1)}{2^{p-1}}x + \frac{p(p-1)(p-2)}{3\cdot 2^{p-2}}x^3,$$

$$R_p^{\pm(p-4)}(x) = \frac{p(p-1)}{2^{p+1}} + \frac{p(p-1)(p-2)}{2^{p-1}}x^2 + \frac{p(p-1)(p-2)(p-3)}{3\cdot 2^{p-1}}x^4,$$

where $p \geqslant 0, 1, 2, 3, 4$ respectively.

Acknowledgements. I would like to thank A.F.M. Tom ter Elst, Matthew Auger, and Radosław Szmytkowski for valuable discussions. I would also like to thank Shenghui Yang previously at Wolfram Research for valuable assistance in generating an algorithm to symbolically compute the logarithmic polynomials. I would very much like to acknowledge two anonymous referees whose comments improved this paper. In particular, one of these referees suggested a proof of Proposition 1, and for this I am grateful. Part of this work was conducted while H. S. Cohl was a National Research Council Research Postdoctoral Associate in the Applied and Computational Mathematics Division at the National Institute of Standards and Technology, Gaithersburg, Maryland, U.S.A.

REFERENCES

[1] M. ABRAMOWITZ, I. A. STEGUN, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Vol. 55 of National Bureau of Standards Applied Mathematics Series, U.S. Government Printing Office, Washington, D.C., (1972).

[2] J. B. BOYLING, *Green's functions for polynomials in the Laplacian*, Zeitschrift für Angewandte Mathematik und Physik, Vol. 47, pp. 485–492, (1996).

[3] H. S. COHL, D. E. DOMINICI, *Generalized Heine's identity for complex Fourier series of binomials*, Proceedings of the Royal Society A, Vol. 467, pp. 333–345, (2011).

[4] H. S. COHL, J. E. TOHLINE, *A Compact Cylindrical Green's Function Expansion for the Solution of Potential Problems*, The Astrophysical Journal, Vol. 527, pp. 86–101, (1999).

[5] I. M. GEL'FAND, G. E. SHILOV, *Generalized functions*, Vol. 1, Academic Press [Harcourt Brace Jovanovich Publishers], New York, Properties and operations, Translated from the Russian by Eugene Saletan, (1964 [1977]).

[6] B. T. HELENBROOK, *Mesh deformation using the biharmonic operator*, International Journal for Numerical Methods in Engineering, Vol. 56, pp. 1007–1021, (2003).

[7] B. J. KIRBY, *Micro- and Nanoscale Fluid Mechanics: Transport in Microfluidic Devices*, Cambridge University Press, Cambridge, (2010).

[8] W. M. LAI, D. RUBIN, E. KREMPL, *Introduction to continuum mechanics, revised edition, SI/metric units*, Pergamon, Oxford, UK, (1978).

[9] S. A. LURIE, V. V. VASILIEV, *The biharmonic problem in the theory of elasticity*, Gordon and Breach Publishers, Luxembourg, (1995).

[10] W. MAGNUS, F. OBERHETTINGER, R. P. SONI, *Formulas and theorems for the special functions of mathematical physics*, Third enlarged edition, Die Grundlehren der mathematischen Wissenschaften, Band 52, Springer-Verlag New York, Inc., New York, (1966).

[11] W. MILLER, JR., *Symmetry and separation of variables*, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, With a foreword by Richard Askey, Encyclopedia of Mathematics and its Applications, Vol. 4, (1977).

[12] J. MONTERDE, H. UGAIL, *On harmonic and biharmonic Bézier surfaces*, Computer Aided Geometric Design, Vol. 21, pp. 697–715, (2004).

[13] M. NICOLESCU, *Les Fonctions Polyharmoniques*, Hermann, Paris, (1936).

[14] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT, C. W. CLARK (Eds.), *NIST handbook of mathematical functions*, Cambridge University Press, Cambridge, (2010).

[15] S. ROMAN, *The umbral calculus*, Vol. 111 of Pure and Applied Mathematics, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, (1984).

[16] L. SCHWARTZ, *Théorie des distributions. Tome I*, Actualités Sci. Ind., no. 1091 = Publ. Inst. Math. Univ. Strasbourg 9, Hermann, Paris, (1950).

[17] S. L. SOBOLEV, *Cubature formulas and modern analysis: an introduction*, Oxonian, New Delhi, (1992).

[18] R. SZMYTKOWSKI, *On the derivative of the associated Legendre function of the first kind of integer order with respect to its degree (with applications to the construction of the associated Legendre*

function of the second kind of integer degree and order), Journal of Mathematical Chemistry, Vol. 49, pp. 1436–1477, (2011).
[19] C. C. TSAI, C. S. CHEN, T. W. HSU, *The method of particular solutions for solving axisymmetric polyharmonic and poly-Helmholtz equations*, Engineering Analysis with Boundary Elements, Vol. 33, pp. 1396–1402, (2009).
[20] H. UGAIL, *Partial Differential Equations for Geometric Design*, Springer-Verlag, London, (2011).
[21] D. VAUTHERIN, *Hartree-Fock Calculations with Skyrme's Interaction. II. Axially Deformed Nuclei*, Physical Review C, Vol. 7, pp. 296–316, (1973).

(Received September 28, 2012)

Howard S. Cohl
Applied and Computational Mathematics Division
National Institute of Standards and Technology
Gaithersburg, Maryland, U.S.A.
e-mail: howard.cohl@nist.gov