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MULTIVALENT FUNCTIONS WITH VARYING ARGUMENTS

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Abstract. Silverman [4] was defined the class of univalent functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ for which $arg(a_k)$ prescribed in such way that $f(z)$ is univalent if and only if $f(z)$ is starlike. In this paper we introduce the subclass of p-valent functions with varying arguments, especially p-valent starlike functions and $p-$ valent convex functions, moreover we give some interesting properties of functions in these classes, including coefficients estimates, distortion theorems and extreme functions.

1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1,2,3,\ldots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in A(p)$ is said to be p-valent starlike of order α if it satisfies the inequality:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leqslant \alpha < p;\ p \in \mathbb{N};\ z \in U), \tag{1.2}$$

or, equivalently,

$$\left|\frac{\frac{zf'(z)}{f(z)} - p}{\frac{zf'(z)}{f(z)} + p - 2\alpha}\right| < 1, \tag{1.3}$$

we denote by $S_p(\alpha)$ the class of all p-valent starlike functions of order α. Also a function $f(z) \in A(p)$ is said to be p-valent convex of order α if it satisfies the inequality:

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (0 \leqslant \alpha < p;\ p \in \mathbb{N};\ z \in U), \tag{1.4}$$

or, equivalently,

$$\left|\frac{1 + \frac{zf''(z)}{f'(z)} - p}{1 + \frac{zf''(z)}{f'(z)} + p - 2\alpha}\right| < 1, \tag{1.5}$$

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we denote by $K_p(\alpha)$ the class of all p-valently convex functions of order α, the classes $S_p(\alpha)$ and $K_p(\alpha)$ were introduced by Patil and Thakare [3] and Owa [2] studied these classes with negative coefficients, further from (1.2) and (1.3), we can see that

$$f(z) \in K_p(\alpha) \Longleftrightarrow \frac{zf'(z)}{p} \in S_p(\alpha) \quad (0 \leqslant \alpha < p;\ p \in \mathbb{N}). \tag{1.6}$$

Silverman [4] defined the class of univalent functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ for which $arg(a_k)$ prescribed in such way that $f(z)$ is univalent if and only if $f(z)$ is starlike. In this paper we introduce the subclass of p-valent functions with varying arguments as follows:

DEFINITION 1. A function $f(z)$ of the form (1.1) is said to be in the class $V_p(\theta_k)$ if $f \in A(p)$ and $arg(a_k) = \theta_k$ for all $k \geqslant p+1$. If furthermore there exist a real number δ such that $\theta_k + (k-p)\delta \equiv \pi \ (mod\ \ 2\pi)$ for all $k \geqslant p+1$, then $f(z)$ is said to be in the class $V_p(\theta_k, \delta)$. The union of $V_p(\theta_k, \delta)$ taken over all possible sequences $\{\theta_k\}$ and all possible real numbers δ is denoted by V_p.

Note that:

(i) $V_p(\theta_k + 2k\pi) = V_p(\theta_k)$, k is an integer;

(ii) $V_p(\pi, 0) = T_p = \left\{ f \in A(p) : f(z) = z^p - \sum_{k=p+1}^{\infty} |a_k| z^k \ (p \in \mathbb{N}) \right\}$, the set of p-valent functions with negative coefficients.

Let $V_p(\alpha)$ denote the subclass of V_p consisting of functions $f(z) \in S_p(\alpha)$, also let $\overline{V_p}(\alpha)$ denote the subclass of V_p consisting of functions $f(z) \in K_p(\alpha)$, which are the classes of p-valent starlike functions with varying arguments and p-valent convex functions with varying arguments, respectively.

We note that $V_1(\alpha) = V^*(\alpha)$, which was introduced and studied by Silverman [4].

In this paper we obtain coefficient bounds for functions in the classes $V_p(\alpha)$ and $\overline{V_p}(\alpha)$, further we obtain distortion bounds and the extreme points for functions in these classes.

2. Coefficient estimates

Unless otherwise mentioned, we assume in the reminder of this paper that $0 \leqslant \alpha < p$, $p \in \mathbb{N}$ and $z \in U$.

We shall need the following lemmas given by Owa [2] and Aouf [1, with $A = -1$ and $B = 1$]:

LEMMA 1. *The sufficient condition for* $f(z)$ *given by* (1.1) *to be in the class* $S_p(\alpha)$ *is that*

$$\sum_{k=p+1}^{\infty} (k-\alpha)\,|a_k| \leqslant p - \alpha. \tag{2.1}$$

LEMMA 2. *The sufficient condition for $f(z)$ given by (1.1) to be in the class $K_p(\alpha)$ is that*

$$\sum_{k=p+1}^{\infty} \left(\frac{k}{p}\right)(k-\alpha)|a_k| \leqslant (p-\alpha). \tag{2.2}$$

THEOREM. ***1.** Let $f(z)$ of the form (1.1), then $f(z) \in V_p(\alpha)$ if and only if*

$$\sum_{k=p+1}^{\infty} (k-\alpha)|a_k| \leqslant p-\alpha. \tag{2.3}$$

Proof. In view of Lemma 1, we need only to show that function $f(z) \in V_p(\alpha)$ satisfies the coefficient inequality (2.3). If $f(z) \in V_p(\alpha)$, then from (1.3), we have

$$\left| \frac{\frac{zf'(z)}{f(z)} - p}{\frac{zf'(z)}{f(z)} + p - 2\alpha} \right| < 1,$$

thus we have

$$\left| \frac{\sum_{k=p+1}^{\infty} (k-p)a_k z^{k-p}}{2(p-\alpha) + \sum_{k=p+1}^{\infty} (k+p-2\alpha)a_k z^{k-p}} \right| < 1. \tag{2.4}$$

Since $f(z) \in V_p$, $f(z)$ lies in the class $V_p(\theta_k, \delta)$ for some sequence $\{\theta_k\}$ and a real number δ such that

$$\theta_k + (k-p)\delta \equiv \pi (mod 2\pi) \qquad (k \geqslant p+1), \tag{2.5}$$

also $z = re^{i\delta}$, we obtain the following

$$\left| \frac{\sum_{k=p+1}^{\infty} (k-p)|a_k| r^{k-p}}{2(p-\alpha) - \sum_{k=p+1}^{\infty} (k+p-2\alpha)|a_k| r^{k-p}} \right| < 1. \tag{2.6}$$

Since Re $\{w(z)\} < |w(z)|$, we have

$$\mathrm{Re}\left\{ \frac{\sum_{k=p+1}^{\infty} (k-p)|a_k| r^{k-p}}{2(p-\alpha) - \sum_{k=p+1}^{\infty} (k+p-2\alpha)|a_k| r^{k-p}} \right\} < 1. \tag{2.7}$$

Letting $r \longrightarrow 1$, then we obtain that

$$\sum_{k=p+1}^{\infty} (k-\alpha)|a_k| \leqslant p-\alpha.$$

Hence, the proof of Theorem 1 is completed. □

COROLLARY 1. *If* $f(z) \in V_p(\alpha)$, *then*

$$|a_k| \leqslant \frac{p-\alpha}{k-\alpha} \quad (k \geqslant p+1). \tag{2.8}$$

The result is sharp for the function $f(z)$ *given by*

$$f(z) = z^p + \frac{p-\alpha}{k-\alpha} e^{i\theta_k} z^k \quad (k \geqslant p+1). \tag{2.9}$$

Using the same technique as used in Theorem 1, we have the following theorem:

THEOREM 2. *Let* $f(z)$ *of the form (1.1), then* $f(z) \in \overline{V_p}(\alpha)$ *if and only if*

$$\sum_{k=p+1}^{\infty} \left(\frac{k}{p}\right)(k-\alpha)\,|a_k| \leqslant (p-\alpha). \tag{2.10}$$

COROLLARY 2. *If* $f(z) \in \overline{V_p}(\alpha)$, *then*

$$|a_k| \leqslant \frac{p\,(p-\alpha)}{k\,(k-\alpha)} \quad (k \geqslant p+1). \tag{2.11}$$

The result is sharp for the function $f(z)$ *given by*

$$f(z) = z^p + \frac{p\,(p-\alpha)}{k\,(k-\alpha)} e^{i\theta_k} z^k \quad (k \geqslant p+1). \tag{2.12}$$

3. Distortion theorems

THEOREM 3. *Let the function $f(z)$ defined by (1.1) be in the class $V_p(\alpha)$. Then*

$$|z|^p - \frac{p-\alpha}{p+1-\alpha}|z|^{p+1} \leqslant |f(z)| \leqslant |z|^p + \frac{p-\alpha}{p+1-\alpha}|z|^{p+1}. \tag{3.1}$$

The result is sharp.

Proof. We employ the same technique as used by Silverman [4]. In view of Theorem 1, since

$$\Phi(k) = k - \alpha, \tag{3.2}$$

is an increasing function of k $(k \geqslant p+1)$, in view of Lemma 1, we have

$$(p+1-\alpha)\sum_{k=p+1}^{\infty}|a_k| \leqslant \sum_{k=p+1}^{\infty}(k-\alpha)|a_k| \leqslant p-\alpha,$$

that is

$$\sum_{k=p+1}^{\infty}|a_k| \leqslant \frac{p-\alpha}{p+1-\alpha}. \tag{3.3}$$

Thus we have

$$|f(z)| \leqslant |z|^p + |z|^{p+1}\sum_{k=p+1}^{\infty}|a_k|,$$

$$|f(z)| \leqslant |z|^p + \frac{p-\alpha}{p+1-\alpha}|z|^{p+1}.$$

Similarly, we get

$$|f(z)| \geqslant |z|^p - |z|^{p+1}\sum_{k=p+1}^{\infty}|a_k|,$$

$$|f(z)| \geqslant |z|^p - \frac{p-\alpha}{p+1-\alpha}|z|^{p+1}.$$

Finally the result is sharp for the function

$$f(z) = z^p + \frac{p-\alpha}{p+1-\alpha}e^{i\theta_{p+1}}z^{p+1}, \tag{3.4}$$

at $z = \pm|z|\,e^{-i\theta_{p+1}}$.

This completes the proof of Theorem 3. □

COROLLARY 3. *Under the hypotheses of Theorem 3, $f(z)$ is included in a disc with center at the origin and radius r_1 given by*

$$r_1 = 1 + \frac{p-\alpha}{p+1-\alpha}. \tag{3.5}$$

THEOREM 4. *Let the function $f(z)$ defined by (1.1) be in the class $V_p(\alpha)$. Then*

$$\left\{p-\frac{(p+1)(p-\alpha)}{p+1-\alpha}|z|\right\}|z|^{p-1} \leqslant |f'(z)| \leqslant \left\{p+\frac{(p+1)(p-\alpha)}{p+1-\alpha}|z|\right\}|z|^{p-1}. \quad (3.6)$$

The result is sharp.

Proof. Similarly for $\Phi(k)$ defined by (3.2) it is clear that $\frac{\Phi(k)}{k}$ is an increasing function of $k(k \geqslant p+1; p \in \mathbb{N})$, in view of Theorem 1, we have

$$\frac{\Phi(p+1)}{p+1}\sum_{k=p+1}^{\infty} k|a_k| \leqslant \sum_{k=p+1}^{\infty}\frac{\Phi(k)}{k}k|a_k| = \sum_{k=p+1}^{\infty}\Phi(k)|a_k| \leqslant p-\alpha,$$

that is

$$\sum_{k=p+1}^{\infty} k|a_k| \leqslant \frac{(p+1)(p-\alpha)}{p+1-\alpha}.$$

Thus we have

$$\begin{aligned}|f'(z)| &\leqslant p|z|^{p-1}+|z|^p\sum_{k=p+1}^{\infty}k|a_k| \leqslant \left\{p+|z|\sum_{k=p+1}^{\infty}k|a_k|\right\}|z|^{p-1}\\ &\leqslant \left\{p+\frac{(p+1)(p-\alpha)}{p+1-\alpha}|z|\right\}|z|^{p-1}.\end{aligned}$$

Similarly

$$\begin{aligned}|f'(z)| &\geqslant p|z|^{p-1}-|z|^p\sum_{k=p+1}^{\infty}k|a_k| \geqslant \left\{p-|z|\sum_{k=p+1}^{\infty}k|a_k|\right\}|z|^{p-1}\\ &\leqslant \left\{p-\frac{(p+1)(p-\alpha)}{p+1-\alpha}|z|\right\}|z|^{p-1}.\end{aligned}$$

Finally, we can see that the assertions of Theorem 4 are sharp for the function $f(z)$ defined by (3.4). This completes the proof of Theorem 4. □

COROLLARY 4. *Under the hypotheses of Theorem 4, $f'(z)$ is included in a disc with center at the origin and radius r_2 given by*

$$r_2 = p+\frac{(p+1)(p-\alpha)}{p+1-\alpha}. \quad (3.7)$$

Using the same technique as used in Theorem 3 and Theorem 4 , in view of Lemma 2, we have the following theorems for functions in the class $\overline{V_p}(\alpha)$:

THEOREM 5. *Let the function $f(z)$ defined by (1.1) be in the class $\overline{V_p}(\alpha)$. Then*

$$|z|^p-\frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}|z|^{p+1} \leqslant |f(z)| \leqslant |z|^p+\frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}|z|^{p+1}. \quad (3.8)$$

The result is sharp for the function

$$f(z)=z^p+\frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}e^{i\theta_{p+1}}z^{p+1}, \tag{3.9}$$

at $z=\pm|z|\,e^{-i\theta_{p+1}}$.

COROLLARY 5. *Under the hypotheses of Theorem 5,* $f(z)$ *is included in a disc with center at the origin and radius* r_3 *given by*

$$r_3=1+\frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}. \tag{3.10}$$

THEOREM 6. *Let the function* $f(z)$ *defined by (1.1) be in the class* $\overline{V_p}(\alpha)$. *Then*

$$\left\{p-\frac{p(p-\alpha)}{p+1-\alpha}|z|\right\}|z|^{p-1}\leqslant\left|f'(z)\right|\leqslant\left\{p+\frac{p(p-\alpha)}{p+1-\alpha}|z|\right\}|z|^{p-1}. \tag{3.11}$$

The result is sharp for the function given by (3.9).

COROLLARY 6. *Under the hypotheses of Theorem 6,* $f'(z)$ *is included in a disc with center at the origin and radius* r_4 *given by*

$$r_4=p+\frac{p(p-\alpha)}{p+1-\alpha}. \tag{3.12}$$

4. Extreme points

THEOREM 7. *Let the function* $f(z)$ *defined by (1.1) be in the class* $V_p(\alpha)$, *with* $arg(a_k)=\theta_k$, *where* $\theta_k+(k-p)\delta\equiv\pi \pmod{2\pi}\,(k\geqslant p+1)$. *Define*

$$f_p(z)=z^p$$

and

$$f_k(z)=z^p+\frac{p-\alpha}{k-\alpha}e^{i\theta_k}z^k \quad (k\geqslant p+1). \tag{4.1}$$

Then $f(z)\in V_p(\alpha)$ *if and only if* $f(z)$ *can expressed in the form* $f(z)=\sum_{k=p}^{\infty}\mu_k f_k(z)$, *where* $\mu_k\geqslant 0$ *and* $\sum_{k=p}^{\infty}\mu_k=1$.

Proof. If $f(z)=\sum_{k=p}^{\infty}\mu_k f_k(z)$ with $\mu_k\geqslant 0$ and $\sum_{k=p}^{\infty}\mu_k=1$, then

$$\sum_{k=p+1}^{\infty}(k-\alpha)\left(\frac{p-\alpha}{k-\alpha}\right)\mu_k=\sum_{k=p+1}^{\infty}(p-\alpha)\mu_k=(p-\alpha)(1-\mu_p)\leqslant(p-\alpha).$$

Hence $f(z) \in V_p(\alpha)$.

Conversely, let the function $f(z)$ defined by (1.1) be in the class $V_p(\alpha)$, define

$$\mu_k = \frac{k-\alpha}{p-\alpha}|a_k| \qquad (k \geqslant p+1) \tag{4.2}$$

and

$$\mu_p = 1 - \sum_{k=p+1}^{\infty} \mu_k.$$

From Theorem 1, $\sum_{k=p+1}^{\infty} \mu_k \leqslant 1$ and so $\mu_p \geqslant 0$. Since $\mu_k f_k(z) = \mu_k z^p + a_k z^k$, then

$$\sum_{k=p}^{\infty} \mu_k f_k(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k = f(z).$$

This completes the proof of Theorem 7. □

Similarly we can get the following theorem.

THEOREM 8. *Let the function $f(z)$ defined by (1.1) be in the class $\overline{V_p}(\alpha)$, with* arg $a_k = \theta_k$, *where* $\theta_k + (k-p)\delta \equiv \pi \ (mod\ 2\pi)$ $(k \geqslant p+1)$. *Define*

$$f_p(z) = z^p$$

and

$$f_k(z) = z^p + \frac{p(p-\alpha)}{k(k-\alpha)} e^{i\theta_k} z^k \quad (k \geqslant p+1). \tag{4.3}$$

Then $f(z) \in \overline{V_p}(\alpha)$ if and only if $f(z)$ can expressed in the form $f(z) = \sum_{k=p}^{\infty} \mu_k f_k(z)$, where $\mu_k \geqslant 0$ and $\sum_{k=p}^{\infty} \mu_k = 1$.

REMARK. Putting $p = 1$ in all the above results, we obtain the corresponding results obtained by Silverman [4].

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