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SOME GROWTH PROPERTIES OF WRONSKIANS USING THEIR RELATIVE ORDER

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Abstract. In the paper we establish some newly developed results based on the growth properties of relative order, relative type and relative weak type of wronskians generated by entire and meromorphic functions.

1. Introduction, definitions and notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$.

The following definitions are well known.

DEFINITION 1. The order ρ_f and lower order λ_f of a meromorphic function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r} .$$

DEFINITION 2. The type σ_f and lower type $\overline{\sigma}_f$ of a meromorphic function f are defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}} \quad \text{and} \quad \overline{\sigma}_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty .$$

In this connection Datta and Jha [1] gave the definition of weak type of a meromorphic function of finite positive lower order in the following way:

DEFINITION 3. [1] The weak type τ_f of a meromorphic function f of finite positive lower order λ_f is defined by

$$\tau_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}} .$$

Similarly one can define the growth indicator $\overline{\tau}_f$ of a meromorphic function f of finite positive lower order λ_f as

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}} .$$

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For an entire function g, the Nevanlinna's characteristic function $T_g(r)$ is defined as $T_g(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|g(re^{i\theta})\right| d\theta$ where $\log^+ x = \max(0, \log x)$ for $x > 0$.

If g is non-constant then $T_g(r)$ is strictly increasing and continuous and its inverse $T_g^{-1} : (T_g(0), \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} T_g^{-1}(s) = \infty$.

Lahiri and Banerjee [4] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

DEFINITION 4. [4] Let f be meromorphic and g be entire. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as

$$\begin{aligned}\rho_g(f) &= \inf\left\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\right\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one [4] if $g(z) = \exp z$.

Similarly one can define the relative lower order of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

Datta and Biswas [2] gave the definition of relative type and relative weak type of a meromorphic function with respect to an entire function g which are as follows:

DEFINITION 5. [2] The relative type $\sigma_g(f)$ of a meromorphic function f with respect to an entire function g are defined as

$$\sigma_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}}, \quad \text{where } 0 < \rho_g(f) < \infty.$$

Similarly one can define the lower relative type $\overline{\sigma}_g(f)$ in the following way

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}}, \quad \text{where } 0 < \rho_g(f) < \infty.$$

DEFINITION 6. [2] The relative weak type $\tau_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ is defined by

$$\tau_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}}.$$

Analogously one can define the growth indicator $\overline{\tau}_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ as

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}}.$$

The following definitions are also well known:

DEFINITION 7. A meromorphic function $a = a(z)$ is called small with respect to f if $T(r,a) = S(r,f)$.

DEFINITION 8. Let $a_1, a_2, ...a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, ...a_k, f)$, the Wronskian determinant of $a_1, a_2, ..., a_k,\ f$ i.e.,

$$L(f) = \begin{vmatrix} a_1 & a_2 & \cdots & a_k & f \\ a_1' & a_2' & \cdots & a_k' & f' \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_1^{(k)} & a_2^{(k)} & \cdots & a_k^{(k)} & f^{(k)} \end{vmatrix} .$$

DEFINITION 9. If $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T_f(r)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T_f(r)}$$

is called the Nevanlinna's deficiency of the value "a".

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) \leqslant 2$ (cf [3], p. 43). If in particular, $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

In the paper we establish some newly developed results based on the growth properties of relative order (relative lower order) relative type (relative lower type) and relative weak type of wronskians generated by entire and meromorphic functions. We do not explain the standard notations and definitions in the theory of entire and meromorphic functions because those are available in [3] and [6].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

LEMMA 1. [5] *Let f be a transcendental meromorphic function having the maximum deficiency sum. Then*

$$\lim_{r\to\infty} \frac{T_{L(f)}(r)}{T_f(r)} = 1 + k - k\delta(\infty; f) .$$

LEMMA 2. [1] *If f be a meromorphic function of regular growth i.e., $\rho_f = \lambda_f$ then*

$$\sigma_f = \overline{\sigma}_f = \tau_f = \overline{\tau}_f .$$

3. Theorems

In this section we present the main results of the paper.

THEOREM 1. *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be a transcendental entire function with $0 < \tau_g \leqslant \overline{\tau}_g < \infty$ and $0 < \overline{\sigma}_g \leqslant \sigma_g < \infty$. Also let $\sum_{a \neq \infty} \delta(a;g) + \delta(\infty;g) = 2$. Then*

$$\max\left\{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\lambda_g}} \cdot \left(\frac{\tau_g}{\overline{\tau}_g}\right)^{\frac{1}{\lambda_g}}, \left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}} \cdot \left(\frac{\overline{\sigma}_g}{\sigma_g}\right)^{\frac{1}{\rho_g}}\right\}$$

$$\leqslant \liminf_{r\to\infty} \frac{T_{L(g)}^{-1} T_{L(f)}(r)}{T_g^{-1} T_f(r)} \leqslant \limsup_{r\to\infty} \frac{T_{L(g)}^{-1} T_{L(f)}(r)}{T_g^{-1} T_f(r)}$$

$$\leqslant \min\left\{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\lambda_g}} \cdot \left(\frac{\overline{\tau}_g}{\tau_g}\right)^{\frac{1}{\lambda_g}}, \left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}} \cdot \left(\frac{\sigma_g}{\overline{\sigma}_g}\right)^{\frac{1}{\rho_g}}\right\},$$

where $L(f) = W\left(a_1, a_2, \ldots. a_{k_1}, f\right)$ and $L(g) = W\left(a_1, a_2, \ldots. a_{k_2}, g\right)$.

Proof. For any $\varepsilon(>0)$ we get from Lemma 1 for all sufficiently large values of r

$$T_{L(f)}(r) \leqslant \{1+k_1-k_1\delta(\infty;f)+\varepsilon\} T_f(r) \tag{1}$$

and

$$T_{L(f)}(r) \geqslant \{1+k_1-k_1\delta(\infty;f)-\varepsilon\} T_f(r). \tag{2}$$

Also from Lemma 1 we get for all sufficiently large values of r that

$$T_{L(g)}(r) \geqslant \{1+k_2-k_2\delta(\infty;g)-\varepsilon\} T_g(r)$$

$$i.e.,\ r \geqslant T_{L(g)}^{-1}\left[\{1+k_2-k_2\delta(\infty;g)-\varepsilon\} T_g(r)\right]$$

$$i.e.,\ T_g^{-1}\left(\frac{r}{1+k_2-k_2\delta(\infty;g)-\varepsilon}\right) \geqslant T_{L(g)}^{-1}(r). \tag{3}$$

and

$$T_{L(g)}(r) \leqslant \{1+k_2-k_2\delta(\infty;g)+\varepsilon\} T_g(r)$$

$$i.e.,\ r \leqslant T_{L(g)}^{-1}\left[\{1+k_2-k_2\delta(\infty;g)+\varepsilon\} T_g(r)\right]$$

$$i.e.,\ T_g^{-1}\left(\frac{r}{1+k_2-k_2\delta(\infty;g)+\varepsilon}\right) \leqslant T_{L(g)}^{-1}(r). \tag{4}$$

Now from (1) and (3) it follows for all sufficiently large values of r,

$$T_{L(g)}^{-1} T_{L(f)}(r) \leqslant T_{L(g)}^{-1}\left[\{1+k_1-k_1\delta(\infty;f)+\varepsilon\} T_f(r)\right]$$

$$i.e.,\ T_{L(g)}^{-1} T_{L(f)}(r) \leqslant T_g^{-1}\left[\left(\frac{1+k_1-k_1\delta(\infty;f)+\varepsilon}{1+k_2-k_2\delta(\infty;g)-\varepsilon}\right) T_f(r)\right]. \tag{5}$$

Again from (2) and (4) it follows for all sufficiently large values of r,

$$T_{L(g)}^{-1}T_{L(f)}(r) \geqslant T_{L(g)}^{-1}\left[\{1+k_1-k_1\delta(\infty;f)-\varepsilon\}T_f(r)\right]$$

$$i.e.,\ T_{L(g)}^{-1}T_{L(f)}(r) \geqslant T_g^{-1}\left[\left(\frac{1+k_1-k_1\delta(\infty;f)-\varepsilon}{1+k_2-k_2\delta(\infty;g)+\varepsilon}\right)T_f(r)\right]. \tag{6}$$

Now for the definition of type and lower type we get for all sufficiently large values of r that

$$T_g\left(\left\{\frac{T_f(r)}{(\sigma_g+\varepsilon)}\right\}^{\frac{1}{\rho_g}}\right) \leqslant T_f(r)$$

$$i.e.,\quad T_g^{-1}T_f(r) \geqslant \left\{\frac{T_f(r)}{(\sigma_g+\varepsilon)}\right\}^{\frac{1}{\rho_g}} \tag{7}$$

and

$$T_g\left(\left\{\left(\frac{1+k_1-k_1\delta(\infty;f)+\varepsilon}{(1+k_2-k_2\delta(\infty;g)-\varepsilon)(\overline{\sigma}_g-\varepsilon)}\right)T_f(r)\right\}^{\frac{1}{\rho_g}}\right)$$

$$\geqslant \left[\left(\frac{1+k_1-k_1\delta(\infty;f)+\varepsilon}{1+k_2-k_2\delta(\infty;g)-\varepsilon}\right)T_f(r)\right]$$

$$i.e.,\ \left[\left(\frac{1+k_1-k_1\delta(\infty;f)+\varepsilon}{(1+k_2-k_2\delta(\infty;g)-\varepsilon)(\overline{\sigma}_g-\varepsilon)}\right)T_f(r)\right]^{\frac{1}{\rho_g}}$$

$$\geqslant T_g^{-1}\left[\left(\frac{1+k_1-k_1\delta(\infty;f)+\varepsilon}{1+k_2-k_2\delta(\infty;g)-\varepsilon}\right)T_f(r)\right]. \tag{8}$$

Therefore from (5) and (8) it follows for all sufficiently large values of r,

$$T_{L(g)}^{-1}T_{L(f)}(r) \leqslant \left[\left(\frac{1+k_1-k_1\delta(\infty;f)+\varepsilon}{(1+k_2-k_2\delta(\infty;g)-\varepsilon)(\overline{\sigma}_g-\varepsilon)}\right)T_f(r)\right]^{\frac{1}{\rho_g}}. \tag{9}$$

Therefore from (7) and (9) it follows for all sufficiently large values of r,

$$\frac{T_{L(g)}^{-1}T_{L(f)}(r)}{T_g^{-1}T_f(r)} \leqslant \frac{\left[\left(\frac{1+k_1-k_1\delta(\infty;f)+\varepsilon}{(1+k_2-k_2\delta(\infty;g)-\varepsilon)(\overline{\sigma}_g-\varepsilon)}\right)T_f(r)\right]^{\frac{1}{\rho_g}}}{\left\{\frac{T_f(r)}{(\sigma_g+\varepsilon)}\right\}^{\frac{1}{\rho_g}}}$$

$$i.e.,\ \frac{T_{L(g)}^{-1}T_{L(f)}(r)}{T_g^{-1}T_f(r)} \leqslant \left(\frac{(1+k_1-k_1\delta(\infty;f)+\varepsilon)(\sigma_g+\varepsilon)}{(1+k_2-k_2\delta(\infty;g)-\varepsilon)(\overline{\sigma}_g-\varepsilon)}\right)^{\frac{1}{\rho_g}}$$

$$i.e.,\ \limsup_{r\to\infty}\frac{T_{L(g)}^{-1}T_{L(f)}(r)}{T_g^{-1}T_f(r)} \leqslant \left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\sigma_g}{\overline{\sigma}_g}\right)^{\frac{1}{\rho_g}}. \tag{10}$$

Similarly from (6) it can also be shown that for all sufficiently large values of r,

$$\liminf_{r\to\infty}\frac{T_{L(g)}^{-1}T_{L(f)}(r)}{T_g^{-1}T_f(r)} \geqslant \left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\overline{\sigma}_g}{\sigma_g}\right)^{\frac{1}{\rho_g}}. \tag{11}$$

Therefore from (10) and (11) we obtain that

$$\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\overline{\sigma}_g}{\sigma_g}\right)^{\frac{1}{\rho_g}} \leqslant \liminf_{r\to\infty}\frac{T_{L(g)}^{-1}T_{L(f)}(r)}{T_g^{-1}T_f(r)}$$

$$\leqslant \limsup_{r\to\infty}\frac{T_{L(g)}^{-1}T_{L(f)}(r)}{T_g^{-1}T_f(r)} \leqslant \left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\sigma_g}{\overline{\sigma}_g}\right)^{\frac{1}{\rho_g}}. \tag{12}$$

Similarly using the weak type one can easily verify that

$$\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\lambda_g}}\cdot\left(\frac{\tau_g}{\overline{\tau}_g}\right)^{\frac{1}{\lambda_g}} \leqslant \liminf_{r\to\infty}\frac{T_{L(g)}^{-1}T_{L(f)}(r)}{T_g^{-1}T_f(r)}$$

$$\leqslant \limsup_{r\to\infty}\frac{T_{L(g)}^{-1}T_{L(f)}(r)}{T_g^{-1}T_f(r)} \leqslant \left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\lambda_g}}\cdot\left(\frac{\overline{\tau}_g}{\tau_g}\right)^{\frac{1}{\lambda_g}}. \tag{13}$$

Thus the theorem follows from (12) and (13). □

COROLLARY 1. *Under the same conditions of Theorem 1 if g is of regular growth then by Lemma 2 one can easily verify that*

$$\lim_{r\to\infty}\frac{T_{L(g)}^{-1}T_{L(f)}(r)}{T_g^{-1}T_f(r)} = \left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}.$$

THEOREM 2. *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be a transcendental entire function with $0<\lambda_g\leqslant\rho_g<\infty$. Also let $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then*

$$\frac{\lambda_g}{\rho_g}\leqslant\liminf_{r\to\infty}\frac{\log T_{L(g)}^{-1}T_{L(f)}(r)}{\log T_g^{-1}T_f(r)}\leqslant\limsup_{r\to\infty}\frac{\log T_{L(g)}^{-1}T_{L(f)}(r)}{\log T_g^{-1}T_f(r)}\leqslant\frac{\rho_g}{\lambda_g}.$$

Proof. From (5) and (6) we get for all sufficiently large values of r that

$$\log T_{L(g)}^{-1}T_{L(f)}(r)\leqslant\log T_g^{-1}\left[\left(\frac{1+k_1-k_1\delta(\infty;f)+\varepsilon}{1+k_2-k_2\delta(\infty;g)-\varepsilon}\right)T_f(r)\right] \tag{14}$$

and

$$\log T_{L(g)}^{-1}T_{L(f)}(r)\geqslant\log T_g^{-1}\left[\left(\frac{1+k_1-k_1\delta(\infty;f)-\varepsilon}{1+k_2-k_2\delta(\infty;g)+\varepsilon}\right)T_f(r)\right]. \tag{15}$$

Now for the definition of order and lower order we get for all sufficiently large values of r that

$$T_g\left(\left\{T_f(r)\right\}^{\frac{1}{\rho_g+\varepsilon}}\right) \leqslant T_f(r)$$

$$i.e., \quad \log T_g^{-1}T_f(r) \geqslant \frac{1}{(\rho_g+\varepsilon)}\log T_f(r). \tag{16}$$

and

$$T_g\left(\left\{\left(\frac{1+k_1-k_1\delta(\infty;f)+\varepsilon}{(1+k_2-k_2\delta(\infty;g)-\varepsilon)(\overline{\sigma}_g-\varepsilon)}\right)T_f(r)\right\}^{\frac{1}{\lambda_g-\varepsilon}}\right) \geqslant \left[\left(\frac{1+k_1-k_1\delta(\infty;f)+\varepsilon}{1+k_2-k_2\delta(\infty;g)-\varepsilon}\right)T_f(r)\right]$$

$$i.e., \left[\left(\frac{1+k_1-k_1\delta(\infty;f)+\varepsilon}{(1+k_2-k_2\delta(\infty;g)-\varepsilon)(\overline{\sigma}_g-\varepsilon)}\right)T_f(r)\right]^{\frac{1}{\lambda_g-\varepsilon}} \geqslant T_g^{-1}\left[\left(\frac{1+k_1-k_1\delta(\infty;f)+\varepsilon}{1+k_2-k_2\delta(\infty;g)-\varepsilon}\right)T_f(r)\right]$$

$$i.e., \frac{1}{(\lambda_g-\varepsilon)}\log T_f(r)+O(1) \geqslant \log T_g^{-1}\left[\left(\frac{1+k_1-k_1\delta(\infty;f)+\varepsilon}{1+k_2-k_2\delta(\infty;g)-\varepsilon}\right)T_f(r)\right]. \tag{17}$$

Therefore from (14) and (17) it follows for all sufficiently large values of r,

$$\log T_{L(g)}^{-1}T_{L(f)}(r) \leqslant \frac{1}{(\lambda_g-\varepsilon)}\log T_f(r)+O(1). \tag{18}$$

Therefore from (16) and (18) it follows for all sufficiently large values of r,

$$\frac{\log T_{L(g)}^{-1}T_{L(f)}(r)}{\log T_g^{-1}T_f(r)} \leqslant \left(\frac{\rho_g+\varepsilon}{\lambda_g-\varepsilon}\right)\cdot\frac{\log T_f(r)+O(1)}{\log T_f(r)}$$

$$i.e., \limsup_{r\to\infty}\frac{\log T_{L(g)}^{-1}T_{L(f)}(r)}{\log T_g^{-1}T_f(r)} \leqslant \frac{\rho_g}{\lambda_g}. \tag{19}$$

Similarly from (15) it can be shown for all sufficiently large values of r,

$$\liminf_{r\to\infty}\frac{\log T_{L(g)}^{-1}T_{L(f)}(r)}{\log T_g^{-1}T_f(r)} \geqslant \frac{\lambda_g}{\rho_g}. \tag{20}$$

Therefore from (19) and (20) we obtain that

$$\frac{\lambda_g}{\rho_g} \leqslant \liminf_{r\to\infty}\frac{\log T_{L(g)}^{-1}T_{L(f)}(r)}{\log T_g^{-1}T_f(r)} \leqslant \limsup_{r\to\infty}\frac{\log T_{L(g)}^{-1}T_{L(f)}(r)}{\log T_g^{-1}T_f(r)} \leqslant \frac{\rho_g}{\lambda_g}.$$

Thus the theorem follows from above. □

COROLLARY 2. *Under the same conditions of Theorem 2 if g is of regular growth then one may get that*

$$\lim_{r\to\infty} \frac{\log T_{L(g)}^{-1} T_{L(f)}(r)}{\log T_g^{-1} T_f(r)} = 1 .$$

THEOREM 3. *If f be a transcendental meromorphic function with the maximum deficiency sum and g be a transcendental entire function of regular growth having non zero finite order and $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$, then the relative order and relative lower order of $L(f)$ with respect to $L(g)$ are same as those of f with respect to g.*

Proof. By Corollary 2 we get that

$$\begin{aligned}
\rho_{L[g]}(L[f]) &= \limsup_{r\to\infty} \frac{\log T_{L(g)}^{-1} T_{L(f)}(r)}{\log r} \\
&= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} \cdot \lim_{r\to\infty} \frac{\log T_{L(g)}^{-1} T_{L(f)}(r)}{\log T_g^{-1} T_f(r)} \\
&= \rho_g(f).1 = \rho_g(f).
\end{aligned}$$

In a similar manner, $\lambda_{L[g]}(L[f]) = \lambda_g(f)$.

Thus the theorem follows. □

THEOREM 4. *Let f be a transcendental meromorphic function with the maximum deficiency sum and g be a transcendental entire function of regular growth having non zero finite type and $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$. Then the relative type and relative lower type of $L(f)$ with respect to $L(g)$ are $\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g if $\rho_g(f)$ is positive finite.*

Proof. From Corollary 1 and Theorem 3 we get that

$$\begin{aligned}
\sigma_{L(g)}(L(f)) &= \limsup_{r\to\infty} \frac{T_{L(g)}^{-1} T_{L(f)}(r)}{r^{\rho_{L[g]}(L[f])}} \\
&= \lim_{r\to\infty} \frac{T_{L(g)}^{-1} T_{L(f)}(r)}{T_g^{-1} T_f(r)} \cdot \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}} \\
&= \left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}} \sigma_g(f).
\end{aligned}$$

Similarly $\overline{\sigma}_{L(g)}(L(f)) = \left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}} .\overline{\sigma}_g(f)$.

This proves the theorem. □

THEOREM 5. *If f be a transcendental meromorphic function with the maximum deficiency sum and g be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$, *then* $\tau_{L(g)}(L(f))$ *and* $\overline{\tau}_{L(g)}(L(f))$ *are* $\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ *times that of f with respect to g i.e.,* $\tau_{L(g)}(L(f))=\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}.\tau_g(f)$ *and* $\overline{\tau}_{L(g)}(L(f))=\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}.\overline{\tau}_g(f)$ *when $\lambda_g(f)$ is positive finite.*

We omit the proof of Theorem 5 because it can be carried out in the line of Theorem 4.

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