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COEFFICIENT ESTIMATES FOR SOME FAMILIES OF BI–BAZILEVIČ FUNCTIONS OF THE MA–MINDA TYPE INVOLVING THE HOHLOV OPERATOR

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Abstract. In this paper, we introduce and investigate a new subclass of the function class Σ of bi-univalent functions of the Bazilevič type defined in the open unit disk, which are associated with the Hohlov operator and satisfy some subordination conditions. Furthermore, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the new subclass introduced here. Several (known or new) consequences of the results are also pointed out.

1. Introduction, definitions and preliminaries

Let $\mathscr{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Further, by $\mathscr{S}$ we shall denote the class of all functions $f(z)$ in $\mathscr{A}$ which are univalent in $\mathbb{U}$ and indeed normalized by

$$f(0) = f'(0) - 1 = 0.$$

Some of the important and well-investigated subclasses of the univalent function class $\mathscr{S}$ include (for example) the class $\mathscr{S}^*(\alpha)$ $(0 \leqq \alpha < 1)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathscr{K}(\alpha)$ $(0 \leqq \alpha < 1)$ of convex functions of order α in $\mathbb{U}$. It is well known that every function $f \in \mathscr{S}$ has an inverse f^{-1}, defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

and

$$f\big(f^{-1}(w)\big) = w \qquad \left(|w| < r_0(f);\, r_0(f) \geqq \frac{1}{4}\right),$$

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where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \qquad (1.2)$$

A function $f \in \mathscr{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1).

An analytic function f is subordinate to another analytic function g, written as follows:

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

provided that there exists an analytic function (that is, Schwarz function) $\omega(z)$ defined on $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \qquad (z \in \mathbb{U})$$

such that (see, for details, [23])

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

Ma and Minda [22] unified various subclasses of starlike and convex functions for which either of the functions

$$\frac{zf'(z)}{f(z)} \qquad \text{and} \qquad 1 + \frac{zf''(z)}{f'(z)}$$

is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ, with

$$\Re\big(\phi(z)\big) > 0 \qquad (z \in \mathbb{U}), \quad \phi(0) = 1 \qquad \text{and} \qquad \phi'(0) > 0,$$

which maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions in $\mathbb{U}$ consists of functions $f \in \mathscr{A}$ satisfying the following subordination condition:

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \qquad (z \in \mathbb{U}).$$

Similarly, the class of Ma-Minda convex functions in $\mathbb{U}$ consists of functions $f \in \mathscr{A}$ satisfying the following subordination condition:

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \qquad (z \in \mathbb{U}).$$

A function f is said to be bi-starlike of Ma-Minda type in $\mathbb{U}$ or bi-convex of Ma-Minda type in $\mathbb{U}$ if both f and f^{-1} are, respectively, Ma-Minda starlike in $\mathbb{U}$ or Ma-Minda convex in $\mathbb{U}$. These function classes are denoted, respectively, by $\mathscr{S}^*_\Sigma(\phi)$ and $\mathscr{K}_\Sigma(\phi)$. In the sequel, it is assumed tacitly that ϕ is an analytic function with positive real part in $\mathbb{U}$ such that

$$\phi(0) = 1 \qquad \text{and} \qquad \phi'(0) > 0,$$

and $\phi(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has a series expansion of the following form:

$$\phi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \qquad (c_1 > 0;\ z \in \mathbb{U}). \tag{1.3}$$

The study of operators plays an important rôle in Geometric Function Theory in Complex Analysis and its related fields. Many derivative and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to better understand the geometric properties of such operators. The convolution or the Hadamard product of two functions $f, g \in \mathscr{A}$ is denoted by $f * g$ and is defined as follows:

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z), \tag{1.4}$$

where $f(z)$ is given by (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

In terms of the Hadamard product (or convolution), the Dziok-Srivastava linear convolution operator involving the generalized hypergeometric function was introduced and studied systematically by Dziok and Srivastava [8, 7] and (subsequently) by many other authors (see, for details, [19] and [31]). In fact, in *Geometric Function Theory*, there are other families of general convolution operators including (for example) the generalized fractional calculus operator and the Srivastava-Wright operator (see [18], [19] and [31]; see also the other closely-related works cited in each of these recent publications). Here, in our present investigation, we recall a much simpler convolution operator $\mathscr{I}_{a,b,c}$ due to Hohlov [14, 15], which indeed is a very specialized case of the widely- (and extensively-) investigated Dziok-Srivastava operator.

For parameters $a, b, c \in \mathbb{C}$ (with $c \neq 0, -1, -2, -3, \cdots$), the Gauss hypergeometric function ${}_2F_1(a, b, c; z)$ is defined as follows:

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \\ &= 1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1}} \frac{z^{n-1}}{(n-1)!} \qquad (z \in \mathbb{U}), \end{aligned} \tag{1.5}$$

where $(\lambda)_n$ is the Pochhammer symbol (or the shifted factorial) defined, in terms of the familiar Gamma function, by

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \cdots\}). \end{cases} \tag{1.6}$$

For positive real values of the parameters a, b and c with $c \neq 0, -1, -2, -3, \cdots$, by using the Gauss hypergeometric function given by (1.5), Hohlov [14, 15] introduced the familiar convolution operator $\mathscr{I}_{a,b,c}$ as follows:

$$\begin{aligned}\mathscr{I}_{a,b;c}f(z) &= z\,{}_2F_1(a,b,c;z) * f(z),\\ &= z + \sum_{n=2}^{\infty} \Lambda_n a_n z^n \qquad (z \in \mathbb{U}),\end{aligned} \tag{1.7}$$

where

$$\Lambda_n = \frac{(a)_{n-1}(b)_{n-1}}{(n-1)!(c)_{n-1}} \qquad (n \in \mathbb{N} \setminus \{1\}). \tag{1.8}$$

Hohlov [14, 15] discussed several interesting geometrical properties exhibited by the operator $\mathscr{I}_{a,b;c}$. Even though the three-parameter family of operators $\mathscr{I}_{a,b;c}$ is a very specialized case of the widely- (and extensively-) investigated Dziok-Srivastava operator (just as we mentioned above), it does contain, as its *further* special cases, most of the known linear derivative or integral operators. In particular, if $b = 1$ in (1.7), then $\mathscr{I}_{a,b;c}$ reduces to the Carlson-Shaffer operator. It is also easily seen that the Hohlov operator $\mathscr{I}_{a,b;c}$ provides a generalization of the Ruscheweyh derivative operator as well as the Bernardi-Libera-Livingston operator.

Recently, especially after its revival by Srivastava *et al.* [33], there has been triggering interest in the study of the bi-univalent function class Σ leading to non-sharp coefficient estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ in (1.1). However, the coefficient problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1,2\})$$

is still an open problem (see [3, 4, 5, 20, 24, 36]). Motivated largely by (and following the work of) Srivastava *et al.* [33], many researchers (see, for example, [6, 9, 11, 12, 13, 21, 32, 37, 38]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the corresponding first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

Several authors have discussed various subfamilies of the well-known Bazilevič functions (see, for details, [10, 35]; see also [1, 2, 16, 17, 25, 29, 30, 39]) of type λ from various viewpoints such as the perspective of convexity, inclusion theorems, radii of starlikeness and convexity, boundary rotational problems, subordination relationships, and so on. It is interesting to note in this connection that the earlier investigations on the subject do not seem to have addressed the problems involving coefficient inequalities and coefficient bounds for these subfamilies of Bazilevič type functions especially when the parameter λ is greater than 1 ($\lambda \in \mathbb{R}$). Thus, motivated primarily by the recent work of Deniz [9], we introduce here a new subfamily of Bazilevič type functions belonging to the function class Σ and involving the Hohlov operator $\mathscr{I}_{a,b;c}$. For this new subfamily of Bazilevič type functions, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Several closely-related function classes are also considered and relevant connections to earlier known results are pointed out.

DEFINITION. Let $h : \mathbb{U} \longrightarrow \mathbb{C}$ be a convex univalent function in $\mathbb{U}$ such that

$$h(0) = 1 \qquad \text{and} \qquad \Re\big(h(z)\big) > 0 \qquad (z \in \mathbb{U}).$$

Suppose also that the function $h(z)$ is given by

$$h(z) = 1 + \sum_{n=1}^{\infty} B_n z^n \qquad (z \in \mathbb{U}). \tag{1.9}$$

A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathscr{B}_{\Sigma}^{a,b;c}(\beta,\lambda;h)$ if the following conditions are satisfied:

$$e^{i\beta}\left(\frac{z^{1-\lambda}\left(\mathscr{I}_{a,b;c}f(z)\right)'}{[\mathscr{I}_{a,b;c}f(z)]^{1-\lambda}}\right) \prec h(z)\cos\beta + i\sin\beta \tag{1.10}$$

and

$$e^{i\beta}\left(\frac{w^{1-\lambda}\left(\mathscr{I}_{a,b;c}g(w)\right)'}{[\mathscr{I}_{a,b;c}g(w)]^{1-\lambda}}\right) \prec h(w)\cos\beta + i\sin\beta, \tag{1.11}$$

where

$$\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \lambda \geqq 0 \qquad \text{and} \qquad z, w \in \mathbb{U}$$

and the function g is given by (1.2).

EXAMPLE 1. If we set

$$h(z) = \frac{1+Az}{1+Bz} =: h_{A,B}(z) \qquad (-1 \leqq B < A \leqq 1),$$

then we have

$$\mathscr{B}_{\Sigma}^{a,b;c}(\beta,\lambda;h) = \mathscr{B}_{\Sigma}^{a,b;c}\left(\beta,\lambda;\frac{1+Az}{1+Bz}\right) =: \mathscr{B}_{\Sigma}^{a,b;c}(\beta,\lambda;h_{A,B}),$$

in which $\mathscr{B}_{\Sigma}^{a,b;c}(\beta,\lambda;h_{A,B})$ denotes the class of functions $f \in \Sigma$ satisfying the following conditions:

$$e^{i\beta}\left(\frac{z^{1-\lambda}\left(\mathscr{I}_{a,b;c}f(z)\right)'}{[\mathscr{I}_{a,b;c}f(z)]^{1-\lambda}}\right) \prec \left(\frac{1+Az}{1+Bz}\right)\cos\beta + i\sin\beta \qquad (z \in \mathbb{U})$$

and

$$e^{i\beta}\left(\frac{w^{1-\lambda}\left(\mathscr{I}_{a,b;c}g(w)\right)'}{[\mathscr{I}_{a,b;c}g(w)]^{1-\lambda}}\right) \prec \left(\frac{1+Aw}{1+Bw}\right)\cos\beta + i\sin\beta \qquad (w \in \mathbb{U}),$$

where $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\lambda \geqq 0$ and the function g is given by (1.2).

EXAMPLE 2. If, in Example 1, we set

$$A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B = -1,$$

that is, if we put

$$h(z) = h_{1-2\alpha,-1} = \frac{1+(1-2\alpha)z}{1-z} =: \mathfrak{h}_{\alpha}(z) \qquad (0 \leqq \alpha < 1),$$

then we get

$$\mathscr{B}_{\Sigma}^{a,b;c}(\beta,\lambda;h)=\mathscr{B}_{\Sigma}^{a,b;c}\left(\beta,\lambda;\frac{1+(1-2\alpha)z}{1-z}\right)=:\mathscr{B}_{\Sigma}^{a,b;c}(\beta,\lambda;\mathfrak{h}_{\alpha}),$$

in which $\mathscr{B}_{\Sigma}^{a,b;c}(\beta,\lambda;\mathfrak{h}_{\alpha})$ denotes the class of functions $f\in\Sigma$ such that

$$\Re\left[e^{i\beta}\left(\frac{z^{1-\lambda}\left(\mathscr{I}_{a,b;c}f(z)\right)'}{[\mathscr{I}_{a,b;c}f(z)]^{1-\lambda}}\right)\right]>\alpha\cos\beta \qquad (z\in\mathbb{U})$$

and

$$\Re\left[e^{i\beta}\left(\frac{w^{1-\lambda}\left(\mathscr{I}_{a,b;c}g(w)\right)'}{[\mathscr{I}_{a,b;c}g(w)]^{1-\lambda}}\right)\right]>\alpha\cos\beta \qquad (w\in\mathbb{U}),$$

where $\beta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $\lambda\geqq 0$ and the function g is given by (1.2).

Upon specializing the parameters λ and a,b,c, one can get the various (presumably new) subclasses of Σ as illustrated in the following examples.

EXAMPLE 3. For $\lambda=0$, we have

$$\mathscr{B}_{\Sigma}^{a,b;c}(\beta,0;h)=:\mathscr{S}_{\Sigma}^{a,b;c}(\beta;h),$$

in which $\mathscr{S}_{\Sigma}^{a,b;c}(\beta;h)$ denotes the class of functions $f\in\Sigma$ given by (1.1) and satisfying the following conditions:

$$e^{i\beta}\left(\frac{z\left(\mathscr{I}_{a,b;c}f(z)\right)'}{\mathscr{I}_{a,b;c}f(z)}\right)\prec h(z)\cos\beta+i\sin\beta \tag{1.12}$$

and

$$e^{i\beta}\left(\frac{w\left(\mathscr{I}_{a,b;c}g(w)\right)'}{\mathscr{I}_{a,b;c}g(w)}\right)\prec h(w)\cos\beta+i\sin\beta \tag{1.13}$$

where $\beta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $z,w\in\mathbb{U}$ and the function g is given by (1.2).

EXAMPLE 4. For $\lambda=1$, we get

$$\mathscr{B}_{\Sigma}^{a,b;c}(\beta,1;h)=:\mathscr{G}_{\Sigma}^{a,b;c}(\beta;h),$$

in which $\mathscr{G}_{\Sigma}^{a,b;c}(\beta;h)$ denotes the class of functions $f\in\Sigma$ given by (1.1) and satisfying the following conditions:

$$e^{i\beta}\left(\mathscr{I}_{a,b;c}f(z)\right)'\prec h(z)\cos\beta+i\sin\beta \tag{1.14}$$

and

$$e^{i\beta}\left(\mathscr{I}_{a,b;c}g(w)\right)'\prec h(w)\cos\beta+i\sin\beta \tag{1.15}$$

where $\beta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $z,w\in\mathbb{U}$ and the function g is given by (1.2).

We note here that, for $a=c$ and $b=1$, the classes $\mathscr{S}_{\Sigma}^{a,b;c}(\beta;h)$ and $\mathscr{G}_{\Sigma}^{a,b;c}(\beta;h)$ would reduce to the interesting subclasses given by Examples 5 and 6 below.

EXAMPLE 5. For $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have

$$\mathscr{S}_{\Sigma}^{a,1;a}(\beta;h) =: \mathscr{S}_{\Sigma}^{*}(\beta;h),$$

in which $\mathscr{S}_{\Sigma}^{*}(\beta;h)$ denotes the class of functions $f \in \Sigma$ given by (1.1) and satisfying the following conditions:

$$e^{i\beta}\left(\frac{zf'(z)}{f(z)}\right) \prec h(z)\cos\beta + i\sin\beta \qquad \text{and} \qquad e^{i\beta}\left(\frac{wg'(w)}{g(w)}\right) \prec h(w)\cos\beta + i\sin\beta,$$

where $z, w \in \mathbb{U}$ and the function g is given by (1.2).

EXAMPLE 6. For $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we get

$$\mathscr{G}_{\Sigma}^{a,1;a}(\beta;h) =: \mathscr{G}_{\Sigma}^{*}(\beta;h),$$

in which $\mathscr{G}_{\Sigma}^{*}(\beta;h)$ denotes the class of functions $f \in \Sigma$ given by (1.1) and satisfying the following conditions:

$$e^{i\beta}\left(f'(z)\right) \prec h(z)\cos\beta + i\sin\beta \qquad \text{and} \qquad e^{i\beta}\left(g'(w)\right) \prec h(w)\cos\beta + i\sin\beta,$$

where $z, w \in \mathbb{U}$ and the function g is given by (1.2).

REMARK 1. For $\beta = 0$, the above-mentioned subclass $\mathscr{S}_{\Sigma}^{*}(\beta;h)$ (see Example 5) was studied in [3] and the subclass $\mathscr{G}_{\Sigma}^{*}(\beta;h)$ (see Example 6) was considered in [33] (see also [34]).

In order to derive our main results in Section 2 involving the estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the Bi-Bazilevič type subclass $\mathscr{B}_{\Sigma}^{a,b;c}(\beta,\lambda;h)$ of the bi-univalent function class Σ, we shall need such coefficient inequalities as those asserted by the following lemmas (especially Lemma 2).

LEMMA 1. (see [27]) *If a function $p \in \mathscr{P}$ is given by*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \mathbb{U}),$$

then

$$|p_k| \leqq 2 \qquad (k \in \mathbb{N}),$$

where $\mathscr{P}$ is the family of all functions p, analytic in $\mathbb{U}$, for which

$$p(0) = 1 \qquad \textit{and} \qquad \Re\big(p(z)\big) > 0 \qquad (z \in \mathbb{U}).$$

LEMMA 2. (see [28]; see also [10]) *Let the function $\varphi(z)$ given by*

$$\varphi(z) = \sum_{n=1}^{\infty} C_n z^n \qquad (z \in \mathbb{U})$$

be convex in $\mathbb{U}$. *Suppose also that the function* $\mathfrak{h}(z)$ *given by*

$$\mathfrak{h}(z) = \sum_{n=1}^{\infty} \mathfrak{h}_n z^n$$

is holomorphic in $\mathbb{U}$. *If*

$$\mathfrak{h}(z) \prec \varphi(z) \qquad (z \in \mathbb{U}),$$

then

$$|\mathfrak{h}_n| \leqq |C_1| \qquad (n \in \mathbb{N}).$$

Mapping and many other properties and characteristics of various families of analytic, univalent and bi-univalent functions, including (for example) the Bi-Bazilevič functions of the Ma-Minda type being considered here, are potentially useful in several problems in mathematical, physical and engineering sciences.

2. Coefficient bounds for the function class $\mathscr{B}_{\Sigma}^{a,b;c}(\beta,\lambda;h)$

We begin by finding the estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ in (1.1) for functions in the class $\mathscr{B}_{\Sigma}^{a,b;c}(\beta,\lambda;h)$.

THEOREM 1. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathscr{B}_{\Sigma}^{a,b;c}(\beta,\lambda;h)$. *Suppose also that* B_1 *is given as in the Taylor-Maclaurin expansion* (1.9) *of the function* $h(z)$. *Then*

$$|a_2| \leqq \sqrt{\frac{2|B_1|\cos\beta}{[(\lambda-1)(\lambda+2)\Lambda_2^2 + 2(\lambda+2)\Lambda_3]}} \tag{2.1}$$

and

$$|a_3| \leqq \frac{|B_1|\cos\beta}{(\lambda+2)\Lambda_3} + \left(\frac{|B_1|\cos\beta}{(\lambda+1)\Lambda_2}\right)^2, \tag{2.2}$$

where the coefficients Λ_n *are given by* (1.8).

Proof. It follows from (1.10) and (1.11) that

$$e^{i\beta}\left(\frac{z^{1-\lambda}\left(\mathscr{I}_{a,b;c}f(z)\right)'}{[\mathscr{I}_{a,b;c}f(z)]^{1-\lambda}}\right) = p(z)\cos\beta + i\sin\beta \tag{2.3}$$

and

$$e^{i\beta}\left(\frac{w^{1-\lambda}\left(\mathscr{I}_{a,b;c}g(w)\right)'}{[\mathscr{I}_{a,b;c}g(w)]^{1-\lambda}}\right) = q(w)\cos\beta + i\sin\beta, \tag{2.4}$$

where the functions

$$p(z) \prec h(z) \qquad (z \in \mathbb{U}) \qquad \text{and} \qquad q(w) \prec h(w) \qquad (w \in \mathbb{U})$$

are in the above-defined class $\mathscr{P}$ and have the following forms:

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \mathbb{U}) \tag{2.5}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + \cdots \qquad (w \in \mathbb{U}), \tag{2.6}$$

respectively. Now, equating the coefficients in (2.3) and (2.4), we get

$$e^{i\beta}(\lambda+1)\Lambda_2\, a_2 = p_1 \cos\beta, \tag{2.7}$$

$$e^{i\beta}\left(\frac{(\lambda-1)(\lambda+2)}{2}\Lambda_2^2\, a_2^2 + (\lambda+2)\Lambda_3\, a_3\right) = p_2 \cos\beta, \tag{2.8}$$

$$-e^{i\beta}(\lambda+1)\Lambda_2 a_2 = q_1 \cos\beta \tag{2.9}$$

and

$$e^{i\beta}\left[\left(2(\lambda+2)\Lambda_3 + \frac{(\lambda-1)(\lambda+2)}{2}\Lambda_2^2\right) a_2^2 - (\lambda+2)\Lambda_3\, a_3\right] = q_2 \cos\beta. \tag{2.10}$$

From (2.7) and (2.9), we find that

$$a_2 = \frac{p_1 e^{-i\beta}\cos\beta}{(\lambda+1)\Lambda_2} = -\frac{q_1 e^{-i\beta}\cos\beta}{(\lambda+1)\Lambda_2}, \tag{2.11}$$

which implies that

$$p_1 = -q_1 \tag{2.12}$$

and

$$2\left[(\lambda+1)\Lambda_2\, a_2\right]^2 = (p_1^2 + q_1^2)\, e^{-2i\beta}\cos^2\beta. \tag{2.13}$$

Upon adding (2.8) and (2.10), if we make use of (2.11) and (2.12), we obtain

$$e^{i\beta}\left[(\lambda-1)(\lambda+2)\Lambda_2^2 + 2(\lambda+2)\Lambda_3\right] a_2^2 = (p_2 + q_2)\cos\beta, \tag{2.14}$$

which yields

$$a_2^2 = \frac{(p_2 + q_2)e^{-i\beta}\ \cos\beta}{\left[(\lambda-1)(\lambda+2)\Lambda_2^2 + 2(\lambda+2)\Lambda_3\right]}. \tag{2.15}$$

Since, by definition, $p(z)$, $q(w) \subset h(\mathbb{U})$, by applying Lemma 2 in conjunction with the Taylor-Maclaurin expansions (1.9), (2.5) and (2.6), we find that

$$|p_n| := \left|\frac{p^{(n)}(0)}{n!}\right| \leqq |B_1| \qquad (n \in \mathbb{N}) \tag{2.16}$$

and

$$|q_n| := \left|\frac{q^{(n)}(0)}{n!}\right| \leqq |B_1| \qquad (n \in \mathbb{N}). \tag{2.17}$$

Thus, by using (2.16) and (2.17) for the coefficients p_2 and q_2, we immediately have

$$|a_2|^2 = \frac{2|B_1|\cos\beta}{\left[(\lambda-1)(\lambda+2)\Lambda_2^2 + 2(\lambda+2)\Lambda_3\right]}, \tag{2.18}$$

which easily yields the bound on $|a_2|$ as asserted in (2.1).

Next, in order to find the bound on $|a_3|$, by subtracting (2.10) from (2.8), we get

$$e^{i\beta}\left[2(\lambda+2)\Lambda_3 a_3 - 2(\lambda+2)\Lambda_3 a_2^2\right] = (p_2 - q_2)\cos\beta. \tag{2.19}$$

It follows from (2.11), (2.12) and (2.19) that

$$a_3 = \frac{(p_2-q_2)e^{-i\beta}\cos\beta}{2(\lambda+2)\Lambda_3} + \frac{(p_1^2+q_1^2)e^{-2i\beta}\cos^2\beta}{2(\lambda+1)^2\Lambda_2^2}.$$

Applying Lemma 2, (2.16) and (2.17) once again for the coefficients p_2 and q_2, we readily get

$$|a_3| \leqq \frac{|B_1|\cos\beta}{(\lambda+2)\Lambda_3} + \left(\frac{|B_1|\cos\beta}{(\lambda+1)\Lambda_2}\right)^2.$$

This completes the proof of Theorem 1. □

Corresponding essentially to Examples 3 to 6, Theorem 1 yields the following corollaries.

COROLLARY 1. *Let the function $f(z)$ given by (1.1) be in the class $\mathscr{S}_{\Sigma}^{a,b;c}(\beta;h)$. Suppose also that B_1 is given as in the Taylor-Maclaurin expansion (1.9) of the function $h(z)$. Then*

$$|a_2| \leqq \sqrt{\frac{|B_1|\cos\beta}{2\Lambda_3 - \Lambda_2^2}} \tag{2.20}$$

and

$$|a_3| \leqq \frac{|B_1|\cos\beta}{2\Lambda_3} + \left(\frac{|B_1|\cos\beta}{\Lambda_2}\right)^2, \tag{2.21}$$

where the coefficients Λ_n are given by (1.8).

COROLLARY 2. *Let the function $f(z)$ given by (1.1) be in the class $\mathscr{G}_{\Sigma}^{a,b;c}(\beta;h)$. Suppose also that B_1 is given as in the Taylor-Maclaurin expansion (1.9) of the function $h(z)$. Then*

$$|a_2| \leqq \sqrt{\frac{|B_1|\cos\beta}{3\Lambda_3}} \tag{2.22}$$

and

$$|a_3| \leqq \frac{|B_1|\cos\beta}{3\Lambda_3} + \left(\frac{|B_1|\cos\beta}{2\Lambda_2}\right)^2, \tag{2.23}$$

where the coefficients Λ_n are given by (1.8).

COROLLARY 3. *Let the function $f(z)$ given by (1.1) be in the class $\mathscr{S}_{\Sigma}^{*}(\beta;h)$. Suppose also that B_1 is given as in the Taylor-Maclaurin expansion (1.9) of the function $h(z)$. Then*

$$|a_2| \leqq \sqrt{|B_1|\cos\beta} \tag{2.24}$$

and

$$|a_3| \leqq \frac{|B_1|\cos\beta}{2} + (|B_1|\cos\beta)^2. \tag{2.25}$$

COROLLARY 4. *Let the function $f(z)$ given by (1.1) be in the class $\mathscr{G}_{\Sigma}^{*}(\beta;h)$. Suppose also that B_1 is given as in the Taylor-Maclaurin expansion (1.9) of the function $h(z)$. Then*

$$|a_2| \leqq \sqrt{\frac{|B_1|\cos\beta}{3}} \tag{2.26}$$

and

$$|a_3| \leqq \frac{|B_1|\cos\beta}{3} + \left(\frac{|B_1|\cos\beta}{2}\right)^2. \tag{2.27}$$

3. Further corollaries and consequences of Theorem 1

Theorem 2 below is an interesting consequence of Theorem 1 corresponding to Example 1.

THEOREM 2. *Let the function $f(z)$ given by (1.1) be in the class $\mathscr{B}_{\Sigma}^{a,b;c}(\beta,\lambda;h_{A,B})$. Then*

$$|a_2| \leqq \sqrt{\frac{2(A-B)\cos\beta}{[(\lambda-1)(\lambda+2)\Lambda_2^2+2(\lambda+2)\Lambda_3]}} \tag{3.1}$$

and

$$|a_3| \leqq \frac{(A-B)\cos\beta}{(\lambda+2)\Lambda_3} + \left(\frac{(A-B)\cos\beta}{(1+\lambda)\Lambda_2}\right)^2, \tag{3.2}$$

where the coefficients Λ_n are given by (1.8).

Now, corresponding essentially to Examples 3 to 6, Theorem 2 can be shown to yield the following corollaries.

COROLLARY 5. *Let the function $f(z)$ given by (1.1) be in the class $\mathscr{S}_{\Sigma}^{a,b;c}(\beta;h_{A,B})$. Then*

$$|a_2| \leqq \sqrt{\frac{(A-B)\cos\beta}{2\Lambda_3-\Lambda_2^2}} \tag{3.3}$$

and

$$|a_3| \leqq \frac{(A-B)\cos\beta}{2\Lambda_3} + \left(\frac{(A-B)\cos\beta}{\Lambda_2}\right)^2, \tag{3.4}$$

where the coefficients Λ_n are given by (1.8).

COROLLARY 6. *Let the function $f(z)$ given by (1.1) be in the class $\mathscr{G}_{\Sigma}^{a,b;c}(\beta;h_{A,B})$. Then*

$$|a_2| \leqq \sqrt{\frac{(A-B)\cos\beta}{3\Lambda_3}} \tag{3.5}$$

and

$$|a_3| \leqq \frac{(A-B)\cos\beta}{3\Lambda_3} + \left(\frac{(A-B)\cos\beta}{2\Lambda_2}\right)^2, \tag{3.6}$$

where the coefficients Λ_n are given by (1.8).

COROLLARY 7. *Let the function $f(z)$ given by (1.1) be in the class $\mathscr{S}^*_\Sigma(\beta;h_{A,B})$. Then*

$$|a_2| \leqq \sqrt{(A-B)\cos\beta} \tag{3.7}$$

and

$$|a_3| \leqq \frac{(A-B)\cos\beta}{2} + [(A-B)\cos\beta]^2. \tag{3.8}$$

COROLLARY 8. *Let the function $f(z)$ given by (1.1) be in the class $\mathscr{G}^*_\Sigma(\beta;h_{A,B})$. Then*

$$|a_2| \leqq \sqrt{\frac{(A-B)\cos\beta}{3}} \tag{3.9}$$

and

$$|a_3| \leqq \frac{(A-B)\cos\beta}{3} + \left(\frac{(A-B)\cos\beta}{2}\right)^2. \tag{3.10}$$

Theorem 3 below would result when we apply Theorem 1 in conjunction with Example 2.

THEOREM 3. *Let the function $f(z)$ given by (1.1) be in the class $\mathscr{B}^{a,b;c}_\Sigma(\beta,\lambda;\mathfrak{h}_\alpha)$. Then*

$$|a_2| \leqq \sqrt{\frac{4(1-\alpha)\cos\beta}{[(\lambda-1)(\lambda+2)\Lambda_2^2+2(\lambda+2)\Lambda_3]}} \tag{3.11}$$

and

$$|a_3| \leqq \frac{2(1-\alpha)\cos\beta}{(\lambda+2)\Lambda_3} + \left(\frac{2(1-\alpha)\cos\beta}{(\lambda+1)\Lambda_2}\right)^2, \tag{3.12}$$

where the coefficients Λ_n are given by (1.8).

Finally, by making use of Theorem 3 together with Examples 3 to 6, we readily arrive at the following corollaries.

COROLLARY 9. *Let the function $f(z)$ given by (1.1) be in the class $\mathscr{S}^{a,b;c}_\Sigma(\beta;\mathfrak{h}_\alpha)$. Then*

$$|a_2| \leqq \sqrt{\frac{2(1-\alpha)\cos\beta}{2\Lambda_3-\Lambda_2^2}} \tag{3.13}$$

and

$$|a_3| \leqq \frac{(1-\alpha)\cos\beta}{\Lambda_3} + \left(\frac{2(1-\alpha)\cos\beta}{\Lambda_2}\right)^2, \tag{3.14}$$

where the coefficients Λ_n are given by (1.8).

COROLLARY 10. *Let the function $f(z)$ given by (1.1) be in the class $\mathscr{G}^{a,b;c}_\Sigma(\beta;\mathfrak{h}_\alpha)$. Then*

$$|a_2| \leqq \sqrt{\frac{2(1-\alpha)\cos\beta}{3\Lambda_3}} \tag{3.15}$$

and

$$|a_3| \leqq \frac{2(1-\alpha)\cos\beta}{3\Lambda_3} + \left(\frac{(1-\alpha)\cos\beta}{\Lambda_2}\right)^2, \tag{3.16}$$

where the coefficients Λ_n *are given by* (1.8).

COROLLARY 11. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathscr{S}_\Sigma^*(\beta;\mathfrak{h}_\alpha)$. *Then*

$$|a_2| \leqq \sqrt{2(1-\alpha)\cos\beta} \tag{3.17}$$

and

$$|a_3| \leqq (1-\alpha)\cos\beta + [2(1-\alpha)\cos\beta]^2. \tag{3.18}$$

COROLLARY 12. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathscr{G}_\Sigma^*(\beta;\mathfrak{h}_\alpha)$. *Then*

$$|a_2| \leqq \sqrt{\frac{2(1-\alpha)\cos\beta}{3}} \tag{3.19}$$

and

$$|a_3| \leqq \frac{2(1-\alpha)\cos\beta}{3} + [(1-\alpha)\cos\beta]^2. \tag{3.20}$$

REMARK 2. In their special cases when $\beta = 0$, the results presented in this paper would lead to various other (new or known) results, some of which for the function class Σ were considered in earlier works (see, for example, [5, 11, 26, 33, 34]).

REMARK 3. Even though the Hohlov operator $I_{a,b;c}$ is a very specialized case of the widely- (and extensively-) investigated Dziok-Srivastava operator, just as we mentioned in Section 1, it does contain (as its *further* special cases) such other relatively simpler linear operators as (for example) the Carlson-Shaffer operator, the Ruscheweyh derivative operator and the generalized Bernardi-Libera-Livingston operator. In particular, for the generalized Bernardi-Libera-Livingston operator $\mathscr{J}_\mu$, it is easily seen that

$$\mathscr{J}_\mu f(z) := \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1} f(t)dt = \mathscr{I}_{1,\mu+1;\mu+2} f(z) \qquad \big(f \in \mathscr{A};\ \Re(\mu) > -1\big). \tag{3.21}$$

Thus, obviously, various other interesting consequences of our general results (which are asserted by Theorems 1, 2 and 3 and Corollaries 1 to 12 above) can be derived by appropriately specializing these results. The details involved may be left as an exercise for the interested reader who may also consider investigating the problem of *further* extending the work presented here to hold true for such substantially more general convolutions operators as (for example) the Dziok-Srivastava operator, the generalized fractional calculus operator and the Srivastava-Wright operator (see, for details, [18], [19] and [31]).

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