SOME INEQUALITIES FOR THE GENERALIZED SINE AND THE GENERALIZED HYPERBOLIC SINE

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Abstract. In this paper, the authors establish some new inequalities for the generalized sine and the generalized hyperbolic sine.

1. Introduction

During the last years, many authors have studied the generalized trigonometric functions introduced by D. Lindqvist in a highly cited papers [9]. Let $1 < p < \infty$, we can define the function as follows:

$$\arcsin_p(x) = \int_0^x \frac{1}{(1-t^p)^{1/p}} \, dt, \quad 0 \leq x \leq 1,$$

(1.1)

and

$$\frac{\pi_p}{2} = \arcsin_p(1) = \int_0^1 \frac{1}{(1-t^p)^{1/p}} \, dt.$$  

(1.2)

The inverse of $\arcsin_p$ on $[0, \pi_p/2]$ is called the generalized sine function and denoted $\sin_p$. By standard extension procedure as the sine function, we get a differentiable function on the whole of $(-\infty, +\infty)$. It is easy to see that the function $\sin_p$ is strictly increasing and concave on $[0, \pi_p/2]$. In the same way, we can define the generalized cosine function, the generalized tangent function and their inverses, and also the corresponding hyperbolic functions.

The generalized cosine function $\cos_p$ is defined as

$$\cos_p(x) = \frac{d}{dx} \sin_p(x), \quad x \in [0, \pi_p/2].$$

(1.3)

It is easy to see that

$$\cos_p(x) = (1 - (\sin_p(x))^p)^{1/p}, \quad x \in [0, \pi_p/2]$$

(1.4)

and

$$\frac{d}{dx} \cos_p(x) = -(\cos_p(x))^{2-p}(\sin_p(x))^{p-1}, \quad x \in [0, \pi_p/2].$$

(1.5)


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The generalized tangent function \( \tan_p \) is defined as
\[
\tan_p(x) = \frac{\sin_p(x)}{\cos_p(x)}, \quad x \in \mathbb{R} \setminus \left\{ k\pi_p + \frac{\pi_p}{2} : k \in \mathbb{Z} \right\}.
\] (1.6)

Similarly, the generalized inverse hyperbolic sine function
\[
\arcsinh_p(x) = \begin{cases} \int_0^x \frac{1}{(1+t^p)^{1/p}} \, dt, & x \geq 0, \\ -\arcsinh_p(-x), & x < 0. \end{cases}
\] (1.7)

The inverse of \( \arcsin_p \) is called the generalized hyperbolic sine function and denoted \( \sinh_p \). The generalized hyperbolic cosine function is defined as
\[
\cosh_p(x) = \frac{d}{dx} \sinh_p(x).
\] (1.8)

For above definition, we easily obtain
\[
(cosh_p(x))^p - |\sinh_p(x)|^p = 1, \quad x \in \mathbb{R},
\] (1.9)
and
\[
\frac{d}{dx} \cosh_p(x) = (\cosh_p(x))^{2-p}(\sinh_p(x))^{p-1}, \quad x \geq 0.
\] (1.10)

For more, the reader may see the references [4], [5], [6], [7], [8].

2. Lemmas

**Lemma 2.1.** Let the nonempty number set \( D \subseteq (0, \infty) \), the mapping \( f : D \rightarrow J \subseteq (0, \infty) \) is a bijective function. Assume that function \( \left( \frac{f(x)}{x^k} \right) \) \( (x \in D, \ k > 0) \) is strictly increasing. Then

1. If \( f(x) \geq y \) for all \( x \in D \), then \( x^k y \leq f(x)(f^{-1}(y))^k \) where \( f^{-1} : J \rightarrow D \) denotes the inverse function of \( f \):
2. If \( f(x) \leq y \) for all \( x \in D \), then \( x^k y \geq f(x)(f^{-1}(y))^k \).

**Proof.** First of all, we prove the first part of Lemma. Since the function \( \frac{f(x)}{x^k} \) is strictly increasing, the function \( f(x) \) must be strictly increasing, too. (In fact, if \( x_1 < x_2 \), then \( \frac{f(x_1)}{x_1^k} < \frac{f(x_2)}{x_2^k} \), so \( f(x_1) < \frac{x_1^k}{x_2^k} f(x_2) < f(x_2) \).) Thus, \( f^{-1}(x) \) is also strictly increasing.

Taking \( t = f^{-1}(y) \), monotonicity property of \( f^{-1}(x) \) implies \( t \leq x \). So \( \frac{f(t)}{t^k} \leq \frac{f(x)}{x^k} \), i.e. \( x^k y \leq f(x)(f^{-1}(y))^k \).

The proof of second part of Lemma is similar to (1). The proof is complete. \( \square \)

**Remark 2.1.** If \( k = 1 \), we obtain Theorem 2.1 of [11].
Lemma 2.2. [4, Lemma 2.2.] For \( p > 1 \) and \( x \in (0,1) \), we have

(1) \( \left( 1 + \frac{x^p}{p(1+p)} \right) x < \arcsin_p(x) < \frac{\pi_p}{2} x; \)
(2) \( z \left( 1 + \frac{\log(1+x^p)}{1+p} \right) < \arcsinh_p(x) < z \left( 1 + \frac{1}{p} \log(1+x^p) \right), \quad z = \left( \frac{x^p}{1+x^p} \right)^{1/p}. \)

3. Main results

Theorem 3.1. For \( p > 1, k \leq 1 \) and \( x \in (0,1) \), then

\[
\frac{x}{\arcsin_p(x)} > \left( \sin_p \left( \frac{\pi_p x}{2} \right) \right)^k.
\]

Proof. Let \( D = (0,1) \) and \( f(x) = \arcsin_p(x) \). Direct computation yields

\[
\left( \frac{f(x)}{x^k} \right)' = \frac{1}{x^{k+1}} \left( \frac{x^k}{(1-x^p)^{1/p}} - k \arcsin_p(x)x^{k-1} \right) = \frac{1}{x^{k+1}} g(x)
\]
where \( g(x) = \frac{x}{(1-x^p)^{1/p}} - k \arcsin_p(x) \) and \( g(0) = 0 \). For \( k \leq 1 \),

\[
g'(x) = \frac{1-k+x^p(1-x^p)^{-1}}{(1-x^p)^{1/p}} > 0
\]
implies \( g(x) > g(0) = 0 \), and hence \( \frac{f(x)}{x^k} \) is strictly increasing.

Taking \( y = \frac{\pi_p x}{2} \), we have

\[
x^k \frac{\pi_p x}{2} > \arcsin_p(x) \left( \sin_p \left( \frac{\pi_p x}{2} \right) \right)^k
\]
which implies inequality (3.1) by using (1) of Lemma 2.1 and (1) of Lemma 2.2.  \( \square \)

Theorem 3.2. If \( f(x) \leq y \) for all \( p > 1, k \leq \frac{1}{2} \) and \( x \in (0,1) \), \( y \in (0, \pi_p/2) \), then

\[
x^k y \geq \arcsinh_p(x) (\sinh_p(y))^k.
\]

Proof. By differentiation and easy computation, we have

\[
\left( \frac{\arcsinh_p(x)}{x} \right)' = \frac{1}{x^{k+1}} \left( \frac{x}{(1+x^p)^{1/p}} - k \arcsinh_p(x) \right) = \frac{1}{x^{k+1}} h(x)
\]
and

\[
h'(x) = \frac{1}{(1+x^p)^{1/p}} \left( 1 - k - \frac{x^p}{1+x^p} \right)
\]
where \( h(x) = \frac{x}{(1+x^p)^{1/p}} - k \arcsinh_p(x) \) and \( h(0) = 0 \).
Putting
\[ \lambda(x) = 1 - k - \frac{x^p}{1 + x^p}, \]  
we have
\[ \lambda'(x) = \frac{-px^{p-1}}{(1 + x^p)^{2/p}} < 0 \]  
which implies \( \lambda(x) > \lambda(1) = \frac{1}{2} - k \geq 0 \), and hence \( \frac{\text{arcsinh}_p(x)}{x^k} \) is strictly increasing. Using Lemma 2.1, we easily obtain inequality (3.5). □

**Remark 3.1.** By Lemma 2.2, we have
\[ \text{arcsinh}_p(x) < \left( \frac{x^p}{1 + x^p} \right)^{1/p} \left( 1 + \frac{1}{p} \log(1 + x^p) \right) \]
\[ < x \left( 1 + \frac{1}{p} \log(1 + x^p) \right) < x \left( 1 + \frac{\log 2}{p} \right) = y. \]
So, (3.5) of Theorem 3.2 becomes
\[ x^{k+1} \left( 1 + \frac{\log 2}{p} \right) > \text{arcsinh}_p(x) \left( \sinh_p \left( x \left( 1 + \frac{\log 2}{p} \right) \right) \right)^k \]  
or
\[ \frac{x}{\text{arcsinh}_p(x)} > \left( \frac{\sinh_p \left( x \left( 1 + \frac{\log 2}{p} \right) \right)}{1 + \frac{\log 2}{p}} \right)^k. \]  

**Theorem 3.3.** Let \( p > 1, q > 1 \) satisfy \( \frac{1}{p} + \frac{1}{q} = 1 \). For any \( x \in (0,1) \), then
\[ \frac{x}{2p} B_{x^2p} \left( \frac{1}{2p}, 1 - \frac{1}{p} \right) \leq \text{arcsinh}_p(x) \text{arcsinh}_p(x) < x^{1+1/q} \left( -\ln(1-x) \right)^{1/p} \]  
where \( B_{x^2p} \left( \frac{1}{2p}, 1 - \frac{1}{p} \right) \) is incomplete beta function.

**Proof.** For the first inequality, it is easy to see that the function \( \frac{1}{(1-t^p)^{1/p}} \) is strictly increasing and \( \frac{1}{(1+t^p)^{1/p}} \) is strictly decreasing on \( t \in (0,1) \). Using integral expression of \( \text{arcsinh}_p(x) \), \( \text{arcsinh}_p(x) \) and Tchebychef’s inequality, we have
\[ \text{arcsinh}_p(x) \text{arcsinh}_p(x) = \int_0^x \frac{1}{(1-t^p)^{1/p}} dt \int_0^x \frac{1}{(1+t^p)^{1/p}} dt \]
\[ \geq \frac{1}{x} \int_0^x \frac{1}{(1-t^p)^{1/p}} dt 2^{p-u} \frac{x}{2p} \int_0^{x^2p} (1-u)^{-1/p} u^{(1/2p)-1} du \]
\[ = \frac{x}{2p} B_{x^2p} \left( \frac{1}{2p}, 1 - \frac{1}{p} \right). \]
For the second inequality, using Hölder’s inequality, we have
\[
\arcsin_p(x) \arcsinh_p(x) = \int_0^x \frac{1}{(1-t^p)^{1/p}} dt \int_0^x \frac{1}{(1+t^p)^{1/p}} dt
\]
\[
\leq \left( \int_0^x \frac{1}{1-t^p} dt \right)^{1/p} \left( \int_0^x \frac{1}{1+t^p} dt \right)^{1/q} \left( \int_0^x \frac{1}{1+t^p} dt \right)^{1/q} \left( \int_0^x \frac{1}{1+t^p} dt \right)^{1/q}
\]
\[
= x^{2/q} \left( \int_0^x \frac{1}{1-t^p} dt \int_0^x \frac{1}{1+t^p} dt \right)^{1/p}
\]
\[
< x^{2/q} \left( \int_0^x \frac{1}{1-t} dt \int_0^x \frac{1}{1+t} dt \right)^{1/p} = x^{1+1/q} (-\ln (1-x))^{1/p}. \quad \square
\]

Finally, we pose an open problem.

**Open Problem 3.1.** For all \( p \in (1, 2] \) and \( x \in (0, \pi/p/2) \), then
\[
\frac{\ln (1 - \sin_p(x))}{\ln \cos_p(x)} < \frac{x + p}{x}. \quad (3.13)
\]

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**References**


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