SOME PROPERTIES OF $p$–VALENT ANALYTIC FUNCTIONS INVOLVING CHO–KWON–SRIVASTAVA INTEGRAL OPERATOR

K. I. NOOR, S. Z. H. BUKHARI, M. ARIF AND M. NAZIR

Abstract. In this paper, we introduce a subclass of $p$-valent functions involving Cho-Kwon-Srivastava integral operator. We study some interesting results including inclusion relations, convolution with convex functions and integral preserving property for this class.

1. Introduction

Let $A_p$ denote the class of functions $f$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \quad p \in \mathbb{N} = \{1, 2, \ldots\},$$

(1.1)

which are $p$-valent analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. For functions $f, g \in A_p$ of the form (1.1), we define the convolution of $f$ and $g$ by

$$f(z) \ast g(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} = g(z) \ast f(z).$$

We consider the following well-known subclasses

$$S^*_p(\delta) = \left\{ f : f \in A_p \text{ and } \Re \frac{zf'(z)}{f(z)} > \delta, \quad 0 \leq \delta < p, \quad z \in E \right\}$$

and

$$C_p(\delta) = \left\{ f : f \in A_p \text{ and } \Re \frac{(zf'(z))'}{f'(z)} > \delta, \quad 0 \leq \delta < p, \quad z \in E \right\}.$$ 

Both of these classes were extensively studied by many authors, for example, see [17, 22]. The operator $\varphi_p(a, c)$ is defined as follows

$$\varphi_p(a, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k}, \quad a \in \mathbb{R}, \quad c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \quad \mathbb{Z}_0^- = \{0, -1, \ldots\}, \quad z \in E,$$

(1.2)

where $(x)_k$ is the shifted factorial defined in term of gamma function by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \left\{ \begin{array}{ll} 1, & k = 0, \quad x \in \mathbb{C} \setminus \{0\} \\ x(x+1)\ldots(x+k-1), & k \in \mathbb{N}, \quad x \in \mathbb{C} \end{array} \right..$$


Keywords and phrases: Convolution, subordination, Cho-Kwon-Srivastava integral operator.
Using convolution, the operator $L_p(a,c)$ can be defined as

$$L_p(a,c)f(z) = \varphi_p(a,c;z) \ast f(z) = z^p + \sum_{k=1}^{\infty} \frac{(a)^k}{(c)^k} a_{p+k} z^{p+k}, \quad z \in E,$$

where $a \in \mathbb{R}$, $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $f \in A_p$ and $\varphi_p(a,c)$ is defined in (1.2). The operator $L_p(a,c)$ was studied by Saitoh [19]. This operator is an extension of the operator $L(a,c)$ introduced by Carlson and Shaffer [1] and fractional derivative operator $D^\lambda_z$, for details, see [15, 16, 21]. Moreover, these operators were studied by various authors, for example [11, 13].

Analogous to the operator $L_p(a,c)$, Cho-Kwon-Srivastava [2], introduced the operator $I^\lambda_p(a,c)$ as below.

$$I^\lambda_p(a,c)f(z) = \varphi_p^\dagger(a,c;z) \ast f(z) = z^p + \sum_{k=1}^{\infty} \frac{(\lambda + p)^k (c)^k}{k!(a)^k} a_{p+k} z^{p+k},$$

where $a, c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\lambda > -p$, $p \in \mathbb{N}$, $f \in A_p$ and $\varphi_p^\dagger(a,c)$ is such that

$$\varphi_p^\dagger(a,c;z) \ast \varphi_p(a,c;z) = \frac{z^p}{(1-z)^{\lambda+p}}, \quad z \in E.$$

From (1.4), the following identities can easily be obtained.

$$z \left( I^\lambda_p (a+1,c)f(z) \right)' = aI^\lambda_p(a,c)f(z) - (a-p)I^\lambda_p(a+1,c)f(z),$$

and

$$z \left( I^\lambda_p (a,c)f(z) \right)' = (\lambda + p)I^{\lambda+1}_p(a,c)f(z) - \lambda I^\lambda_p(a,c)f(z).$$

For (1.5) and (1.6), we refer [2]. We also observe that $I^1_p(p+1,1)f(z) = f(z)$ and $I^1_p(p,1)f(z) = \frac{zf'(z)}{p}$. The operator $I^\lambda_p(v+2,1)$, $\lambda > -1$, $v > -2$ was studied in [3]. For some details, see [7, 10, 14].

For any two analytic functions $f$ and $g$, we say that $f$ is subordinate to $g$ and write $f(z) \prec g(z)$, if there exists a Schwarz function $w$ analytic in $E$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$ such that $f(z) = g(w(z))$. Furthermore, if $g$ is univalent in $E$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$. Let $\widetilde{N}$ be the class of analytic functions $\prec$ such that $\prec(0) = 1$, which are univalent and convex in $E$ and satisfy the condition $\text{Re } \prec(z) > 0$, $z \in E$.

As in [12], using the principle of subordination between analytic functions, we define the following.

**Definition 1.1.** Let $f \in A_p$ for $a,c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta > 0$, $0 \leq \delta < p$ and $\prec \in \widetilde{N}$. Then $f \in Q^n_p,_{\lambda,\beta,\delta}(a,c;\prec(z))$ if and only if

$$\frac{1}{p - \delta} \left\{ \left( z(I^\lambda_p(a,c)f(z))' + \beta z^2 (I^\lambda_p(a,c)f(z))'' \right) - \delta \right\} \prec \prec(z), \quad z \in E.$$
We have the following special cases.

(i) For $\beta = 0$, we obtain the class $Q^{\lambda,0,\delta}_{p}(a,c;\varpi(z))$, introduced in [2] and further studied in [20].

(ii) By taking $\beta = 0$, $a = p + 1$, $c = 1 = \lambda$ in (1.7), we obtain the class $S^*_p(\delta;\varpi(z))$ and $\beta = 0$, $a = p$, $c = 1 = \lambda$, gives us the class $C_p(\delta;\varpi(z))$. These classes were studied by Ma and Minda [8]. For details, also see [6].

(iii) For $0 \leq \beta \leq 1$, $a = p + 1$, $c = 1 = \lambda$ and $\varpi(z) = \frac{1+z}{1-z}$, we have the class $T_{p}(\delta)$ which has been studied in [5].

Furthermore, for different choices of parameters being involved, we obtain many other known subclasses of the class $A_p$.

2. Preliminary results

The following lemmas are of prime importance in the proof of our main results.

**Lemma 2.1.** [4] Let $\tau \neq 0$ and $\nu$ be complex numbers and let $\varpi \in \tilde{N}$ such that $\Re(\tau \varpi(z) + \nu) > 0$, $z \in E$. If $h(z) = 1 + b_1z + b_2z^2 + \ldots$, is analytic in $E$ with $h(0) = 1$, then

$$h(z) + \frac{zh'(z)}{\varpi h(z) + \nu} \prec \varpi(z), \ z \in E \implies h \prec \varpi(z).$$

**Lemma 2.2.** [18] Let $f \in S^*$ and $g \in C$. Then for any analytic function $\chi$

$$\frac{f \ast \chi g}{f \ast g}(E) \subseteq \overline{\varnothing\chi}(E),$$

where $\overline{\varnothing\chi}(E)$ denotes the closed convex hull of $\chi(E)$.

**Lemma 2.3.** [9] Let $H \in S^*$ defined in $E$ with $H(0) = 0$ and let $h(z) = b + b_nz^n + b_{n+1}z^{n+1} + \ldots$, with $b \neq 0$ be analytic in $E$ and satisfy

$$\frac{zh'(z)}{h(z)} \prec H(z), \ z \in E.$$

Then

$$h(z) \prec q(z) = b \exp \left[ \frac{1}{n} \int_0^z \frac{t^{-1}H(t)dt}{h(t)} \right], \ z \in E,$$

where $q$ is the best dominant in the sense if $h(z) \prec q(z)$, then $q(z) \prec q(z)$. 

3. Main results

**Theorem 3.1.** Let $f \in A_p$ for $a,c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta \geq 0$, $0 \leq \delta < p$ and $z \in \tilde{N}$ with $\text{Re } \varpi(z) > \left(1 - \frac{a}{p-\delta}\right)$. Then

$$Q_p^{\lambda,\beta,\delta}(a,c;\varpi(z)) \subset Q_p^{\lambda,\beta,\delta}(a+1,c;\varpi(z)), z \in E.$$

**Proof.** For $f \in A_p$, $a,c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta \geq 0$, $0 \leq \delta < p$ and $z \in \tilde{N}$, consider

$$H(z) = \frac{z\left(I_p^{\lambda}(a+1,c)f(z)'+\beta z^2 \left(I_p^{\lambda}(a+1,c)f(z)\right)''\right)}{(1-\beta)I_p^{\lambda}(a+1,c)f(z)+\beta z \left(I_p^{\lambda}(a+1,c)f(z)\right)'}.$$  \hspace{1cm} (3.1)

where $H(z) = (p-\delta)h(z)+\delta$ is analytic in $E$ with $H(0) = 1$. Using (1.5) in (3.1), we have

$$H(z)+a-p = \frac{(1-\beta)I_p^{\lambda}(a,c)f(z)+\beta z \left(I_p^{\lambda}(a,c)f(z)\right)'}{(1-\beta)I_p^{\lambda}(a+1,c)f(z)+\beta z \left(I_p^{\lambda}(a+1,c)f(z)\right)'}.$$  \hspace{1cm} (3.2)

Differentiating (3.2) logarithmically and then using (3.1), we obtain

$$H(z)+\frac{zh'(z)}{H(z)+a-p} = \frac{z\left(I_p^{\lambda}(a,c)f(z)'+\beta z^2 \left(I_p^{\lambda}(a,c)f(z)\right)''\right)}{(1-\beta)I_p^{\lambda}(a,c)f(z)+\beta z \left(I_p^{\lambda}(a,c)f(z)\right)'}.$$  \hspace{1cm} (3.3)

Using (1.7) in the above equation and then replacing $H$ by $(p-\delta)h(z)+\delta$, we can write

$$h(z)+\frac{zh'(z)}{(p-\delta)h(z)+\delta+a-p} < \varpi(z), z \in E.$$

Lemma 2.1 and (3.3) yield $h(z) < \varpi(z)$ for $\text{Re } \varpi(z) > \left(1 - \frac{a}{p-\delta}\right)$, that is $f \in Q_p^{\lambda,\beta,\delta}(a+1,c;\varpi(z))$. Thus

$$Q_p^{\lambda,\beta,\delta}(a,c;\varpi(z)) \subset Q_p^{\lambda,\beta,\delta}(a+1,c;\varpi(z)), z \in E.$$

**Theorem 3.2.** Let $f \in Q_p^{\lambda,\beta,\delta}(a,c;\varpi(z))$ and let $a,c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta \geq 0$, $0 \leq \delta < p$ and $z \in \tilde{N}$ with $\text{Re } \varpi(z) > \left(1 - \frac{p-\beta}{p-\delta}\right)$. Then

$$Q_p^{\lambda+1,\beta,\delta}(a,c;\varpi(z)) \subset Q_p^{\lambda,\beta,\delta}(a,c;\varpi(z)), z \in E.$$

**Proof.** Let $f \in Q_p^{\lambda+1,\beta,\delta}(a,c;\varpi(z))$ for the conditions on the parametres described above in the theorem. Then by (1.7), we have

$$\frac{1}{p-\delta}\left\{ \frac{z\left(I_p^{\lambda+1}(a,c)f(z)'+\beta z^2 \left(I_p^{\lambda+1}(a,c)f(z)\right)''\right)}{(1-\beta)I_p^{\lambda+1}(a,c)f(z)+\beta z \left(I_p^{\lambda+1}(a,c)f(z)\right)'} - \delta \right\} < \varpi(z).$$  \hspace{1cm} (3.4)
Set
\[
h(z) = \frac{1}{p-\delta} \left\{ z\left( I_p^\lambda (a,c) f(z) \right)' + \beta z^2 \left( I_p^\lambda (a,c) f(z) \right)'' - \delta \right\}, \tag{3.5}
\]
where \( h \) is analytic in \( E \) with \( h(0) = 1 \). Using (1.6) in (3.5) and simplifying, we can write
\[
H(z) + \beta = (p+\lambda) \frac{(1-\beta) I_p^{\lambda+1} (a,c) f(z) + \beta z \left( I_p^{\lambda+1} (a,c) f(z) \right)''}{(1-\beta) I_p^\lambda (a,c) f(z) + \beta z \left( I_p^\lambda (a,c) f(z) \right)'}.
\tag{3.6}
\]
where \( H(z) = (p-\delta)h(z) + \delta \). Logarithmic differentiation of (3.6) and an application of (3.5) yield
\[
\frac{H(z) + zH'(z)}{H(z) + \beta} = \frac{z \left( I_p^\lambda (a,c) f(z) \right)' + \beta z^2 \left( I_p^\lambda (a,c) f(z) \right)''}{(1-\beta) I_p^\lambda (a,c) f(z) + \beta z \left( I_p^\lambda (a,c) f(z) \right)'}.
\]
Replacing \( H(z) \) by \((p-\delta)h(z) + \delta \) in the above equation, simplifying and then using (3.4), we have
\[
h(z) + \frac{z h'(z)}{(p-\delta) h(z) + \delta + \beta} \prec \varphi(z), z \in E. \tag{3.7}
\]
Applying Lemma 2.1 to (3.7), we have \( h \prec \varphi \) for \( \text{Re} \varphi(z) > \left( 1 - \frac{p-\beta}{p-\delta} \right) \). That is \( f \in Q_p^{\lambda+1,\beta,\delta}(a,c;\varphi(z)) \). Thus
\[
Q_p^{\lambda+1,\beta,\delta}(a,c;\varphi(z)) \subset Q_p^{\lambda,\beta,\delta}(a,c;\varphi(z)), z \in E.
\]
The operator \( \mathcal{F}_\eta : A_p \to A_p \) is defined as follows.
\[
\mathcal{F}_\eta(z) = \mathcal{F}_\eta(f(z)) = \frac{\eta + p}{z^\eta} \int_0^z t^{\eta-1} f(t) \, dt, \eta > -p, z \in E. \tag{3.8}
\]
By taking \( p = 1 \) and \( \eta \in \mathbb{N} \) in (3.6), we have the well known Bernardi operator. \( \square \)

We now prove:

**Theorem 3.3.** Let \( f \in Q_p^{\lambda,\beta,\delta}(a,c;\varphi(z)) \) for \( a,c \in \mathbb{R}\setminus\mathbb{Z}^-_0, \lambda > -p, p \in \mathbb{N}, \beta \geq 0, 0 \leq \delta < p \) and \( \varphi \in \mathbb{N} \). Then the function \( \mathcal{F}_\eta \) defined by (3.8) is also in the class \( Q_p^{\lambda,\beta,\delta}(a,c;\varphi(z)) \) for \( \text{Re} \varphi(z) > \left( 1 - \frac{p-\eta-\beta}{p-\delta} \right) \) and \( z \in E \).

**Proof.** By simplifying (3.8) and using the operator \( I_p^\lambda (a,c) \), for \( z \in E \), we can write
\[
z \left( I_p^\lambda (a,c) \mathcal{F}_\eta(z) \right)' = (p+\eta) I_p^\lambda (a,c) f - \eta I_p^\lambda (a,c) \mathcal{F}_\eta(z). \tag{3.9}
\]
We take
\[
h = \frac{1}{p-\delta} \left\{ z \left( I_p^\lambda (a,c) \mathcal{F}_\eta(z) \right)' + \beta z^2 \left( I_p^\lambda (a,c) \mathcal{F}_\eta(z) \right)'' \right\} - \delta \tag{3.10}.
\]
where $h$ is analytic in $E$ with $h(0) = 1$. Equations (3.9) and (3.10) yield

$$\frac{(p - \delta) h(z) + \delta + \eta}{p + \eta} = \frac{(1 - \beta) I_p^\lambda (a,c) f(z) + \beta z (I_p^\lambda (a,c) f(z))^\prime}{(1 - \beta) I_p^\lambda (a,c) \mathcal{F}_p (z) + \beta z (I_p^\lambda (a,c) \mathcal{F}_p (z))^\prime}.$$

Using technique similar to that of Theorem 3.2, we obtain the required result. □

**Theorem 3.4.** Let $f \in A_p$, $a, c \in \mathbb{R} \setminus \mathbb{Z}_0^+$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta \geq 0$, $0 \leq \delta < p$ and $\kappa \in \tilde{\mathbb{N}}$. Then $f \in Q_p^{\lambda, \beta, \delta} (a,c; \kappa (z))$ if and only if

$$G(z) = (1 - \beta) f(z) + \beta z f(z) \in Q_p^{\lambda, 0, \delta} (a,c; \kappa (z)), \ z \in E.$$

**Proof.** Using the operator $I_p^\lambda (a,c)$, we can write

$$I_p^\lambda (a,c) G(z) = (1 - \beta) I_p^\lambda (a,c) f(z) + \beta z (I_p^\lambda (a,c) f(z))^\prime, \ z \in E. \quad (3.11)$$

Logarithmic differentiation of (3.11) yields

$$\frac{z (I_p^\lambda (a,c) G(z))^\prime}{I_p^\lambda (a,c) G(z)} = \frac{z (I_p^\lambda (a,c) f(z))^\prime + \beta z^2 (I_p^\lambda (a,c) f(z))^\prime}{(1 - \beta) I_p^\lambda (a,c) f(z) + \beta z (I_p^\lambda (a,c) f(z))^\prime}. \quad (3.12)$$

From (3.12), we obtain the desired result. □

**Theorem 3.5.** Let $f \in A_p$, $a, c \in \mathbb{R} \setminus \mathbb{Z}_0^+$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta \geq 0$, $0 \leq \delta < p$ and $\kappa \in \tilde{\mathbb{N}}$. Then for $f \in Q_p^{\lambda, \beta, \delta} (a,c; \kappa (z))$ and $\Psi (z) \in C$, we have

$$[z^{p-1} \Psi (z)] * f(z) \in Q_p^{\lambda, \beta, \delta} (a,c; \kappa (z)), \ z \in E.$$

**Proof.** Let $a, c \in \mathbb{R} \setminus \mathbb{Z}_0^+$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta \geq 0$, $0 \leq \delta < p$, $\kappa \in \tilde{\mathbb{N}}$ and $f \in Q_p^{\lambda, \beta, \delta} (a,c; \kappa (z))$. Then by using (1.7), we have

$$\frac{z (I_p^\lambda (a,c) f(z))^\prime + \beta z^2 (I_p^\lambda (a,c) f(z))^\prime}{(1 - \beta) I_p^\lambda (a,c) f(z) + \beta z (I_p^\lambda (a,c) f(z))^\prime} = \kappa (z), \quad (3.13)$$

where $\chi (z) = ((p - \delta) (\kappa (w(z)) + \delta)), w$ is analytic in $E$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$.

Let $G(z) = [z^{p-1} \Psi (z)] * f(z)$ for $f \in Q_p^{\lambda, \beta, \delta} (a,c; \kappa (z))$ and $\Psi \in C$. By using the properties of the convolution and (3.13), we can write

$$\frac{z (I_p^\lambda (a,c) G(z))^\prime + \beta z^2 (I_p^\lambda (a,c) G(z))^\prime}{(1 - \beta) I_p^\lambda (a,c) G(z) + \beta z (I_p^\lambda (a,c) G(z))^\prime} = \frac{z^{p-1} \Psi (z) * g \chi (z)}{z^{p-1} \Psi (z) * g (z)}.$$
where \( g(z) = (1 - \beta) I_p^\lambda (a,c) f(z) + \beta z (I_p^\lambda (a,c) f(z))' \). We take

\[
F(z) = \frac{1}{p - \delta} \left\{ \frac{z(I_p^\lambda (a,c) G(z))' + \beta z^2 (I_p^\lambda (a,c) G(z))''}{(1 - \beta) I_p^\lambda (a,c) G(z) + \beta z (I_p^\lambda (a,c) G(z))'} - \delta \right\}
\]

\[
= \frac{1}{p - \delta} \left( \frac{\Psi(z) * z^{1-p} g(z) ((p - \delta) \lambda (w(z)) + \delta)}{\Psi(z) * z^{1-p} g(z)} - \delta \right). 
\tag{3.14}
\]

Since \( z^{1-p} \left((1 - \beta) I_p^\lambda (a,c) f + \beta z (I_p^\lambda (a,c) f)'ight) \in S^* \) and \( \Psi \in C \), so application of Lemma 2.1 in (3.14) yields the required result. □

**Theorem 3.6.** Let \( a,c \in \mathbb{R} \setminus \mathbb{Z}_0 \), \( \lambda > -p \), \( p \in \mathbb{N} \), \( \beta \geq 0 \), \( 0 \leq \delta < p \) and \( \lambda \in \bar{\mathbb{N}} \). Then \( f \in Q_p^{\lambda, \beta, \delta} (a,c; \lambda (z)) \) if and only if

\[
f(z) = \sum_{k=0}^{\infty} \frac{k! (a)_k}{(\lambda + p)_k (c)_k [1 - \beta + \beta (p + k)]} z^{p+k} \ast \left[ z^p \exp \left( \frac{z H(t)}{t} dt \right) \right], \quad z \in E,
\]

where \( H(t) = ((p - \delta) \lambda (w(t)) - 1) \), is analytic in \( E \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) for \( z \in E \).

**Proof.** Let \( f \in Q_p^{\lambda, \beta, \delta} (a,c; \lambda (z)) \) for \( a,c \in \mathbb{R} \setminus \{0, -1, \ldots\} \), \( \lambda > -p \), \( p \in \mathbb{N} \), \( \beta \geq 0 \), \( 0 \leq \delta < p \) and \( \lambda \in \bar{\mathbb{N}} \). Then by using the definition of the class \( Q_p^{\lambda, \beta, \delta} (a,c; \lambda (z)) \) and simplifying, we can write

\[
\frac{(I_p^\lambda (a,c) f(z))' + \beta z (I_p^\lambda (a,c) f(z))''}{(1 - \beta) I_p^\lambda (a,c) f(z) + \beta z (I_p^\lambda (a,c) f(z))'} - \frac{p}{z} = \frac{(p - \delta) \lambda (w(z)) - 1}{z},
\]

where \( w \) is analytic in \( E \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) for \( z \in E \). Thus

\[
(1 - \beta) I_p^\lambda (a,c) f(z) + \beta z (I_p^\lambda (a,c) f(z)) = z^p \exp \left( \frac{z H(t)}{t} dt \right),
\]

for \( H(t) = ((p - \delta) \lambda (w(t)) - 1) \). This implies that

\[
f(z) = \sum_{k=0}^{\infty} \frac{k! (a)_k}{(\lambda + p)_k (c)_k [1 - \beta + \beta (p + k)]} z^{p+k} \ast \left[ z^p \exp \left( \frac{z H(t)}{t} dt \right) \right]. \tag{□}
\]

**Corollary 3.1.** A function \( f \) belongs to the class \( Q_p^{\lambda, \beta, \delta} (a,c;A,B) \) for \( a,c \in \mathbb{R} \setminus \mathbb{Z}_0 \), \( \lambda > -p \), \( p \in \mathbb{N} \), \( \beta \geq 0 \), \( 0 \leq \delta < p \) and \(-1 \leq B < A \leq 1\), if and only if

\[
f(z) = \sum_{k=0}^{\infty} \frac{k! (a)_k}{(\lambda + p)_k (c)_k [1 - \beta + \beta (p + k)]} z^{p+k} \ast \left[ z^p \exp \left( \frac{z H_1(t)}{t} dt \right) \right],
\]
where $H_1(t) = \frac{(p-\delta)((A-B)w(t))}{(1+Bw(t))}$, $w$ is analytic in $E$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$.

**Theorem 3.7.** Let $f \in A_p$, $a,c \in \mathbb{R}\setminus\mathbb{Z}_0^-$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta > 0$, $0 \leq \delta < p$ and $\varkappa \in \tilde{N}$. Then for $f \in Q_p^{\gamma,\delta}(a,c;\varkappa(z))$,\n
\[
\frac{(1-\beta)I_p^{\lambda}(a,c)f(z) + \beta z(I_p^{\lambda}(a,c)f(z))'}{z^p} < \exp\int_0^z \frac{((p-\delta)(\varkappa(t)-1)dt}{t}, z \in E.
\]

**Proof.** We consider

\[
F(z) = z^{-p}\left[(1-\beta)I_p^{\lambda}(a,c)f(z) + \beta z\left(I_p^{\lambda}(a,c)f(z)\right)\right] \quad (3.15)
\]

\[
= 1 + b_1z + b_2z^2 + \ldots, \quad z \in E.
\]

By taking logarithmic differentiation of (3.15) and then using (1.7), we obtain

\[
\frac{zF'(z)}{F(z)} < H(z) = ((p-\delta)(\varkappa(z)-1), z \in E.
\]

The function $H(z) = ((p-\delta)(\varkappa(z)-1))$ is starlike because $\varkappa(z)$ is convex univalent in $E$. Using Lemma 2.3, we have

\[
\frac{(1-\beta)I_p^{\lambda}(a,c)f(z) + \beta z\left(I_p^{\lambda}(a,c)f(z)\right)'}{z^p} < \exp\int_0^z \frac{H(t)}{t}dt, \quad z \in E,
\]

where $H(t) = ((p-\delta)(\varkappa(t)-1))$. □

**Corollary 3.2.** Let $f \in Q_p^{\gamma,\delta}(a,c;\varkappa(z))$, where $a,c \in \mathbb{R}\setminus\mathbb{Z}_0^-$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta > 0$, $0 \leq \delta < p$ and $\varkappa \in \tilde{N}$. Then

\[
\sum_{k=1}^{\infty} \frac{(\lambda+p)_k(c)_k(1-\beta+\beta(p+k))}{k!(a)_k}a_{p+k}z^k < \exp\int_0^z \frac{((p-\delta)(\varkappa(t)-1))}{t}dt, \quad z \in E.
\]

The proof is simple and straightforward.

**Corollary 3.3.** Let $f \in Q_p^{\gamma,\delta}(a,c;A,B)$ where $a,c \in \mathbb{R}\setminus\mathbb{Z}_0^-$, $0 \leq \delta < p$, $\lambda > -p$, $p \in \mathbb{N}$, $\beta > 0$ and $-1 \leq B < A \leq 1$. Then

\[
\frac{(1-\beta)I_p^{\lambda}(a,c)f(z) + \beta z\left(I_p^{\lambda}(a,c)f(z)\right)'}{z^p} < (1+Bz)^{(A-B)(p-\delta)}B, \quad z \in E.
\]
REFERENCES


(Received February 8, 2013)

K. I. Noor
COMSATS Institute of Information Technology, Islamabad, Pakistan
e-mail: khalidanoor@hotmail.com

S. Z. H. Bukhari
Mirpur University of Science and Technology, Mirpur, Pakistan
e-mail: fatmijaku@hotmail.com

M. Arif
Abdul Wali Khan University Mardan, Khyber Pakhtunkhwa, Pakistan
e-mail: marifmaths@yahoo.com

M. Nazir
Mirpur University of Science and Technology, Mirpur, Pakistan
e-mail: cuaidians16@yahoo.com

Journal of Classical Analysis
www.ele-math.com
jca@ele-math.com