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AN INEQUALITY FOR MATRIX OPERATORS AND ITS APPLICATIONS

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Abstract. In this paper, we prove a simple inequality which plays important role in the summability theory, matrix operators theory, approximation theory, and also provides great convenience in computations. As a corollary, we give the well known results of $[1,2,5]$ under some simpler conditions, and a very short and different.proofs of results in $[6,7]$.

1. Introduction

Let $A=(a_{nv})$ be an infinite matrix of complex numbers and X,Y be two non-empty subsets of the space s of all sequences with complex numbers. We say that the matrix A is a transformation from X into Y, if $A(x)=(A_n(x))$ exists for every sequence $x=(x_v)\in X$ and is in Y, where

$$A_n(x)=\sum_{v=0}^{\infty} a_{nv}x_v, \ \ n=0,1,\dots .$$

By (X,Y) we denote the class of all such matrices. The following sequence spaces are very important in summability theory and matrix operators; we write ℓ_∞, c, c_0 and bv for the set of all bounded, convergent, null and bounded variation sequences, and

$$\begin{aligned}
\ell_k &= \left\{x=(x_v)\in s : \sum_{v=0}^{\infty}|x_v|^k<\infty,\ k>0\right\},\\
bv^k &= \left\{x=(x_v)\in s : \sum_{v=0}^{\infty}|x_v-x_{v-1}|^k<\infty,\ x_0=0,\ k\geqslant 1\right\},\\
bv_0 &= \{x=(x_v)\in bv : x\in c_0\},\\
m_s &= \left\{x=(x_v)\in s : \left(\sum_{v=0}^{n}x_v\right)\in \ell_\infty\right\}\\
c_s &= \left\{x=(x_v)\in s : \left(\sum_{v=0}^{n}x_v\right)\in c\right\},\\
(c_0)_s &= \left\{x=(x_v)\in s : \left(\sum_{v=0}^{n}x_v\right)\in c_0\right\},
\end{aligned}$$

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Throughout the paper N and k^* will denote two finite subsets of all nonnegative integers N_0 and the cojugate index of k for $k>1$; i.e., $\frac{1}{k^*}+\frac{1}{k}=1$, respectively, also

$$U_p[A]=\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}|a_{nv}|\right)^{p_v} \tag{1.1}$$

and

$$L_p[A]=\sup_N\sum_{v=0}^{\infty}\left|\sum_{n\in N}a_{nv}\right|^{p_v}. \tag{1.2}$$

The following results on matrix operators are well known (see [1, 2, 5]), which contain the most important ancestor results in the field.

THEOREM 1.1. *$A\in(X,\ell)$ if and only if $L_1[A]<\infty$, where $X=\ell_\infty,c,c_0$.*

THEOREM 1.2. *Let $k>1$. Then, $A\in(X,\ell_k)$ if and only if*

$$L_k\left[A^T\right]<\infty,$$

where $X=\ell_\infty,c,c_0$.

THEOREM 1.3. *Let $k>1$. Then, $A\in(X,\ell_k)$ if and only if*

$$L_k\left[(a_{nv}-a_{n-1,v})^T\right]<\infty,$$

where $X=c_s,(c_0)_s$.

THEOREM 1.4. *Let $k>1$. Then, $A\in((\ell_\infty)_s,\ell_k)$ if and only if $\lim_v a_{nv}=0$ for $n=0,1,\ldots$ and*

$$L_k\left[(a_{nv}-a_{n-1,v})^T\right]<\infty.$$

THEOREM 1.5. *Let $k>1$. Then, $A\in(\ell_k,\ell)$ if and only if*

$$L_{k^*}[A]<\infty.$$

THEOREM 1.6. *Let $k>1$. Then, $A\in(\ell_k,b_v)$ if and only if*

$$L_{k^*}[(a_{nv}-a_{n-1,v})]<\infty.$$

THEOREM 1.7. *Let $k>1$. Then, $A\in\left(\ell_k,b_{v_0}\right)$ if and only if $\lim_n a_{nv}=0$ for $v=0,1,\ldots$ and*

$$L_{k^*}[(a_{nv}-a_{n-1,v})]<\infty$$

THEOREM 1.8. *Let* $k>1$. *Then,* $A\in\left(bv^{k},\ell\right)$ *if and only if*

$$L_{k^*}\left[\left(\sum_{j=v}^{\infty}a_{nj}\right)\right]<\infty$$

and

$$\sup_{v}\left(v^{1/k^*}\left|\sum_{j=v}^{\infty}a_{nj}\right|\right)<\infty \text{ for all } n. \tag{1.3}$$

THEOREM 1.9. *Let* $k>1$. *Then,* $A\in\left(bv^{k},bv\right)$ *if and only if condition (1.3) is satisfied and*

$$L_{k^*}\left[\left(\sum_{j=v}^{\infty}\left(a_{nj}-a_{n-1,j}\right)\right)\right]<\infty.$$

2. The main results

In this paper, we prove a simple inequality which plays important role in the summability theory, matrix operators theory, approximation theory, and also provides great convenience in computations. As a corollary, we give the above theorems under some simpler conditions, and a short and different proofs of results in $[6,7]$.

THEOREM 2.1. *Suppose that* $A=(a_{nv})$ *is an infinite matrix with complex numbers and* (p_v) *is a bounded sequence of positive numbers. If*

$$U_p[A]<\infty \tag{2.1}$$

or

$$L_p[A]<\infty \tag{2.2}$$

then

$$\frac{1}{4C^2}U_p[A]\leqslant L_p[A]\leqslant U_p[A],$$

where $C=max\left\{1,2^{H-1}\right\}$ *and* $H=\sup_v p_v$.

Proof. We first note that

$$|a_v+b_v|^{p_v}\leqslant C\left(|a_v|^{p_v}+|b_v|^{p_v}\right),$$

[3]. Now, if $U_p[A]<\infty$, then it is clear that the right side of the inequality is valid since

$$\left|\sum_{n\in N}a_{nv}\right|\leqslant\sum_{n=0}^{\infty}|a_{nv}|$$

for all finite subset $N \subset N_0$. For the converse, assume first that a_{nv} $(n, v = 0, 1, ...)$ are real numbers and $L_p[A] < \infty$.Then,

$$\sum_{v=0}^{\infty} \left| \sum_{n \in N} a_{nv} \right|^{p_v} \leqslant L_p[A]$$

for any finite subset $N \subset N_0$. Now write $N_+ = \{n, v \in N : a_{nv} \geqslant 0\}$ and $N_- = \{n, v \in N : a_{nv} < 0\}$. Then, we have

$$\begin{aligned}
\sum_{v=0}^{\infty} \left(\sum_{n \in N} |a_{nv}| \right)^{p_v} &= \sum_{v=0}^{\infty} \left(\sum_{n \in N_+} a_{nv} + \sum_{n \in N_-} (-a_{nv}) \right)^{p_v} \\
&\leqslant C \left[\sum_{v=0}^{\infty} \left(\sum_{n \in N_+} a_{nv} \right)^{p_v} + \sum_{v=0}^{\infty} \left(\sum_{n \in N_-} (-a_{nv}) \right)^{p_v} \right] \\
&\leqslant 2CL_p[A].
\end{aligned}$$

So, if a_{nv} is complex number, $a_{nv} = a_{nv}^{(1)} + i a_{nv}^{(2)}$ say, then, it follows that

$$\begin{aligned}
\sum_{v=0}^{\infty} \left(\sum_{n \in N}^{\infty} |a_{nv}| \right)^{p_v} &\leqslant C \left[\sum_{v=0}^{\infty} \left(\sum_{n \in N}^{\infty} \left| a_{nv}^{(1)} \right| \right)^{p_v} + \sum_{v=0}^{\infty} \left(\sum_{n \in N}^{\infty} \left| a_{nv}^{(2)} \right| \right)^{p_v} \right] \\
&\leqslant 4C^2 L_p[A]
\end{aligned}$$

since

$$\sum_{v=0}^{\infty} \left| \sum_{n \in N}^{\infty} a_{nv}^{(1)} \right|^{p_v} \leqslant L_p[A] \text{ and } \sum_{v=0}^{\infty} \left| \sum_{n \in N}^{\infty} a_{nv}^{(2)} \right|^{p_v} \leqslant L_p[A].$$

This completes the proof.

Theorem 2.1 includes many results as special case. We give some examples. Take $p_v = k \geqslant 1$ for all v. Then, the above theorems can be stated as follows. □

COROLLARY 2.2. *$A \in (X, \ell)$ if and only if $U_1[A] < \infty$, where $X = \ell_\infty, c, c_0$.*

COROLLARY 2.3. *Let $k \geqslant 1$. Then $A \in (X, \ell_k)$ if and only if*

$$U_k\left[A^T\right] < \infty,$$

where $X = \ell_\infty, c, c_0$.

COROLLARY 2.4. *Let $k \geqslant 1$. Then $A \in (X, \ell_k)$ if and only if*

$$U_k\left[(a_{nv} - a_{n-1,v})^T\right] < \infty,$$

where $X = c_s, (c_0)_s$.

COROLLARY 2.5. *Let $k \geqslant 1$. Then $A \in ((\ell_\infty)_s, \ell_k)$ if and only if $\lim_v a_{nv} = 0$ $(n = 0, 1, ...)$ and*

$$U_k\left[(a_{nv} - a_{n-1,v})^T\right] < \infty.$$

COROLLARY 2.6. *Let* $k>1$. *Then* $A\in(\ell_k,\ell)$ *if and only if*

$$U_{k^*}[A]<\infty.$$

COROLLARY 2.7. *Let* $k>1$. *Then* $A\in(\ell_k,bv)$ *if and only if*

$$U_{k^*}[(a_{nv}-a_{n-1,v})]<\infty.$$

COROLLARY 2.8. *Let* $k>1$. *Then* $A\in(\ell_k,bv_0)$ *if and only if* $\lim_n a_{nv}=0$ $(v=0,1,\ldots)$ *and*

$$U_{k^*}[(a_{nv}-a_{n-1,v})]<\infty.$$

COROLLARY 2.9. *Let* $k>1$. *Then,* $A\in\left(bv^k,\ell\right)$ *if and only if condition (1.3) is satisfied and*

$$U_{k^*}\left[\left(\sum_{j=v}^{\infty}a_{nj}\right)\right]<\infty.$$

COROLLARY 2.10. *Let* $k>1$. *Then,* $A\in\left(bv^k,bv\right)$ *if and only if condition (1.3) is satisfied and*

$$U_{k^*}\left[\left(\sum_{j=v}^{\infty}\left(a_{nj}-a_{n-1,j}\right)\right)\right]<\infty.$$

Now, using a different technique we give other applications of which proofs are very short and different. A series $\sum a_v$ is summable $\left|\overline{N},p_n\right|_k$ (see [4]) iff $\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|y_n|^k<\infty$, where

$$y_n=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}\lambda_v a_v,\ y_0=\lambda_0a_0.$$

Following the lines in Theorem 2.1 of [6], $\sum\lambda_v a_v$ is summable $\left|\overline{N},q_n\right|$ whenever $\sum a_v$ is is summable $\left|\overline{N},p_n\right|_k$ iff $A\in(\ell_k,\ell)$, where

$$a_{nv}=\begin{cases}\frac{q_n}{Q_nQ_{n-1}}\left[\frac{P_v}{p_v}\Delta(Q_{v-1}\lambda_v)+Q_v\lambda_{v+1}\right](P_v/p_v)^{(1/k)-1}, & 0\leqslant v\leqslant n-1\\ \frac{q_nP_n}{Q_np_n}(P_n/p_n)^{(1/k)-1}\lambda_n, & v=n\\ 0, & v>n\end{cases}$$

But by Theorem 2.1, it is equivalent to

$$\sum_{v=0}^{\infty}\left(\sum_{n=v}^{\infty}|a_{nv}|\right)^{k^*}=\sum_{v=0}^{\infty}\frac{p_v}{P_v}\left(\frac{q_vP_v}{Q_vp_v}|\lambda_v|+\left|\frac{P_v}{p_v}\Delta(Q_{v-1}\lambda_v)+Q_v\lambda_{v+1}\right|\sum_{n=v+1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\right)^{k^*}<\infty$$

which holds if and only if

$$\sum_{v=0}^{\infty}\frac{p_v}{P_v}\left(\frac{q_vP_v}{Q_vp_v}|\lambda_v|\right)^{k^*}<\infty \tag{2.3}$$

and

$$\sum_{v=0}^{\infty} \frac{p_v}{P_v} \left| \frac{P_v}{p_v} \Delta\lambda_v + \lambda_{v+1} \right|^{k^*} < \infty. \tag{2.4}$$

Therefore the following main result in [6] is obtained from Theorem 2.1.

COROLLARY 2.11. *The series $\sum \lambda_v a_v$ is summable $\left|\overline{N}, q_n\right|$ whenever $\sum a_v$ is is summable $\left|\overline{N}, p_n\right|_k$, $k > 1$, if and only if the conditions (2.3) and (2.4) are satisfied.*

As in [7], $\sum \lambda_v a_v$ is summable $|R, q_n|$ whenever $\sum a_v$ is summable $|R, p_n|_k$ iff $B \in (\ell_k, \ell)$, where

$$b_{nv} = \begin{cases} \frac{q_n}{Q_n Q_{n-1}} \left[\frac{P_v}{p_v} \Delta(Q_{v-1}\lambda_v) + Q_v \lambda_{v+1} \right] v^{1/k-1}, & 0 \leqslant v \leqslant n-1 \\ \frac{q_n P_n}{Q_n p_n} n^{1/k-1} \lambda_n, & v = n \\ 0, & v > n \end{cases}$$

equivalently, by Theorem 2.1,

$$\sum_{v=0}^{\infty} \left(\sum_{n=v}^{\infty} |b_{nv}| \right)^{k^*} = \sum_{v=0}^{\infty} \frac{1}{v} \left(\frac{q_v P_v}{Q_v p_v} |\lambda_v| + \left| \frac{P_v}{p_v} \Delta(Q_{v-1}\lambda_v) + Q_v \lambda_{v+1} \right| \sum_{n=v+1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \right)^{k^*} < \infty.$$

So we have the following main theorem in [7].

COROLLARY 2.12. *$\sum \lambda_v a_v$ is summable $|R, q_n|$ whenever $\sum a_v$ is summable $|R, p_n|_k$, $k > 1$, if and only if*

$$\left\{ v^{-1/k^*} \left(\frac{q_v P_v}{Q_v p_v} |\lambda_v| + \left| \frac{P_v}{p_v} \Delta\lambda_v + \lambda_{v+1} \right| \right) \right\} \in \ell_{k^*}.$$

REFERENCES

[1] M. STIEGLITZ UND H. TIETZ, *Matrixtransformationen von Folgenräumen. Eine Ergebnisübersicht*, Math Z. **154** (1977), 1–16.

[2] A. WILANSKY, *Summability Through Functional Analysis*, Elsevier Science Publishing Company, New York (1984).

[3] I. J. MADDOX, *Elements of Functional Analysis*, Cambridge University Press, Cambridge (1970).

[4] H. BOR, *A note on two summability methods*, Proc. Amer. Math. Soc. **98** (1986) 81–84.

[5] E. MALKOWSKY, V. RAKOČEVIĆ, S. ŽIVKOVIĆ, *Matrix transformations between the sequence space bv^p and certain BK spaces*, Bull. Cl. Sci. Math. Nat. Sci. Math. 27 (2002), 33–46.

[6] M. A. SARIGÖL, *Characterization of absolute summability factors*, J. Math. Anal. Appl. **195** (1995), 537–545.

[7] M. A. SARIGÖL, *On two absolute Riesz summability factors of infinite series*, Proc. Amer. Math. Soc. **118** (1993), 485–488

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