Journal of Classical Analysis
Volume 3, Number 1 (2013), 45–57 doi:10.7153/jca-03-04

EXACT EVALUATION OF SOME HIGHLY OSCILLATORY INTEGRALS

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Abstract. In this note a general result is proved that can be used to evaluate exactly a class of highly oscillatory integrals. The result can also be used to give a procedure to numerically evaluate some oscillatory integrals.

1. Introduction

The first of the ten \$100, 100-digit challenges [11, 5], proposed to calculate the integral $\int_0^1 x^{-1}\cos(x^{-1}\log x)\,dx$ to ten digits. This was a real challenge since the integrand oscillates infinitely often inside the interval of integration.

In this note, a general result is proved that will allow us to determine exactly the value of some highly oscillating integrals. To give you the flavour of what we will prove, here is one of the simplest integrals evaluated by the methods of this paper:

$$\int_0^{\pi/2} \frac{dx}{1+8\sin^2(\tan x)} = \frac{\pi}{6}\left(\frac{2e^2+1}{2e^2-1}\right), \tag{1.1}$$

where the graph of the integrand is depicted in Figure 1.

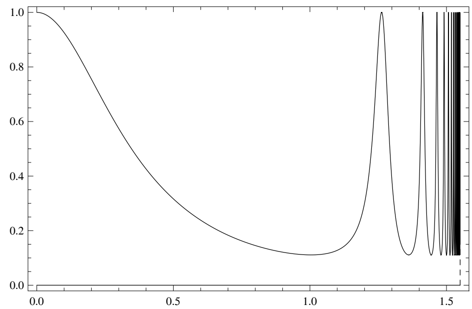

Figure 1: *The graph of the function* $x \mapsto \frac{1}{1+8\sin^2(\tan x)}$ *on* $\left[0, \frac{\pi}{2}\right)$.

In section 2 we will prove our main results and in section 3 we will give some detailed examples and applications.

Mathematics subject classification (2010): 11B68, 26A42, 30D10, 33B30, 42B05.
Keywords and phrases: Fourier series, analytic functions, power series expansion, Bernoulli polynomials, polylogarithms, Bessel functions.

2. The main results

First, let us set the framework of our investigation. Our starting point will be a 2π-periodic locally integrable function $f \in L^1(\mathbb{T})$. We are interested in the evaluation of the Cauchy principal value $\mathscr{H}f(z)$ of the integral $\int_{\mathbb{R}} \frac{f(x)}{x-z}\,dx$ where z is a *non-real* complex number. Recall that $\mathscr{H}f(z)$ is defined as follows:

$$\mathscr{H}f(z) = \lim_{a\to\infty} \int_{-a}^{a} \frac{f(x)}{x-z}\,dx. \tag{2.1}$$

The existence of the limit in (2.1) is not obvious. It will follow from Proposition 2.2.

Our approach is simple; we consider the Fourier series expansion of f:

$$S[f] = \sum_{n\in\mathbb{Z}} C_n(f) e_n \tag{2.2}$$

where $e_n(x) = e^{inx}$ and the $C_n(f)$'s are the exponential Fourier coefficients of f. Then we prove that for $f \in L^1(\mathbb{T})$, we have

$$\mathscr{H}f(z) = \sum_{n\in\mathbb{Z}} C_n(f)\mathscr{H}e_n(z), \qquad \text{for } \Im z \neq 0. \tag{2.3}$$

Note that in this general setting, the Fourier series $S[f]$ of f does not necessarily converge to f neither in the L^1-norm, nor pointwise, (even, there exits a function f in $L^1(\mathbb{T})$ such that its Fourier series diverges everywhere.) Moreover, the only general result concerning the family of Fourier coefficients $(C_n(f))_{n\in\mathbb{Z}}$ is the Riemann-Lebesgue lemma: $\lim_{n\to\pm\infty} C_n(f) = 0$.

Clearly, the $\mathscr{H}e_n(z)$'s play an important role in this investigation, they are explicitly calculated in the next lemma.

LEMMA 2.1. *For an integer n and a complex number z with $\Im z \neq 0$ we have*

$$\mathscr{H}e_n(z) = i\pi\big(\operatorname{sgn}(n) + \operatorname{sgn}(\Im z)\big)\, e^{inz} \tag{2.4}$$

where $\operatorname{sgn}(x)$ *is the sign of x with the convention* $\operatorname{sgn}(0) = 0$.

Proof. Indeed, the evaluation of $\mathscr{H}e_n(z)$ is standard. First, note that for $n = 0$ the function $x \mapsto \operatorname{Log}(x-z)$ (where Log the principal branch of the logarithm) is a primitive of $x \mapsto 1/(x-z)$, and $\mathscr{H}e_0(z)$ can be determined directly:

$$\int_{-a}^{a} \frac{dx}{x-z} = \log\frac{|a-z|}{|a+z|} + i(\operatorname{Arg}(a-z) - \operatorname{Arg}(-a-z))$$

But $\lim\limits_{a\to\infty} \operatorname{Arg}(a-z) = 0$ and $\lim\limits_{a\to\infty} \operatorname{Arg}(-a-z) = -\pi\operatorname{sgn}(\Im z)$. This proves (2.4) for $n = 0$.

Now, for $n>0$ and $a>|z|$ we have, by the Cauchy integral formula, that

$$\int_{-a}^{a}\frac{e^{inx}}{x-z}\,dx+\int_{\gamma_a}\frac{e^{in\xi}}{\xi-z}\,d\xi=2i\pi e^{inz}\chi_{P^+}(z) \tag{2.5}$$

where γ_a is the positively oriented semi-circle of diameter $[-a,a]$ contained in the upper half plane $P^+=\{w:\Im w>0\}$, and χ_{P^+} is the characteristic function of P^+. This implies, by a well-known argument [8, Lemma 4.8b], that

$$\left|\int_{-a}^{a}\frac{e^{inx}}{x-z}\,dx-2i\pi e^{inz}\chi_{P^+}(z)\right|\leqslant\frac{\pi}{n(a-|z|)}.$$

Letting a tend to $+\infty$ we conclude of the validity of (2.4) for $n>0$.

Finally, the case $n<0$ follows from the fact that $\mathscr{H}e_n(z)=-\mathscr{H}e_{-n}(-z)$. $\square$

In the next proposition, the question of existence of $\mathscr{H}f(z)$ for an arbitrary $f\in L^1(\mathbb{T})$ and $\Im z\neq 0$ is answered.

PROPOSITION 2.2. *Consider $f\in L^1(\mathbb{T})$. For every complex number z with $\Im z\neq 0$, the limit*

$$\lim_{a\to\infty}\int_{-a}^{a}\frac{f(x)}{x-z}\,dx$$

does exist, and consequently $\mathscr{H}f(z)$ is well-defined.

Proof. First, let us consider the particular case where we suppose that $\int_{\mathbb{T}}f(x)dx=0$. Under this condition, the function F defined by $F(t)=\int_0^t f(x)\,dx$ becomes a continuous, 2π-periodic function. In particular, F is bounded on $\mathbb{R}$.

An integration by parts shows that, for $X>0$, we have

$$\int_0^X\frac{f(x)}{x-z}\,dx=\frac{F(X)}{X-z}+\int_0^X\frac{F(x)}{(x-z)^2}\,dx. \tag{2.6}$$

This version of "integration by parts" is a direct application of Fubini's theorem, see for instance [3, Theorem 5.2.3]. Now, if $M=\sup_{\mathbb{R}}|F|$, we have

$$\forall x\in\mathbb{R},\qquad\left|\frac{F(x)}{(x-z)^2}\right|\leqslant\frac{M}{(x-\Re z)^2+(\Im z)^2}$$

and consequently, the integral $\int_{\mathbb{R}}\frac{F(x)}{(x-z)^2}\,dx$ is absolutely convergent. This implies, according to (2.6), the convergence of the considered integral, and that

$$\int_{-\infty}^{\infty}\frac{f(x)}{x-z}\,dx=\int_{-\infty}^{\infty}\frac{F(x)}{(x-z)^2}\,dx.$$

Now, let us come to the general case. Consider the 2π-periodic function $g\in L^1(\mathbb{T})$ defined by $g(x)=f(x)-\frac{1}{2\pi}\int_{\mathbb{T}}f(t)\,dt$. Since $\int_{\mathbb{T}}g(x)dx=0$, we conclude that $\mathscr{H}g(z)$ exists according to the previous case. But also we know that $\lim\limits_{a\to\infty}\int_{-a}^{a}\frac{dx}{x-z}$ does exist, (this is the case $n=0$ of (2.4)) Combining these two results proves the proposition. $\square$

Now, we can prove our main result, where we find a formula that allows the determination of $\mathscr{H} f(z)$, and proves (2.3).

THEOREM 2.3. *Consider* $f \in L^1(\mathbb{T})$.

i. *For a non-real complex number* z*, we have*

$$\mathscr{H} f(z) = \frac{1}{2}\int_{\mathbb{T}} \cot\left(\frac{x-z}{2}\right) f(x)\,dx \tag{2.7}$$

where $\mathscr{H} f(z)$ *is defined by* (2.1).

ii. *If* $\left(C_n(f)\right)_{n\in\mathbb{Z}}$ *are the exponential Fourier coefficients of* f *then, for every* $z \in \mathbb{C}$ *with* $\Im z \neq 0$*, we have*

$$\mathscr{H} f(z) = \sum_{n\in\mathbb{Z}} C_n(f)\mathscr{H} e_n(z), \tag{2.8}$$

where $e_n(x) = e^{inx}$*. More precisely, we have:*

$$\mathscr{H} f(z) = \begin{cases} i\pi\left(C_0(f) + 2\sum_{n=1}^{\infty} C_n(f)e^{inz}\right) & \text{if } \Im z > 0, \\ -i\pi\left(C_0(f) + 2\sum_{n=1}^{\infty} C_{-n}(f)e^{-inz}\right) & \text{if } \Im z < 0. \end{cases} \tag{2.9}$$

Proof. Recall that, (see [2, pages 187-190],)

$$\cot(\xi) = \frac{1}{\xi} + \sum_{n=1}^{\infty}\left(\frac{1}{\xi - \pi n} + \frac{1}{\xi + \pi n}\right) = \lim_{n\to\infty}\sum_{k=-n}^{n-1}\frac{1}{\xi + \pi k},$$

with normal convergence on every compact set K contained in $\mathbb{C}\setminus\pi\mathbb{Z}$. Applying this to the compact segment $K = \left\{\frac{x-z}{2} : x \in [0, 2\pi]\right\}$, and recalling that $f \in L^1(\mathbb{T})$, we conclude that

$$\begin{aligned}
\int_0^{2\pi} \cot\left(\frac{x-z}{2}\right) f(x)\,dx &= \lim_{n\to\infty}\sum_{k=-n}^{n-1}\left(\int_0^{2\pi}\frac{2f(x)}{x + 2\pi k - z}\,dx\right) \\
&= 2\lim_{n\to\infty}\sum_{k=-n}^{n-1}\left(\int_{2\pi k}^{2\pi(k+1)}\frac{f(x)}{x-z}\,dx\right) \\
&= 2\lim_{n\to\infty}\int_{-2\pi n}^{2\pi n}\frac{f(x)}{x-z}\,dx \\
&= 2\mathscr{H} f(z)
\end{aligned}$$

where we used Proposition 2.2. This is (i).

On the other hand, consider, for a non-real complex number z, the continuous 2π-periodic function g_z defined by

$$g_z(x) = -\pi\cot\left(\frac{x+z}{2}\right).$$

Since, g_z is twice continuously differentiable, we conclude that the $C_n(g_z) = O(|n|^{-2})$, (see [9, Ch. I, Sec. 4],) and consequently

$$\sum_{n\in\mathbb{Z}} |C_n(g_z)| < +\infty.$$

Also, we have $|C_n(f)| \leqslant \|f\|_1$ for every $n \in \mathbb{Z}$, therefore

$$\sum_{n\in\mathbb{Z}} |C_n(g_\lambda)|\,|C_n(f)| < +\infty.$$

This implies that the continuous 2π-periodic function $g_z * f$, which is the convolution product of g_z and f defined by

$$g_z * f(t) = \frac{1}{2\pi}\int_{\mathbb{T}} g_z(t-x)f(x)\,dx,$$

is equal to its Fourier series expansion. In particular,

$$g_z * f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(x)g_z(-x)\,dx = \sum_{n\in\mathbb{Z}} C_n(f)C_n(g_z)$$

that is,

$$\frac{1}{2}\int_0^{2\pi} \cot\left(\frac{x-z}{2}\right) f(x)\,dx = \sum_{n\in\mathbb{Z}} C_n(f)C_n(g_z).$$

Equivalently

$$\mathscr{H} f(z) = \sum_{n\in\mathbb{Z}} C_n(f)C_n(g_z).$$

Because this conclusion is valid for every $f \in L^1(\mathbb{T})$ then choosing $f = e_k$, for some $k \in \mathbb{Z}$, shows that $\mathscr{H} e_k(z) = C_k(g_z)$.

The final assertion in the theorem follows from Lemma 2.1. This proves (ii). □

Noting that

$$\cot\left(\frac{x-iz}{2}\right) - \cot\left(\frac{x+iz}{2}\right) = \frac{2i\sinh z}{\cosh z - \cos x}$$
$$\cot\left(\frac{x-iz}{2}\right) + \cot\left(\frac{x+iz}{2}\right) = \frac{2\sin x}{\cosh z - \cos x}$$

and combining $\mathscr{H} f(iz)$ and $\mathscr{H} f(-iz)$ we obtain the next corollary.

COROLLARY 2.4. *Consider $f \in L^1(\mathbb{T})$ and a complex number z with $\Re z > 0$, then*

$$\int_{-\infty}^{\infty} \frac{f(x)}{x^2+z^2}\,dx = \frac{\sinh z}{2z}\int_{\mathbb{T}} \frac{f(x)}{\cosh z - \cos x}\,dx, \tag{2.10}$$

$$= \frac{\pi}{z}\sum_{n\in\mathbb{Z}} C_n(f)e^{-|n|z}, \tag{2.11}$$

and

$$\int_{-\infty}^{\infty} \frac{xf(x)}{x^2+z^2}\,dx = \frac{1}{2}\int_{\mathbb{T}} \frac{\sin x}{\cosh z - \cos x} f(x)\,dx, \tag{2.12}$$

$$= i\pi \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n) C_n(f) e^{-|n|z}, \tag{2.13}$$

where $\left(C_n(f)\right)_{n\in\mathbb{Z}}$ *are the exponential Fourier coefficients of* f*.*

In particular, this result takes a beautiful form when f is even or odd:

COROLLARY 2.5. *Consider* $f \in L^1(\mathbb{T})$ *and a complex number* z *with* $\Re z > 0$.

i. *If* f *is even and its Fourier series expansion is* $S[f](x) = \frac{a_0}{2} + \sum_{n\geqslant 1} a_n \cos nx$*, then*

$$\int_0^{\infty} \frac{f(x)}{x^2+z^2}\,dx = \frac{\pi}{2z}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-nz}\right).$$

ii. *If* f *is odd and its Fourier series expansion is* $S[f](x) = \sum_{n\geqslant 1} b_n \sin nx$*, then*

$$\int_0^{\infty} \frac{xf(x)}{x^2+z^2}\,dx = \frac{\pi}{2}\sum_{n=1}^{\infty} b_n e^{-nz}.$$

Theorem 2.3 and Corollaries 2.4 and 2.5 are interesting even when we only seek numerical evaluation of the considered integrals, since the integrands on the left are not oscillatory, and the series have geometric convergence.

For example, consider the function f from $L^1(\mathbb{T})$ defined by

$$f(t) = \sum_{n=2}^{\infty} \frac{\cos(nt)}{\ln n},$$

(see [9, Theorem 4.1].) The series defining f is very slowly convergent, and evaluating f to a high precision is a difficult task, so the numerical evaluation of the integral (2.11) using standard methods for oscillatory integrals is clearly hard. But according to Corollary 2.5 we have

$$I = \int_0^{\infty} \frac{f(x)}{1+x^2}\,dx = \frac{\pi}{2}\sum_{n=2}^{\infty} \frac{e^{-n}}{\ln n}$$

and the evaluation of this series to a very high precision is very easy, for instance, using the first 115 terms, we obtain

$$I = 0.40850\,78850\,46648\,98588\,85437\,84734\,12731\,60971\,32032\,06994$$

to 50 exact digits. Another numerical example will be presented in the next section.

There is a case where our results take a more practical form; where we obtain a closed form for the considered integrals. Let us change our point of view. Let $\mathbb{D} = D(0,1)$ be the open unit disk in the complex plane, and let $G : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ be a continuous function on the closed unit disk $\overline{\mathbb{D}}$, that is analytic in $\mathbb{D}$. We are interested in determining a "closed form" for the value of the $\mathscr{H}\tilde{G}(z)$ where $\tilde{G}$ is the 2π-periodic continuous functions defined by $\tilde{G}(x) = G(e^{ix})$. This is done in the next theorem:

THEOREM 2.6. *Let $G : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ be a continuous function on the closed unit disk $\overline{\mathbb{D}}$, that is analytic in $\mathbb{D}$. For every z with $\Im z \neq 0$, one has*

$$\mathscr{H}\tilde{G}(z) = i\pi\big(2\chi_{P^+}(z)G(e^{iz}) - G(0)\big) \tag{2.14}$$

where $\tilde{G}(x) = G(e^{ix})$, and χ_{P^+} is the characteristic function of the upper half plane: $P^+ = \{w : \Im w > 0\}$.

Proof. Note that G has a power series expansion $G(z) = \sum_{n=0}^{\infty} c_n z^n$ (with radius of convergence greater or equal to 1.) This implies that the Fourier series expansion of the continuous, 2π-periodic function $\tilde{G}$ is given by:

$$S[\tilde{G}](x) = \sum_{n=0}^{\infty} c_n e^{inx}.$$

Thus, the conclusion of the theorem follows immediately from Theorem 2.3. □

Similarly, from Corollary 2.4 we obtain the next result:

COROLLARY 2.7. *Let $G : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ be a continuous function on the closed unit disk which is analytic on $\mathbb{D}$. For every complex number z with $\Re z > 0$, one has*

$$\int_{-\infty}^{\infty} \frac{G(e^{ix})}{x^2 + z^2}\, dx = \frac{\pi}{z} G(e^{-z}), \tag{2.15}$$

and

$$\int_{-\infty}^{\infty} \frac{xG(e^{ix})}{x^2 + z^2}\, dx = i\pi\left(G(e^{-z}) - G(0)\right), \tag{2.16}$$

3. Examples and applications

EXAMPLE 3.1. Consider the 2π-periodic function $f(x) = 1/(1 + 8\sin^2 x)$, it is straightforward to see that

$$\begin{aligned}\frac{1}{1+8\sin^2 x} &= \frac{-e^{2ix}}{(e^{2ix}-2)(2e^{2ix}-1)}\\ &= \frac{1}{3}\left(\frac{1}{2e^{2ix}-1} - \frac{2}{e^{2ix}-2}\right) = \frac{1}{3}\sum_{n\in\mathbb{Z}} 2^{-|n|} e^{2inx}.\end{aligned}$$

Thus, applying Corollary 2.4 with $z=1$ we obtain

$$\int_{-\infty}^{\infty}\frac{f(x)}{1+x^2}\,dx=\frac{\pi}{3}\sum_{n\in\mathbb{Z}}\left(2e^2\right)^{-|n|}=\frac{\pi}{3}\cdot\frac{2e^2+1}{2e^2-1},$$

which is formula (1.1) of our example in the introduction (after the substitution $x=\tan t$).

This example can easily be generalized by considering the analytic function G defined on the domain $\Omega=\mathbb{C}\setminus\{e^{2\alpha}\}$, by

$$G(z)=\frac{e^{2\alpha}+z}{e^{2\alpha}-z}. \tag{3.1}$$

where α is some positive real. Here, it is straightforward to check that

$$G(e^{ix})=\frac{\sinh 2\alpha+i\sin x}{\cosh 2\alpha-\cos x}, \tag{3.2}$$

so, from Corollary 2.7, we conclude that, for $\lambda>0$, we have

$$\int_{-\infty}^{\infty}\frac{G(e^{ix})}{4\lambda^2+x^2}dx=\frac{\pi}{2\lambda}\cdot\frac{e^{2\alpha}+e^{-2\lambda}}{e^{2\alpha}-e^{-2\lambda}}, \tag{3.3}$$

$$\int_{-\infty}^{\infty}\frac{xG(e^{ix})}{4\lambda^2+x^2}dx=i\pi\cdot\frac{2e^{-2\lambda}}{e^{2\alpha}-e^{-2\lambda}}. \tag{3.4}$$

Formula (3.2) implies, using parity and making the change of variables $x\leftarrow 2x$ in (3.3):

$$\int_0^{\infty}\frac{1}{\sinh^2\alpha+\sin^2 x}\cdot\frac{dx}{\lambda^2+x^2}=\frac{\pi}{\lambda\sinh 2\alpha}\cdot\frac{e^{2\alpha}+e^{-2\lambda}}{e^{2\alpha}-e^{-2\lambda}}. \tag{3.5}$$

Similarly, from (3.4), we find that, for $\lambda>0$

$$\int_0^{\infty}\frac{x\sin x}{(\cosh 2\alpha-\cos x)(\lambda^2+x^2)}\,dx=\frac{\pi}{e^{2\alpha+\lambda}-1}. \tag{3.6}$$

The integrals (3.5) and (3.6) are listed in [6, formulæ 3.792(10) and 3.792(13), page 450], with the notation $\sinh(\alpha)=\mu$ or $\cosh(2\alpha)=\mu$ respectively.

EXAMPLE 3.2. In our second example we consider the even function $f\in L^1(\mathbb{T})$ defined by

$$f(x)=-\frac{1}{2}\ln\sin^2\left(\frac{x}{2}\right),\quad \text{for } x\notin 2\pi\mathbb{Z}. \tag{3.7}$$

It is well-known, (see for instance [12, Ch. 3, Sec. 14],) that f has the Fourier series expansion:

$$S[f](x)=\ln 2+\sum_{n=1}^{\infty}\frac{\cos nx}{n}.$$

Thus, applying Corollary 2.5 we obtain that, for $\lambda > 0$, we have

$$\int_0^\infty \frac{\ln(\sin^2(x/2))}{4\lambda^2+x^2}\,dx = -\frac{\pi}{2\lambda}\left(\ln 2+\sum_{n=1}^\infty \frac{e^{-2n\lambda}}{n}\right) = -\frac{\pi}{2\lambda}\left(\ln 2-\ln(1-e^{-2\lambda})\right).$$

The change of variables $x \leftarrow 2x$ yields:

$$\int_0^\infty \frac{\ln(\sin^2 x)}{\lambda^2+x^2}\,dx = \frac{\pi}{\lambda}\ln\left(\frac{1-e^{-2\lambda}}{2}\right), \qquad \text{for } \lambda > 0. \tag{3.8}$$

EXAMPLE 3.3. (*Numerical Quadrature*) In this third example we aim to illustrate the use of Theorem 2.3 and Corollary 2.4 to calculate a numerical approximation of a highly oscillatory integral. Consider the 1-periodic even function f defined by $f(x) = \exp(\{x\}(1-\{x\}))$ where $\{x\}$ is the fractional part of x. We propose to calculate numerically the value of the integral:

$$I = \int_0^\infty \frac{f(x)}{1+x^2}\,dx = \pi\int_{\mathbb{R}} \frac{1}{4\pi^2+x^2} f\left(\frac{x}{2\pi}\right)dx$$

According to Corollary 2.4 we have

$$\begin{aligned} I &= \frac{\sinh(2\pi)}{4}\int_{\mathbb{T}} \frac{1}{\cosh(2\pi)-\cos(x)} f\left(\frac{x}{2\pi}\right)dx \\ &= \frac{\pi\sinh(2\pi)}{2}\int_0^1 \frac{e^{t(1-t)}}{\cosh(2\pi)-\cos(2\pi t)}\,dt \\ &= \frac{\sqrt[4]{e}\,\pi\sinh(2\pi)}{2}\int_0^1 \frac{e^{-x^2/4}}{\cosh(2\pi)+\cos(\pi x)}\,dx \end{aligned}$$

where we used the change of variables $x \leftarrow 2t-1$ in the last line. To calculate the resulting I to 51 digits we can use the trapezoidal rule of numerical quadrature, after transforming the integrand in two steps: First, we map integral over $[0,1]$ onto the real line by the transformation $x = 1/(1+e^{-u})$. Then, we accelerate the decay of the integrand by a sinh transformation [13]. This yields

$$I = 1.86040\,77510\,49016\,03858\,77820\,00396\,84086\,80005\,27974\,88674.$$

The calculation took less than one twentieth of a second using Wolfram Mathematica® on a 2.83 GHz personal computer.

Our next example is a generalization of an old problem.

EXAMPLE 3.4. (*A Generalization of A Problem of Narayana Aiyar*) Let $t_1, t_2, \ldots, t_n$ and a be positive real numbers, such that

$$0 < t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n < a.$$

Consider the meromorphic function G defined by

$$G(z) = \frac{1}{(a - t_1 z)(a - t_2 z)\cdots(a - t_n z)}. \tag{3.9}$$

Clearly, G is analytic in the domain $\Omega = \mathbb{C} \setminus \left\{\frac{a}{t_k} : 1 \leqslant k \leqslant n\right\}$ that contains the closed unit disk. For a given real x, let

$$\phi_k(x) = \arctan\left(\frac{t_k \sin x}{a - t_k \cos x}\right), \qquad \rho_k(x) = \sqrt{a^2 - 2t_k a \cos x + t_k^2}. \tag{3.10}$$

To simplify the notation, we will simply write ϕ_k and ρ_k to denote $\phi_k(x)$ and $\rho_k(x)$ respectively. It is clear that

$$a - t_k e^{ix} = \rho_k e^{-i\phi_k}, \qquad \text{for } 1 \leqslant k \leqslant n. \tag{3.11}$$

Thus,

$$G(e^{ix}) = \frac{e^{i(\phi_1 + \cdots + \phi_n)}}{\rho_1 \cdots \rho_n}. \tag{3.12}$$

Using Theorem 2.6, with $z = -i$, we obtain

$$\int_{-\infty}^{\infty} \frac{G(e^{ix})}{x + i}\, dx = -i\pi G(0) = -\frac{i\pi}{a^n}.$$

Taking imaginary parts, we conclude that

$$\int_0^{\infty} \frac{\cos(\phi_1 + \cdots + \phi_n) - x\sin(\phi_1 + \cdots + \phi_n)}{\rho_1 \cdots \rho_n} \frac{dx}{1 + x^2} = \frac{\pi}{2a^n}. \tag{3.13}$$

The evaluation of the integral (3.13), when $a = 1$ and $t_k = kr$ for some $0 < r < 1/n$, is an unsolved problem proposed by Narayana Aiyar in the beginning of the twentieth century [4], while the generalization, corresponding to $a > 0$ and $t_k = kr$ for some $0 < r < a/n$, is a problem proposed by M. D. Hirchhorn [7].

Note that Corollary 2.7 yields more precise results, valid for $\Re z > 0$, namely:

$$\int_0^{\infty} \frac{\cos(\phi_1 + \cdots + \phi_n)}{\rho_1 \cdots \rho_n} \cdot \frac{dx}{x^2 + z^2} = \frac{\pi}{2z} \prod_{k=1}^{n} \frac{1}{a - t_k e^{-z}}, \tag{3.14}$$

and

$$\int_0^{\infty} \frac{\sin(\phi_1 + \cdots + \phi_n)}{\rho_1 \cdots \rho_n} \cdot \frac{x\,dx}{x^2 + z^2} = \frac{\pi}{2} \left(\prod_{k=1}^{n} \frac{1}{a - t_k e^{-z}} - \frac{1}{a^n}\right), \tag{3.15}$$

where $t_1, \ldots, t_n$ are real numbers from the interval $(0, a)$ and the ϕ_k's and ρ_k's are defined by (3.10).

EXAMPLE 3.5. (*Bernoulli Polynomials and Polylogarithms*) In this example we consider the sequence $(B_m)_{m\geqslant 0}$ of *Bernoulli polynomials*. They can be defined via the generating function

$$\frac{te^{tx}}{e^t-1}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}, \tag{3.16}$$

or recursively by $B_0=1$ and

$$\forall n>0, \qquad B_n'=nB_{n-1}, \quad \text{and} \quad \int_0^1 B_n(t)\,dt=0. \tag{3.17}$$

In particular,

$$B_1(X)=X-\frac{1}{2}, \quad B_2(X)=X^2-X+\frac{1}{6}, \qquad B_3(X)=X(X-\frac{1}{2})(X-1). \tag{3.18}$$

Now, denote by $\widetilde{B}_m$ the 2π-periodic function defined by

$$\widetilde{B}_m(x)=B_m\left(\left\{\frac{x}{2\pi}\right\}\right), \quad \text{where } \{u\} \text{ is the fractional part of } u. \tag{3.19}$$

For $m\geqslant 1$, the Fourier series expansion of $\widetilde{B}_m$, is well-known and easy to find (using the recursive definition (3.17), see, for example [1, Ch. 23].) We have

$$S[\widetilde{B}_{2m-1}](x)=\frac{(-1)^m2(2m-1)!}{(2\pi)^{2m-1}}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^{2m-1}}, \tag{3.20}$$

$$S[\widetilde{B}_{2m}](x)=\frac{(-1)^m2(2m)!}{(2\pi)^{2m}}\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^{2m}}. \tag{3.21}$$

Applying Corollary 2.5 we conclude that, for $m\geqslant 1$ and $\Re z>0$, we have

$$\int_0^{\infty}\frac{x}{x^2+z^2}\widetilde{B}_{2m-1}(x)\,dx=\frac{(-1)^m(2m-1)!}{2(2\pi)^{2m-2}}\sum_{n=1}^{\infty}\frac{e^{-nz}}{n^{2m-1}},$$

$$\int_0^{\infty}\frac{1}{x^2+z^2}\widetilde{B}_{2m}(x)\,dx=\frac{(-1)^m(2m)!}{2z(2\pi)^{2m-1}}\sum_{n=1}^{\infty}\frac{e^{-nz}}{n^{2m}}.$$

The change of variables $x\leftarrow 2\pi x$ and $z\leftarrow 2\pi z$ yields the next result, for $m\geqslant 1$ and $\Re z>0$:

$$\int_0^{\infty}\frac{xB_{2m-1}(\{x\})}{x^2+z^2}\,dx=(-1)^m\frac{(2m-1)!}{2(2\pi)^{2m-2}}\operatorname{Li}_{2m-1}(e^{-2\pi z}), \tag{3.22}$$

$$\int_0^{\infty}\frac{B_{2m}(\{x\})}{x^2+z^2}\,dx=(-1)^m\frac{(2m)!}{2(2\pi)^{2m-1}}\frac{\operatorname{Li}_{2m}(e^{-2\pi z})}{z}, \tag{3.23}$$

where the function Li_k is the polylogarithm of order k. It is defined on open unit disk by the series $\sum_{n=1}^{\infty}z^n/n^k$. (For an extensive account of the polylogarithms see [10].)

In particular, since $B_1(X) = X - \frac{1}{2}$, we see that for $\lambda > 0$, we have

$$\int_0^\infty \frac{x(\{x\}-1/2)}{\lambda^2+x^2}\,dx = \frac{1}{2}\ln(1-e^{-2\pi\lambda}). \tag{3.24}$$

Also, from the expressions of B_2 and B_3 we conclude that, for $\lambda > 0$, we have

$$\int_0^\infty \frac{\{x\}(1-\{x\})}{\lambda^2+x^2}\,dx = \frac{\pi}{12\lambda} + \frac{1}{2\pi\lambda}\operatorname{Li}_2(e^{-2\pi\lambda}). \tag{3.25}$$

$$\int_0^\infty \frac{x\{x\}(\{x\}-1)(\{x\}-1/2)}{\lambda^2+x^2}\,dx = \frac{3}{4\pi^2}\operatorname{Li}_3(e^{-2\pi\lambda}). \tag{3.26}$$

Adding one forth of (3.24) to (3.26) we see that, for $\lambda > 0$, we have

$$\int_0^\infty \frac{x(\{x\}-1/2)^3}{\lambda^2+x^2}\,dx = \frac{1}{8}\ln(1-e^{-2\pi\lambda}) + \frac{3}{4\pi^2}\operatorname{Li}_3(e^{-2\pi\lambda}). \tag{3.27}$$

EXAMPLE 3.6. (*A link to Bessel functions*) Let $(J_n)_{n\in\mathbb{Z}}$ be the family of Bessel functions of the first kind. It is well-known that the generating function of this family is given by

$$e^{\frac{1}{2}\xi(t-1/t)} = \sum_{n\in\mathbb{Z}} t^n J_n(\xi), \quad \text{for } t \neq 0, \tag{3.28}$$

(see [1, formula 9.1.41].) Using Corollary 2.5 we conclude that, for $\xi \in \mathbb{C}$ and $\Re z > 0$ we have

$$\int_{\mathbb{R}} \frac{e^{i\xi\cos x}}{z^2+x^2}\,dx = \frac{\pi}{z}\left(J_0(\xi) + 2\sum_{n=1}^{\infty} i^n J_n(\xi)e^{-nz}\right). \tag{3.29}$$

Acknowledgement.

The author is very grateful to Professor Jörg Waldvogel for his helpful suggestions and valuable comments that greatly improved the content and the form of this article.

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(Received February 27, 2013)

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