Journal of Classical Analysis
Volume 3, Number 1 (2013), 85–90 doi:10.7153/jca-03-07

SOME INEQUALITIES FOR THE GENERALIZED SINE AND THE GENERALIZED HYPERBOLIC SINE

LI YIN AND LI-GUO HUANG

Abstract. In this paper, the authors establish some new inequalities for the generalized sine and the generalized hyperbolic sine.

1. Introduction

During the last years, many authors have studied the generalized trigonometric functions introduced by D. Lindqvist in a highly cited papers [9]. Let $1 < p < \infty$, we can define the function as follows:

$$\operatorname{arcsin}_p(x) = \int_0^x \frac{1}{(1-t^p)^{1/p}} dt, \quad 0 \leqslant x \leqslant 1, \tag{1.1}$$

and

$$\frac{\pi_p}{2} = \operatorname{arcsin}_p(1) = \int_0^1 \frac{1}{(1-t^p)^{1/p}} dt. \tag{1.2}$$

The inverse of arcsin_p on $[0, \pi_p/2]$ is called the generalized sine function and denoted $\sin_p$. By standard extension procedure as the sine function, we get a differentiable function on the whole of $(-\infty, +\infty)$. It is easy to see that the function $\sin_p$ is strictly increasing and concave on $[0, \pi_p/2]$. In the same way, we can define the generalized cosine function, the generalized tangent function and their inverses, and also the corresponding hyperbolic functions.

The generalized cosine function $\cos_p$ is defined as

$$\cos_p(x) = \frac{d}{dx} \sin_p(x), \quad x \in [0, \pi_p/2]. \tag{1.3}$$

It is easy to see that

$$\cos_p(x) = (1 - (\sin_p(x))^p)^{1/p}, \quad x \in [0, \pi_p/2] \tag{1.4}$$

and

$$\frac{d}{dx} \cos_p(x) = -(\cos_p(x))^{2-p} (\sin_p(x))^{p-1}, \quad x \in [0, \pi_p/2]. \tag{1.5}$$

Mathematics subject classification (2010): 26A51, 26D05, 33B10.
Keywords and phrases: Generalized sine, generalized hyperbolic sine, inverse function, inequalities.
Research of the first author is supported by NSF of Shandong Province under grant numbers ZR2011AL001, and by the Science Foundation of Binzhou University under grant BZXYL1303.

The generalized tangent function $\tan_p$ is defined as

$$\tan_p(x) = \frac{\sin_p(x)}{\cos_p(x)}, \quad x \in \mathbb{R}\backslash\{k\pi_p + \tfrac{\pi_p}{2} : k \in \mathbb{Z}\}. \tag{1.6}$$

Similarly, the generalized inverse hyperbolic sine function

$$\operatorname{arcsinh}_p(x) = \begin{cases} \displaystyle\int_0^x \frac{1}{(1+t^p)^{1/p}} dt\,,\ x \geqslant 0, \\ -\operatorname{arcsinh}_p(-x)\,,\ x < 0. \end{cases} \tag{1.7}$$

The inverse of $\arcsin_p$ is called the generalized hyperbolic sine function and denoted $\sinh_p$. The generalized hyperbolic cosine function is defined as

$$\cosh_p(x) = \frac{d}{dx}\sinh_p(x). \tag{1.8}$$

For above definition, we easily obtain

$$(\cosh_p(x))^p - \left|\sinh_p(x)\right|^p = 1\,,\ x \in \mathbb{R}, \tag{1.9}$$

and

$$\frac{d}{dx}\cosh_p(x) = (\cosh_p(x))^{2-p}(\sinh_p(x))^{p-1}\,,\ x \geqslant 0. \tag{1.10}$$

For more, the reader may see the references [4], [5], [6], [7], [8].

2. Lemmas

LEMMA 2.1. *Let the nonempty number set* $D \subseteq (0,\infty)$, *the mapping* $f : D \longrightarrow J \subseteq (0,\infty)$ *is a bijective function. Assume that function* $\frac{f(x)}{x^k}$ $(x \in D,\ k > 0)$ *is strictly increasing. Then*

(1) *If* $f(x) \geqslant y$ *for all* $x \in D$, *then* $x^k y \leqslant f(x)(f^{-1}(y))^k$ *where* $f^{-1} : J \longrightarrow D$ *denotes the inverse function of* f;

(2) *If* $f(x) \leqslant y$ *for all* $x \in D$, *then* $x^k y \geqslant f(x)(f^{-1}(y))^k$.

Proof. First of all, we prove the first part of Lemma. Since the function $\frac{f(x)}{x^k}$ is strictly increasing, the function $f(x)$ must be strictly increasing, too. (In fact, if $x_1 < x_2$, then $\frac{f(x_1)}{x_1^k} < \frac{f(x_2)}{x_2^k}$, so $f(x_1) < \frac{x_1^k}{x_2^k} f(x_2) < f(x_2)$.) Thus, $f^{-1}(x)$ is also strictly increasing.

Taking $t = f^{-1}(y)$, monotonicity property of $f^{-1}(x)$ implies $t \leqslant x$. So $\frac{f(t)}{t^k} \leqslant \frac{f(x)}{x^k}$, i.e. $x^k y \leqslant f(x)(f^{-1}(y))^k$.

The proof of second part of Lemma is similar to (1). The proof is complete. □

REMARK 2.1. If $k = 1$, we obtain Theorem 2.1 of [11].

LEMMA 2.2. [4, Lemma 2.2.] *For* $p > 1$ *and* $x \in (0,1)$, *we have*
(1) $\left(1+\frac{x^p}{p(1+p)}\right)x < \operatorname{arcsin}_p(x) < \frac{\pi_p}{2}x;$
(2) $z\left(1+\frac{\log(1+x^p)}{1+p}\right) < \operatorname{arcsinh}_p(x) < z\left(1+\frac{1}{p}\log(1+x^p)\right),\ z=\left(\frac{x^p}{1+x^p}\right)^{1/p}.$

3. Main results

THEOREM 3.1. *For* $p > 1,\ k \leqslant 1$ *and* $x \in (0,1)$, *then*

$$\frac{x}{\operatorname{arcsin}_p(x)} > \frac{\left(\sin_p\left(\frac{\pi_p x}{2}\right)\right)^k}{\frac{\pi_p}{2}x^k}. \tag{3.1}$$

Proof. Let $D=(0,1)$ and $f(x)=\operatorname{arcsin}_p(x)$. Direct computation yields

$$\left(\frac{f(x)}{x^k}\right)' = \frac{1}{x^{2k}}\left(\frac{x^k}{(1-x^p)^{1/p}} - \operatorname{arcsin}_p(x)kx^{k-1}\right) = \frac{1}{x^{k+1}}g(x) \tag{3.2}$$

where $g(x)=\frac{x}{(1-x^p)^{1/p}} - k\operatorname{arcsin}_p(x)$ and $g(0)=0$. For $k \leqslant 1$,

$$g'(x) = \frac{1-k+x^p(1-x^p)^{-1}}{(1-x^p)^{1/p}} > 0 \tag{3.3}$$

implies $g(x) > g(0) = 0$, and hence $\frac{f(x)}{x^k}$ is strictly increasing.

Taking $y=\frac{\pi_p x}{2}$, we have

$$x^k\frac{\pi_p x}{2} > \operatorname{arcsin}_p(x)\left(\sin_p\left(\frac{\pi_p x}{2}\right)\right)^k \tag{3.4}$$

which implies inequality (3.1) by using (1) of Lemma 2.1 and (1) of Lemma 2.2. □

THEOREM 3.2. *If* $f(x) \leqslant y$ *for all* $p > 1,\ k \leqslant \frac{1}{2}$ *and* $x \in (0,1),\ y \in (0,\pi_p/2)$, *then*

$$x^k y \geqslant \operatorname{arcsinh}_p(x)\left(\sinh_p(y)\right)^k. \tag{3.5}$$

Proof. By differentiation and easy computation, we have

$$\left(\frac{\operatorname{arcsinh}_p(x)}{x}\right)' = \frac{1}{x^{k+1}}\left(\frac{x}{(1+x^p)^{1/p}} - k\operatorname{arcsinh}_p(x)\right) = \frac{1}{x^{k+1}}h(x) \tag{3.6}$$

and

$$h'(x) = \frac{1}{(1+x^p)^{1/p}}\left(1-k-\frac{x^p}{1+x^p}\right) \tag{3.7}$$

where $h(x)=\frac{x}{(1+x^p)^{1/p}} - k\operatorname{arcsinh}_p(x)$ and $h(0)=0$.

Putting

$$\lambda(x) = 1 - k - \frac{x^p}{1+x^p}, \tag{3.8}$$

we have

$$\lambda'(x) = \frac{-px^{p-1}}{(1+x^p)^{2/p}} < 0 \tag{3.9}$$

which implies $\lambda(x) > \lambda(1) = \frac{1}{2} - k \geqslant 0$, and hence $\frac{\operatorname{arcsinh}_p(x)}{x^k}$ is strictly increasing. Using Lemma 2.1, we easily obtain inequality (3.5). $\square$

REMARK 3.1. By Lemma 2.2, we have

$$\begin{aligned}\operatorname{arcsinh}_p(x) &< \left(\frac{x^p}{1+x^p}\right)^{1/p}\left(1+\frac{1}{p}\log(1+x^p)\right) \\ &< x\left(1+\frac{1}{p}\log(1+x^p)\right) < x\left(1+\frac{\log 2}{p}\right) = y.\end{aligned}$$

So, (3.5) of Theorem 3.2 becomes

$$x^{k+1}\left(1+\frac{\log 2}{p}\right) > \operatorname{arcsinh}_p(x)\left(\sinh_p\left(x\left(1+\frac{\log 2}{p}\right)\right)\right)^k \tag{3.10}$$

or

$$\frac{x}{\operatorname{arcsinh}_p(x)} > \frac{\left(\sinh_p\left(x\left(1+\frac{\log 2}{p}\right)\right)\right)^k}{\left(1+\frac{\log 2}{p}\right)x^k}. \tag{3.11}$$

THEOREM 3.3. *Let $p > 1$, $q > 1$ satisfy $1/p + 1/q = 1$. For any $x \in (0,1)$, then*

$$\frac{x}{2p}B_{x^{2p}}\left(\frac{1}{2p}, 1-\frac{1}{p}\right) \leqslant \arcsin_p(x)\operatorname{arcsinh}_p(x) < x^{1+1/q}(-\ln(1-x))^{1/p} \tag{3.12}$$

where $B_{x^{2p}}\left(\frac{1}{2p}, 1-\frac{1}{p}\right)$ is incomplete beta function.

Proof. For the first inequality, it is easy to see that the function $\frac{1}{(1-t^p)^{1/p}}$ is strictly increasing and $\frac{1}{(1+t^p)^{1/p}}$ is strictly decreasing on $t \in (0,1)$. Using integral expression of $\arcsin_p(x), \operatorname{arcsinh}_p(x)$ and Tchebychef's inequality, we have

$$\begin{aligned}\arcsin_p(x)\operatorname{arcsinh}_p(x) &= \int_0^x \frac{1}{(1-t^p)^{1/p}}dt \int_0^x \frac{1}{(1+t^p)^{1/p}}dt \\ &\geqslant \frac{1}{x}\int_0^x \frac{1}{(1-t^{2p})^{1/p}}dt \overset{t^{2p}=u}{=} \frac{x}{2p}\int_0^{x^{2p}} (1-u)^{-1/p}u^{(1/2p)-1}du \\ &= \frac{x}{2p}B_{x^{2p}}\left(\frac{1}{2p}, 1-\frac{1}{p}\right).\end{aligned}$$

For the second inequality, using Hölder's inequality, we have

$$\operatorname{arcsin}_p(x)\operatorname{arcsinh}_p(x) = \int_0^x \frac{1}{(1-t^p)^{1/p}}dt \int_0^x \frac{1}{(1+t^p)^{1/p}}dt$$
$$\leqslant \left(\int_0^x \frac{1}{1-t^p}dt\right)^{1/p} \left(\int_0^x 1^q dt\right)^{1/q} \left(\int_0^x \frac{1}{1+t^p}dt\right)^{1/p} \left(\int_0^x 1^q dt\right)^{1/q}$$
$$= x^{2/q}\left(\int_0^x \frac{1}{1-t^p}dt \int_0^x \frac{1}{1+t^p}dt\right)^{1/p}$$
$$< x^{2/q}\left(\int_0^x \frac{1}{1-t}dt \int_0^x 1dt\right)^{1/p} = x^{1+1/q}(-\ln(1-x))^{1/p}. \quad \square$$

Finally, we pose an open problem.

OPEN PROBLEM 3.1. *For all* $p \in (1,2]$ *and* $x \in (0, \pi_p/2)$, *then*

$$\frac{\ln(1-\sin_p(x))}{\ln\cos_p(x)} < \frac{x+p}{x}. \tag{3.13}$$

Acknowledgements.

The authors appreciate the referee for his helpful and valuable comments on this manuscript.

REFERENCES

[1] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Topics in special functions*, Papers on Analysis: A volume dedicated to Olli Martio on the occasion of his 60th birthday, ed. by J. Heinonen, T. Kilpeläinen, and P. Koskela, Report 83, Univ. Jyväskylä (2001), 5–26, available online at `http://arxiv.org/abs/0712.3856`.

[2] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Special functions of quasiconformal theory*, Expo. Math., **7** (1989), 97–136.

[3] G. D. ANDERSON, M. VUORINEN AND X.-H. ZHANG, *Topics in special functions III*, available online at `http://arxiv.org/abs/1209.1696`.

[4] B. A. BHAYO AND M. VUORINEN, *Power mean inequality of generalized trigonometric functions*, available online at `http://arxiv.org/abs/1209.0873`.

[5] B. A. BHAYO AND M. VUORINEN, *On generalized trigonometric functions with two parameters*, J. Approx. Theory **164** (2012), no. 10, 1415–1426, available online at `http://arxiv.org/abs/1112.0483`.

[6] B. A. BHAYO AND M. VUORINEN, *Inequalities for eigenfunctions of the p-Laplacian*, available online at `http://arxiv.org/abs/1101.3911`.

[7] W.-D. JIANG, M.-K. WANG, Y.-M. CHU, Y.-P. JIANG, AND F. QI, *Convexity of the generalized sine function and the generalized hyperbolic sine function*, J. Approx. Theory, **174** (2013), 1–9, available online at `http://arxiv.org/abs/1301.3264`.

[8] R. KLÉN, M. VUORINEN AND X.-H. ZHANG, *Inequalities for the generalized trigonometric and hyperbolic functions*, J. Math. Anal. Appl., **409** (2014), no. 1, 521–529, available online at `http://arxiv.org/abs/1210.6749`.

[9] P. LINDQVIST, *Some remarkable sine and cosine functions*, Ricerche di Matematica Vol. **XLIV** (1995), 269–290.

[10] D. S. MITRINOVIĆ, *Analytic inequalities*, Springer Verlag, New York, 1970.

[11] J. SÁNDOR, *On certain inequalities for hyperbolic and trigonometric functions*, J. Math. Inequal. **7** (2013), no. 3, 241–245.

(Received May 4, 2013)

L. Yin
Department of Mathematics and Information Science
Binzhou University
Binzhou City, Shandong Province
256603, China
e-mail: yinli_79@163.com

L. G. Huang
Department of Mathematics and Information Science
Binzhou University
Binzhou City, Shandong Province
256603, China
e-mail: liguoh123@sina.com