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SOME PROPERTIES OF p–VALENT ANALYTIC FUNCTIONS INVOLVING CHO–KWON–SRIVASTAVA INTEGRAL OPERATOR

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Abstract. In this paper, we introduce a subclass of p-valent functions involving Cho-Kwon-Srivastava integral operator. We study some interesting results including inclusion relations, convoltion with convex functions and integral preserving property for this class.

1. Introduction

Let A_p denote the class of functions f of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \quad p \in \mathbb{N} = \{1,2,...\}, \tag{1.1}$$

which are p-valent analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. For functions f, $g \in A_p$ of the form (1.1), We define the convolution of f and g by

$$f(z) * g(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} = g(z) * f(z).$$

We consider the following well-known subclasses

$$S_p^*(\delta) = \left\{ f : f \in A_p \text{ and } \operatorname{Re} \frac{zf'(z)}{f(z)} > \delta, \ 0 \leqslant \delta < p, \ z \in E \right\}$$

and

$$C_p(\delta) = \left\{ f : f \in A_p \text{ and } \operatorname{Re} \frac{(zf'(z))'}{f'(z)} > \delta, \ 0 \leqslant \delta < p, \ z \in E \right\}.$$

Both of these classes were extensively studied by many authors, for example, see $[17,22]$. The operator $\varphi_p(a,c)$ is defined as follows

$$\varphi_p(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k}, \ a \in \mathbb{R}, \ c \in \mathbb{R} \backslash \mathbb{Z}_0^-, \ \mathbb{Z}_0^- = \{0,-1,...\}, \ z \in E, \tag{1.2}$$

where $(x)_k$ is the shifted factorial defined in term of gamma function by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1, & k = 0, \ x \in \mathbb{C} \backslash \{0\} \\ x(x+1)....(x+k-1), & k \in \mathbb{N}, \ x \in \mathbb{C} \end{cases}.$$

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Using convolution, the operator $L_p(a,c)$ can be defined as

$$L_p(a,c)f(z) = \varphi_p(a,c;z) * f(z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{p+k} z^{p+k},\ z \in E, \tag{1.3}$$

where $a \in \mathbb{R}$, $c \in \mathbb{R} \backslash \mathbb{Z}_0^-$, $f \in A_p$ and $\varphi_p(a,c)$ is defined in (1.2). The operator $L_p(a,c)$ was studied by Saitoh $[19]$. This operator is an extension of the operator $L(a,c)$ introduced by Carlson and Shaffer $[1]$ and fractional derivative operator D_z^λ, for details, see $[15,16,21]$. Moreover, these operators were studied by various authors, for example $[11,13]$.

Analogous to the operator $L_p(a,c)$, Cho-Kwon-Srivastava $[2]$, introduced the operator $I_p^\lambda(a,c)$ as below.

$$I_p^\lambda(a,c)f(z) = \varphi_p^\dagger(a,c;z) * f(z) = z^p + \sum_{k=1}^{\infty} \frac{(\lambda+p)_k (c)_k}{k!(a)_k} a_{p+k} z^{p+k}, \tag{1.4}$$

where $a,c \in \mathbb{R} \backslash \mathbb{Z}_0^-$, $\lambda > -p$, $p \in \mathbb{N}$, $f \in A_p$ and $\varphi_p^\dagger(a,c)$ is such that

$$\varphi_p^\dagger(a,c;z) * \varphi_p(a,c;z) = \frac{z^p}{(1-z)^{\lambda+p}},\ z \in E.$$

From (1.4), the following identities can easily be obtained.

$$z\left(I_p^\lambda(a+1,c)f(z)\right)' = aI_p^\lambda(a,c)f(z) - (a-p)I_p^\lambda(a+1,c)f(z), \tag{1.5}$$

and

$$z\left(I_p^\lambda(a,c)f(z)\right)' = (\lambda+p)I_p^{\lambda+1}(a,c)f(z) - \lambda I_p^\lambda(a,c)f(z). \tag{1.6}$$

For (1.5) and (1.6), we refer $[2]$. We also observe that $I_p^1(p+1,1)f(z) = f(z)$ and $I_p^1(p,1)f(z) = \frac{zf'(z)}{p}$. The operator $I_1^\lambda(\nu+2,1)$, $\lambda > -1$, $\nu > -2$ was studied in $[3]$. For some details, see $[7,10,14]$.

For any two analytic functions f and g, we say that f is subordinate to g and write $f(z) \prec g(z)$, if there exists a Schwarz function w analytic in E with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$ such that $f(z) = g(w(z))$. Futhermore, if g is univalent in E, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$.Let $\widetilde{N}$ be the class of analytic functions $\varkappa$ such that $\varkappa(0) = 1$, which are univalent and convex in E and satisfy the condition $\operatorname{Re} \varkappa(z) > 0$, $z \in E$.

As in $[12]$, using the principle of subordination between analytic functions, we define the following.

DEFINITION 1.1. Let $f \in A_p$ for $a,c \in \mathbb{R} \backslash \mathbb{Z}_0^-$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta \geqslant 0$, $0 \leqslant \delta < p$ and $\varkappa \in \widetilde{N}$. Then $f \in Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z))$ if and only if

$$\frac{1}{p-\delta}\left\{\frac{z\left(I_p^\lambda(a,c)f(z)\right)' + \beta z^2\left(I_p^\lambda(a,c)f(z)\right)''}{(1-\beta)I_p^\lambda(a,c)f(z) + \beta z\left(I_p^\lambda(a,c)f(z)\right)'} - \delta\right\} \prec \varkappa(z),\ z \in E. \tag{1.7}$$

We have the following special cases.

(i) For $\beta = 0$, we obtain the class $Q_p^{\lambda,0,\delta}(a,c;\varkappa(z))$, introduced in [2] and further studied in [20].

(ii) By taking $\beta = 0$, $a = p+1$, $c = 1 = \lambda$ in (1.7), we obtain the class $S_p^*(\delta;\varkappa(z))$ and $\beta = 0$, $a = p$, $c = 1 = \lambda$, gives us the class $C_p(\delta;\varkappa(z))$. These classes were studied by Ma and Minda [8]. For details, also see [6].

(iii) For $0 \leqslant \beta \leqslant 1$, $a = p+1$, $c = 1 = \lambda$ and $\varkappa(z) = \dfrac{1+z}{1-z}$, we have the class $T_\beta(p,\delta)$ which has been studied in [5].

Furthermore, for different choices of parameters being involved, we obtain many other known subclasses of the class A_p.

2. Preliminary results

The following lemmas are of prime importance in the proof of our main results.

LEMMA 2.1. [4] *Let $\tau \neq 0$ and υ be complex numbers and let $\varkappa \in \widetilde{N}$ such that* $\operatorname{Re}(\tau\varkappa(z)+\upsilon) > 0$, $z \in E$. *If $h(z) = 1 + b_1 z + b_2 z^2 + \ldots$, is analytic in E with $h(0) = 1$, then*

$$h(z) + \frac{zh'(z)}{\tau h(z)+\upsilon} \prec \varkappa(z),\ z \in E \implies h \prec \varkappa(z).$$

LEMMA 2.2. [18] *Let $f \in S^*$ and $g \in C$. Then for any analytic function χ*

$$\frac{f * \chi g}{f * g}(E) \subseteq \overline{co}\chi(E),$$

where $\overline{co}\chi(E)$ denotes the closed convex hull of $\chi(E)$.

LEMMA 2.3. [9] *Let $H \in S^*$ defined in E with $H(0) = 0$ and let $h(z) = b + b_n z^n + b_{n+1} z^{n+1} + \ldots..$ with $b \neq 0$ be analytic in E and satisfy*

$$\frac{zh'(z)}{h(z)} \prec H(z),\ z \in E.$$

Then

$$h(z) \prec q(z) = b\exp\left[\frac{1}{n}\int_0^z t^{-1}H(t)dt\right],\ z \in E,$$

where q is the best dominant in the sense if $h(z) \prec q_1(z)$, then $q(z) \prec q_1(z)$.

3. Main results

THEOREM 3.1. *Let* $f \in A_p$ *for* $a, c \in \mathbb{R} \backslash \mathbb{Z}_0^-$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta \geqslant 0$, $0 \leqslant \delta < p$ *and* $\varkappa \in \widetilde{N}$ *with* $\operatorname{Re} \varkappa(z) > \left(1 - \frac{a}{p-\delta}\right)$. *Then*

$$Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z)) \subset Q_p^{\lambda,\beta,\delta}(a+1,c;\varkappa(z)),\ z \in E.$$

Proof. For $f \in A_p$, $a, c \in \mathbb{R} \backslash \mathbb{Z}_0^-$, $\lambda > -p$, $p \in N$, $\beta \geqslant 0$, $0 \leqslant \delta < p$ and $\varkappa(z) \in \widetilde{N}$, consider

$$H(z) = \frac{z\left(I_p^\lambda(a+1,c)f\right)' + \beta z^2\left(I_p^\lambda(a+1,c)f\right)''}{(1-\beta)I_p^\lambda(a+1,c)f + \beta z\left(I_p^\lambda(a+1,c)f\right)'}, \tag{3.1}$$

where $H(z) = (p-\delta)h(z) + \delta$ is analytic in E with $H(0) = 1$. Using (1.5) in (3.1), we have

$$H(z) + a - p = \frac{(1-\beta)I_p^\lambda(a,c)f(z) + \beta z\left(I_p^\lambda(a,c)f(z)\right)'}{(1-\beta)I_p^\lambda(a+1,c)f(z) + \beta z\left(I_p^\lambda(a+1,c)f(z)\right)'}. \tag{3.2}$$

Differentiating (3.2) logarithmically and then using (3.1), we obtain

$$H(z) + \frac{zH'(z)}{H(z)+a-p} = \frac{z\left(I_p^\lambda(a,c)f(z)\right)' + \beta z^2\left(I_p^\lambda(a,c)f(z)\right)''}{(1-\beta)I_p^\lambda(a,c)f(z) + \beta z\left(I_p^\lambda(a,c)f(z)\right)'}.$$

Using (1.7) in the above equation and then replacing H by $(p-\delta)h+\delta$, we can write

$$h(z) + \frac{zh'(z)}{(p-\delta)h(z)+\delta+a-p} \prec \varkappa(z),\ z \in E. \tag{3.3}$$

Lemma 2.1 and (3.3) yield $h(z) \prec \varkappa(z)$ for $\operatorname{Re}\varkappa(z) > \left(1 - \frac{a}{p-\delta}\right)$, that is $f \in Q_p^{\lambda,\beta,\delta}(a+1,c;\varkappa(z))$. Thus

$$Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z)) \subset Q_p^{\lambda,\beta,\delta}(a+1,c;\varkappa(z)),\ z \in E.$$

THEOREM 3.2. *Let* $f \in Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z))$ *and let* $a, c \in \mathbb{R} \backslash \mathbb{Z}_0^-$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta \geqslant 0$, $0 \leqslant \delta < p$ *and* $\varkappa \in \widetilde{N}$ *with* $\operatorname{Re}\varkappa(z) > \left(1 - \frac{p-\beta}{p-\delta}\right)$. *Then*

$$Q_p^{\lambda+1,\beta,\delta}(a,c;\varkappa(z)) \subset Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z)),\ z \in E.$$

Proof. Let $f \in Q_p^{\lambda+1,\beta,\delta}(a,c;\varkappa(z))$ for the conditions on the parametres described above in the theorem. Then by (1.7), we have

$$\frac{1}{p-\delta}\left\{\frac{z\left(I_p^{\lambda+1}(a,c)f(z)\right)' + \beta z^2\left(I_p^{\lambda+1}(a,c)f(z)\right)''}{(1-\beta)I_p^{\lambda+1}(a,c)f(z) + \beta z\left(I_p^{\lambda+1}(a,c)f(z)\right)'} - \delta\right\} \prec \varkappa(z). \tag{3.4}$$

Set

$$h(z)=\frac{1}{p-\delta}\left\{\frac{z\left(I_p^{\lambda}(a,c)f(z)\right)'+\beta z^2\left(I_p^{\lambda}(a,c)f(z)\right)''}{(1-\beta)I_p^{\lambda}(a,c)f(z)+\beta z\left(I_p^{\lambda}(a,c)f(z)\right)'}-\delta\right\}, \tag{3.5}$$

where h is analytic in E with $h(0)=1$. Using (1.6) in (3.5) and simplifying, we can write

$$H(z)+\beta=(p+\lambda)\frac{(1-\beta)I_p^{\lambda+1}(a,c)f(z)+\beta z\left(I_p^{\lambda+1}(a,c)f(z)\right)''}{(1-\beta)I_p^{\lambda}(a,c)f(z)+\beta z\left(I_p^{\lambda}(a,c)f(z)\right)'}, \tag{3.6}$$

where $H(z)=(p-\delta)h(z)+\delta$. Logarithmic differentiation of (3.6) and an application of (3.5) yield

$$H(z)+\frac{zH'(z)}{H(z)+\beta}=\frac{z\left(I_p^{\lambda}(a,c)f(z)\right)'+\beta z^2\left(I_p^{\lambda}(a,c)f(z)\right)''}{(1-\beta)I_p^{\lambda}(a,c)f(z)+\beta z\left(I_p^{\lambda}(a,c)f(z)\right)'}.$$

Replacing $H(z)$ by $(p-\delta)h(z)+\delta$ in the above equation, simplifying and then using (3.4), we have

$$h(z)+\frac{zh'(z)}{(p-\delta)h(z)+\delta+\beta}\prec\varkappa(z), z\in E. \tag{3.7}$$

Applying Lemma 2.1 to (3.7), we have $h\prec\varkappa(z)$ for $\operatorname{Re}\varkappa(z)>\left(1-\frac{p-\beta}{p-\delta}\right)$. That is $f\in Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z))$. Thus

$$Q_p^{\lambda+1,\beta,\delta}(a,c;\varkappa(z))\subset Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z)),\ z\in E.$$

The operator $\mathscr{F}_\eta : A_{p\to}A_p$ is defined as follows.

$$\mathscr{F}_\eta(z)=\mathscr{F}_\eta(f(z))=\frac{\eta+p}{z^\eta}\int_0^z t^{\eta-1}f(t)\,dt, \eta>-p, z\in E. \tag{3.8}$$

By taking $p=1$ and $\eta\in\mathbb{N}$ in (3.6), we have the well known Bernardi operator. □

We now prove:

THEOREM 3.3. *Let $f\in Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z))$ for $a,c\in\mathbb{R}\backslash\mathbb{Z}_0^-$, $\lambda>-p$, $p\in\mathbb{N}$, $\beta\geqslant 0$, $0\leqslant\delta<p$ and $\varkappa\in\widetilde{N}$. Then the function $\mathscr{F}_\eta$ defined by* (3.8) *is also in the class $Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z))$ for* $\operatorname{Re}\varkappa(z)>\left(1-\frac{p-\eta}{p-\delta}\right)$ *and $z\in E$.*

Proof. By simplifying (3.8) and using the operator $I_p^{\lambda}(a,c)$, for $z\in E$, we can write

$$z\left(I_p^{\lambda}(a,c)\mathscr{F}_\eta(z)\right)'=(p+\eta)I_p^{\lambda}(a,c)f-\eta I_p^{\lambda}(a,c)\mathscr{F}_\eta(z). \tag{3.9}$$

We take

$$h=\frac{1}{p-\delta}\left\{\frac{z\left(I_p^{\lambda}(a,c)\mathscr{F}_\eta(z)\right)'+\beta z^2\left(I_p^{\lambda}(a,c)\mathscr{F}_\eta(z)\right)''}{(1-\beta)I_p^{\lambda}(a,c)\mathscr{F}_\eta+\beta z\left(I_p^{\lambda}(a,c)\mathscr{F}_\eta\right)'}-\delta\right\}, \tag{3.10}$$

where $h$is analytic in E with $h(0)=1$. Equations (3.9) and (3.10) yield

$$\frac{(p-\delta)h(z)+\delta+\eta}{p+\eta}=\frac{(1-\beta)I_p^\lambda(a,c)f(z)+\beta z\left(I_p^\lambda(a,c)f(z)\right)'}{(1-\beta)I_p^\lambda(a,c)\mathscr{F}_\eta(z)+\beta z\left(I_p^\lambda(a,c)\mathscr{F}_\eta(z)\right)'}.$$

Using technique similar to that of Theorem 3.2, we obtain the required result. □

THEOREM 3.4. *Let* $f\in A_p,\ a,c\in\mathbb{R}\backslash\mathbb{Z}_0^-,\ \lambda>-p,\ p\in\mathbb{N},\ \beta\geqslant 0,\ 0\leqslant\delta<p$ *and* $\varkappa\in\widetilde{N}$. *Then* $f\in Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z))$ *if and only if*

$$G(z)=(1-\beta)f(z)+\beta zf\prime(z)\in Q_p^{\lambda,0,\delta}(a,c;\varkappa(z)),\ z\in E.$$

Proof. Using the operator $I_p^\lambda(a,c)$, we can write

$$I_p^\lambda(a,c)G(z)=(1-\beta)I_p^\lambda(a,c)f(z)+\beta z\left(I_p^\lambda(a,c)f(z)\right)',\ z\in E. \tag{3.11}$$

Logarithmic differentiation of (3.11) yields

$$\frac{z\left(I_p^\lambda(a,c)G(z)\right)'}{I_p^\lambda(a,c)G(z)}=\frac{z\left(I_p^\lambda(a,c)f(z)\right)'+\beta z^2\left(I_p^\lambda(a,c)f(z)\right)''}{(1-\beta)I_p^\lambda(a,c)f(z)+\beta z\left(I_p^\lambda(a,c)f(z)\right)'}. \tag{3.12}$$

From (3.12), we obtain the desired result. □

THEOREM 3.5. *Let* $f\in A_p,\ a,c\in\mathbb{R}\backslash\mathbb{Z}_0^-,\ \lambda>-p,\ p\in\mathbb{N},\ \beta\geqslant 0,\ 0\leqslant\delta<p$ *and* $\varkappa\in\widetilde{N}$. *Then for* $f\in Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z))$ *and* $\Psi(z)\in C$, *we have*

$$\left[z^{p-1}\Psi(z)\right]*f(z)\in Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z)),\ z\in E.$$

Proof. Let $a,c\in\mathbb{R}\backslash\mathbb{Z}_0^-,\ \lambda>-p,\ p\in\mathbb{N},\ \beta\geqslant 0,\ 0\leqslant\delta<p,\ \varkappa\in\widetilde{N}$ and $f\in Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z))$. Then by using (1.7), we have

$$\frac{z\left(I_p^\lambda(a,c)f(z)\right)'+\beta z^2\left(I_p^\lambda(a,c)f(z)\right)''}{(1-\beta)I_p^\lambda(a,c)f(z)+\beta z\left(I_p^\lambda(a,c)f(z)\right)'}=\varkappa(z), \tag{3.13}$$

where $\chi(z)=((p-\delta)(\varkappa(w(z))+\delta)$, w is analytic in E with $w(0)=0$ and $|w(z)|<1$ for $z\in E$.

Let $G(z)=\left[z^{p-1}\Psi(z)\right]*f(z)$ for $f\in Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z))$ and $\Psi\in C$. By using the properties of the convolution and (3.13), we can write

$$\frac{z\left(I_p^\lambda(a,c)G(z)\right)'+\beta z^2\left(I_p^\lambda(a,c)G(z)\right)''}{(1-\beta)I_p^\lambda(a,c)G(z)+\beta z\left(I_p^\lambda(a,c)G(z)\right)'}=\frac{z^{p-1}\Psi(z)*g\chi(z)}{z^{p-1}\Psi(z)*g(z)},$$

where $g(z)=(1-\beta)I_p^\lambda(a,c)f(z)+\beta z\left(I_p^\lambda(a,c)f(z)\right)'$. We take

$$F(z)=\frac{1}{p-\delta}\left\{\frac{z\left(I_p^\lambda(a,c)G(z)\right)'+\beta z^2(I_p^\lambda(a,c)G(z))''}{(1-\beta)I_p^\lambda(a,c)G(z)+\beta z\left(I_p^\lambda(a,c)G(z)\right)'}-\delta\right\}$$
$$=\frac{1}{p-\delta}\left(\frac{\Psi(z)*z^{1-P}g(z)((p-\delta)\varkappa(w(z))+\delta)}{\Psi(z)*z^{1-P}g(z)}-\delta\right). \quad (3.14)$$

Since $z^{1-P}\left((1-\beta)I_p^\lambda(a,c)f+\beta z\left(I_p^\lambda(a,c)f\right)'\right)\in S^*$ and $\Psi\in C$, so application of Lemma 2.1 in (3.14) yields the required result. □

THEOREM 3.6. *Let $a,c\in\mathbb{R}\backslash\mathbb{Z}_0^-$, $\lambda>-p$, $p\in\mathbb{N}$, $\beta\geqslant 0$, $0\leqslant\delta<p$ and $\varkappa\in\widetilde{N}$. Then $f\in Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z))$ if and only if*

$$f(z)=\left[\sum_{k=0}^{\infty}\frac{k!(a)_k}{(\lambda+p)_k(c)_k[1-\beta+\beta(p+k)]}z^{p+k}\right]\star\left[z^p\exp\int_0^z\frac{H(t)}{t}dt\right],\ z\in E,$$

where $H(t)=((p-\delta)(\varkappa(w(t))-1)$, w is analytic in E with $w(0)=0$ and $|w(z)|<1$ for $z\in E$.

Proof. Let $f\in Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z))$ for $a,c\in\mathbb{R}\backslash\{0,-1,\ldots\}$, $\lambda>-p$, $p\in\mathbb{N}$, $\beta\geqslant 0$, $0\leqslant\delta<p$ and $\varkappa\in\widetilde{N}$. Then by using the definition of the class $Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z))$ and simplifying, we can write

$$\frac{\left(I_p^\lambda(a,c)f(z)\right)'+\beta z\left(I_p^\lambda(a,c)f(z)\right)''}{(1-\beta)I_p^\lambda(a,c)f(z)+\beta z\left(I_p^\lambda(a,c)f(z)\right)'}-\frac{p}{z}=\frac{(p-\delta)(\varkappa(w)-1)}{z},$$

where w is analytic in E with $w(0)=0$ and $|w(z)|<1$ for $z\in E$. Thus

$$(1-\beta)I_p^\lambda(a,c)f(z)+\beta z\left(I_p^\lambda(a,c)f(z)\right)=z^p\exp\int_0^z\frac{H(t)}{t}dt,$$

for $H(t)=((p-\delta)(\varkappa(w(t))-1)$. This implies that

$$f(z)=\left[\sum_{k=0}^{\infty}\frac{k!(a)_k}{(\lambda+p)_k(c)_k[1-\beta+\beta(p+k)]}z^{p+k}\right]\star\left[z^p\exp\int_0^z\frac{H(t)}{t}dt\right]. \quad □$$

COROLLARY 3.1. *A function f belongs to the class $Q_p^{\lambda,\beta,\delta}(a,c;A,B)$ for $a,c\in\mathbb{R}\backslash\mathbb{Z}_0^-$, $\lambda>-p$, $p\in\mathbb{N}$, $\beta\geqslant 0$, $0\leqslant\delta<p$ and $-1\leqslant B<A\leqslant 1$, if and only if*

$$f(z)=\left[\sum_{k=0}^{\infty}\frac{k!(a)_k}{(\lambda+p)_k(c)_k[1-\beta+\beta(p+k)]}z^{p+k}\right]*\left[z^p\exp\int_0^z\frac{H_1(t)}{t}dt\right],$$

where $H_1(t) = \dfrac{(p-\delta)((A-B)w(t)}{(1+Bw(t))}$, *w is analytic in E with* $w(0) = 0$ *and* $|w(z)| < 1$ *for* $z \in E$.

THEOREM 3.7. *Let* $f \in A_p$, $a,c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta \geqslant 0$, $0 \leqslant \delta < p$ *and* $\varkappa \in \widetilde{N}$. *Then for* $f \in Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z))$,

$$\frac{(1-\beta)I_p^\lambda(a,c)f(z) + \beta z\left(I_p^\lambda(a,c)f(z)\right)'}{z^p} \prec \exp\int_0^z \frac{((p-\delta)(\varkappa(t)-1)dt}{t},\ z \in E.$$

Proof. We consider

$$F(z) = z^{-p}\left[(1-\beta)I_p^\lambda(a,c)f(z) + \beta z\left(I_p^\lambda(a,c)f(z)\right)'\right] \tag{3.15}$$
$$= 1 + b_1 z + b_2 z^2 + \ldots,\ z \in E.$$

By taking logarithmic diffentiation of (3.15) and then using (1.7), we obtain

$$\frac{zF\prime(z)}{F(z)} \prec H(z) = ((p-\delta)(\varkappa(z)-1), z \in E.$$

The function $H(z) = ((p-\delta)(\varkappa(z)-1)$ is starlike because $\varkappa(z)$ is convex univalent in E. Using Lemma 2.3, we have

$$\frac{(1-\beta)I_p^\lambda(a,c)f(z) + \beta z\left(I_p^\lambda(a,c)f(z)\right)'}{z^p} \prec \exp\int_0^z \frac{H(t)}{t}dt,\ z \in E,$$

where $H(t) = ((p-\delta)(\varkappa(t)-1)$. □

COROLLARY 3.2. *Let* $f \in Q_p^{\lambda,\beta,\delta}(a,c;\varkappa(z))$, *where* $a,c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta \geqslant 0$, $0 \leqslant \delta < p$ *and* $\varkappa \in \widetilde{N}$. *Then*

$$\sum_{k=1}^{\infty} \frac{(\lambda+p)_k(c)_k(1-\beta+\beta(p+k))}{k!(a)_k} a_{p+k} z^k \prec \exp\int_0^z \frac{((p-\delta)(\varkappa(t)-1)}{t}dt,\ z \in E.$$

The proof is simple and straight forward.

COROLLARY 3.3. *Let* $f \in Q_p^{\lambda,\beta,\delta}(a,c;A,B)$ *where* $a,c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $0 \leqslant \delta < p$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta \geqslant 0$ *and* $-1 \leqslant B < A \leqslant 1$. *Then*

$$\frac{(1-\beta)I_p^\lambda(a,c)f(z) + \beta z\left(I_p^\lambda(a,c)f(z)\right)'}{z^p} \prec (1+Bz)^{(A-B)(p-\delta)B},\ z \in E.$$

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