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MAJORIZATION RESULTS FOR SOME CLASSES OF MULTIVALENT FUNCTIONS

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Abstract. In this paper we obtain some majorization results for some classes of multivalent functions defined by certain differential operator.

1. Introduction

Let $A(p,j)$ be the class of functions which are analytic and p-valent in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z^p + \sum_{k=p+j}^{\infty} a_k z^k \quad (p, j \in \mathbb{N} = \{1, 2, \ldots\}). \tag{1.1}$$

For $g \in A(p,j)$, given by $g(z) = z^p + \sum_{k=p+j}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=p+j}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.2}$$

For $f \in A(p,j)$, we have (see [7])

$$f^{(q)}(z) = \delta(p,q) z^{p-q} + \sum_{k=p+j}^{\infty} \delta(k,q) a_k z^{k-q} \; (q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; p > q), \tag{1.3}$$

where

$$\delta(x,y) = \frac{x!}{(x-y)!} = \begin{cases} 1 & (y = 0) \\ x(x-1)\ldots(x-y+1) & (y \neq 0). \end{cases}$$

For $f \in A(p,j)$, Aouf ([4] and [5]) defined the operator $D_p^n f^{(q)}$ as follows:

$$D_p^0 f^{(q)}(z) = f^{(q)}(z);$$
$$D_p^1 f^{(q)}(z) = D_p f^{(q)}(z) = \frac{z}{(p-q)} (f^{(q)}(z))' = \frac{z}{(p-q)} f^{(q+1)}(z)$$

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and (in general):

$$\begin{aligned} D_p^n f^{(q)}(z) &= D_p(D_p^{(n-1)} f^{(q)}(z)) \\ &= \delta(p,q)z^{p-q} + \sum_{k=p+j}^{\infty} \delta(k,q) \left(\frac{k-q}{p-q}\right)^n a_k z^{k-q} \\ &(p, j \in \mathbb{N};\ q,n \in \mathbb{N}_0;\ p > q). \end{aligned} \tag{1.4}$$

We note that, for $q = 0$, $D_p^n f^{(0)}(z) = D_p^n f(z)$, where the operator D_p^n was introduced and studied by Kamali and Orhan [11] and Aouf and Mostafa [6] which reduces to the Salagean operator D^n for $p = 1$ (see [17]).

From (1.4), one can easily verify that

$$z\left(D_p^n f^{(q)}(z)\right)' = (p-q) D_p^{n+1} f^{(q)}(z). \tag{1.5}$$

For two analytic functions f, $g \in A(p,j)$, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence (see [13]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

If f and g are analytic functions in U, then f majorized by g in U and written

$$f(z) << g(z), \tag{1.6}$$

if there exists a function ϕ, analytic in U, such that (see [12]):

$$|\phi(z)| \leqslant 1 \text{ and } f(z) = \phi(z) g(z) \quad (z \in U). \tag{1.7}$$

It is noted that the notation of majorization is closely related to the concept of quasi-subordination between analytic functions.

DEFINITION 1. For $\gamma \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$, $-1 \leqslant B < A \leqslant 1$, $p \in \mathbb{N}$, $n,q \in \mathbb{N}_0$, $p > q$ and $|\gamma(A-B) + B(p-q)| \leqslant p-q$, a function $f \in A(p,j)$ is said to be in the class $S_{p,q}^n(\gamma, A, B)$ of p-valently functions in U, if and only if

$$1 + \frac{1}{\gamma}\left(\frac{z\left(D_p^n f^{(q)}(z)\right)'}{D_p^n f^{(q)}(z)} - p + q\right) \prec \frac{1+Az}{1+Bz}, \tag{1.8}$$

where $D_p^n f^{(q)}$ is given by (1.4).

Specializing the parameters n, p, q, A, B and γ, we have the following classes:

i) $S_{p,0}^n(\gamma, A, B) = S_p^n(\gamma, A, B) = \left\{ f \in A(p,j) : 1 + \frac{1}{\gamma}\left(\frac{z\left(D_p^n f(z)\right)'}{D_p^n f(z)} - p\right) \prec \frac{1+Az}{1+Bz} \right\};$

ii) $S_{p,q}^{n}(\gamma,1,-1)=S_{p,q}^{n}(\gamma)=\left\{f\in A(p,j):\operatorname{Re}\left[1+\frac{1}{\gamma}\left(\frac{z\left(D_p^n f^{(q)}(z)\right)'}{D_p^n f^{(q)}(z)}-p+q\right)\right]>0\right\}$;

iii) $S_{p,0}^{0}((p-\alpha)\cos\lambda e^{-i\lambda},1,-1)=S_p^{\lambda}(\alpha)$
$=\left\{f\in A(p,j):\operatorname{Re}\left(e^{i\lambda}\frac{zf'(z)}{f(z)}\right)>\alpha\cos\lambda\ (|\lambda|<\frac{\pi}{2};0\leqslant\alpha<p)\right\}$
(see Srivastava et al. [18]);

iv) $S_{p,0}^{1}((p-\alpha)\cos\lambda e^{-i\lambda},1,-1)=C_p^{\lambda}(\alpha)$
$=\left\{f\in A(p,j):\operatorname{Re}\left\{e^{i\lambda}(1+\frac{zf''(z)}{f'(z)})\right\}>\alpha\cos\lambda\ (|\lambda|<\frac{\pi}{2};0\leqslant\alpha<p)\right\}$
(see Srivastava et al. [18]);

v) $S_{p,q}^{n}((p-q-\alpha)\cos\lambda e^{-i\lambda},1,-1)=S_{p,q}^{n,\lambda}(\alpha)$
$=\left\{f\in A(p,j):\operatorname{Re}\left(e^{i\lambda}\frac{z\left(D_p^n f^{(q)}(z)\right)'}{D_p^n f^{(q)}(z)}\right)>\alpha\cos\lambda\ (|\lambda|<\frac{\pi}{2};0\leqslant\alpha<p-q)\right\}$;

vi) $S_{p,q}^{0}(\gamma,1,-1)=S_{p,q}(\gamma)$ (see Altintas and Srivastava [2]);

vii) $S_{p,0}^{n}(\gamma,1,-1)=S_n(p,\gamma)$ (see Akbulut et al. [3]);

viii) $S_{1,0}^{0}(\gamma,1,-1)=S(\gamma)$ (see Nasr and Aouf [14]);

ix) $S_{1,0}^{1}(\gamma,1,-1)=C(\gamma)$ (see Nasr and Aouf [14] and Wiatrowski [19]);

x) $S_{1,0}^{0}(1-\alpha,1,-1)=S^{*}(\alpha)$ $(0\leqslant\alpha<1)$ (see Robertson [16]).

Majorization problems for the class $S^{*}=S^{*}(0)$ had been investigated by MacGregogor [12]. Recently Altintas et al. [1] investigated a majorization problem for the class $S(\gamma)$. Very recently Goyal and Goswami [10] generalized these results for the fractional operator (see also, Goswami and Aouf [8]). In this paper we investigated majorization problems for the class $S_{p,q}^{n}(\gamma,A,B)$ and some of its subclasses.

2. Main results

Unless otherwise mentioned, we assume that $\gamma\in\mathbb{C}^{*}$, $-1\leqslant B<A\leqslant 1$, $p\in\mathbb{N}$, $n,q\in\mathbb{N}_0$ and $p>q$.

THEOREM 1. *Let the function $f\in A(p,j)$ and $g\in S_{p,q}^{n}(\gamma,A,B)$. If $D_p^n f^{(q)}$ is majorized by $D_p^n g^{(q)}$, then*

$$\left|D_p^{n+1}f^{(q)}(z)\right|\leqslant\left|D_p^{n+1}g^{(q)}(z)\right|\quad(|z|\leqslant r_0),\tag{2.1}$$

where $r_0=r_0(p,q,\gamma,A,B)$ is the smallest positive root of the equation

$$|\gamma(A-B)+B(p-q)|r^3-(2|B|+p-q)r^2-[2+|\gamma(A-B)+B(p-q)|]r+p-q=0.\tag{2.2}$$

Proof. Since $g \in S^n_{p,q}(\gamma,A,B)$, then it follows from (1.8) that

$$1+\frac{1}{\gamma}\left(\frac{z\left(D^n_p g^{(q)}(z)\right)'}{D^n_p g^{(q)}(z)}-p+q\right)=\frac{1+Aw(z)}{1+Bw(z)}, \tag{2.3}$$

where $w(z)=c_1 z+c_2 z^2+\ldots \in P$ and P denotes the well known class of bounded analytic functions in U which satisfy $w(0)=0$ and $|w(z)|\leqslant 1$.

From (2.3), we have:

$$\frac{z\left(D^n_p g^{(q)}(z)\right)'}{D^n_p g^{(q)}(z)}=\frac{p-q+[\gamma(A-B)+B(p-q)]w(z)}{(1+Bw(z))}. \tag{2.4}$$

Also from (1.5) and (2.4), we have

$$\left|D^n_p g^{(q)}(z)\right|\leqslant\frac{(p-q)(1+|B|\,|z|)}{p-q-|\gamma(A-B)+B(p-q)|\,|z|}\left|D^{n+1}_p g^{(q)}(z)\right|. \tag{2.5}$$

Since, $D^n_p f^{(q)}$ is majorized by $D^n_p g^{(q)}$, then we have

$$D^n_p f^{(q)}(z)=\phi(z)D^n_p g^{(q)}(z). \tag{2.6}$$

Differentiating (2.6) with respect to z and then multiplying by z, we get

$$z\left(D^n_p f^{(q)}(z)\right)'=z\phi'(z)D^n_p g^{(q)}(z)+\phi(z)z\left(D^n_p g^{(q)}(z)\right)'. \tag{2.7}$$

Noting that the Schwarz function ϕ satisfies (see [15])

$$\left|\phi'(z)\right|\leqslant\frac{1-|\phi(z)|^2}{1-|z|^2}, \tag{2.8}$$

and using (1.5), (2.5) and (2.8) in (2.7), we have

$$\left|D^{n+1}_p f^{(q)}(z)\right|\leqslant\left\{|\phi(z)|+\frac{|z|\left(1-|\phi(z)|^2\right)}{1-|z|^2}\,\frac{(1+B|z|)}{p-q-|\gamma(A-B)+B(p-q)||z|}\right\}\left|D^{n+1}_p g^{(q)}(z)\right|. \tag{2.9}$$

Setting $|z|=r$ and $|\phi(z)|=\rho$ $(0\leqslant\rho\leqslant 1)$, (2.9) reduces to

$$\left|D^{n+1}_p f^{(q)}(z)\right|\leqslant\frac{\Psi(r,\rho)}{(1-r^2)\,[p-q-|\gamma(A-B)+B(p-q)|\,r]}\left|D^{n+1}_p g^{(q)}(z)\right|, \tag{2.10}$$

where

$$\Psi(r,\rho)=\rho(1-r^2)\,[p-q-|\gamma(A-B)+B(p-q)|\,r]+r(1-\rho^2)(1+r\,|B|)$$

takes its maximum value at $\rho = 1$ with $r = r_0(p,q,\gamma,A,B)$ given by (2.2). Furthermore, if $0 \leqslant \sigma \leqslant r_0(p,q,\gamma,A,B)$, the the function $\Phi(\rho)$ defined by

$$\Phi(\rho) = \rho(1-\sigma^2)\left[p-q-|\gamma(A-B)+B(p-q)|\sigma\right] + \sigma(1-\rho^2)(1+\sigma|B|)$$

is an increasing function on $0 \leqslant \rho \leqslant 1$, so that

$$\Phi(\rho) \leqslant \Phi(1) = (1-\sigma^2)\left[p-q-|\gamma(A-B)+B(p-q)|\sigma\right], \\ 0 \leqslant \rho \leqslant 1; 0 \leqslant \sigma \leqslant r_0(p,q,\gamma,A,B).$$

Then, setting $\rho = 1$ in (2.10), we conclude that (2.1) holds true for $|z| \leqslant r_0(p,q,\gamma,A,B)$. This completes the proof of Theorem 1. $\square$

Putting $q = 0$ in Theorem 1, we have the following corollary:

COROLLARY 1. *Let the function $f \in A(p,j)$ and $g \in S_p^n(\gamma,A,B)$. If $D_p^n f$ is majorized by $D_p^n g$ then*

$$\left|D_p^{n+1} f(z)\right| \leqslant \left|D_p^{n+1} g(z)\right| \quad (|z| \leqslant r_1),$$

where $r_1 = r_1(p,\gamma,A,B)$ is the smallest positive root of the equation:

$$|\gamma(A-B)+Bp|r^3 - [2|B|+p]r^2 - [2+|\gamma(A-B)+Bp|]r + p = 0.$$

Putting $A = 1$ and $B = -1$ in Theorem 1, (2.2) becomes

$$|2\gamma+q-p|r^3 - (2+p-q)r^2 - [2+|2\gamma+q-p|]r + p - q = 0, \tag{2.11}$$

which has $r = -1$ one of its roots and the other two roots are given by

$$|2\gamma+q-p|r^2 - [|2\gamma+q-p|+2+p-q]r + p - q = 0.$$

Since it may find the smallest postive root of (2.11), we have the following corollary:

COROLLARY 2. *Let the function $f \in A(p,j)$ and $g \in S_{p,q}^n(\gamma)$. If $D_p^n f^{(q)}$ is majorized by $D_p^n g^{(q)}$, then*

$$\left|D_p^{n+1} f^{(q)}(z)\right| \leqslant \left|D_p^{n+1} g^{(q)}(z)\right| \quad (|z| \leqslant r_2),$$

where $r_2 = r_2(p,q,\gamma)$ is given by

$$r_2 = \frac{\xi - \{\xi^2 - 4(p-q)|2\gamma+q-p|\}^{\frac{1}{2}}}{2|2\gamma+q-p|},$$

where $\xi = |2\gamma+q-p| + 2 + p - q$.

REMARK 1. Putting $n = q = 0$ and $p = 1$ in Corollary 2, we obtain the result of Goswami et al. [9, Corollary 2.3] which for $\gamma = 1$ reduces to the result of MacGregor [12].

Putting $q = 0$, $\gamma = (p-\alpha)\cos\lambda e^{-i\lambda}$ ($|\lambda| < \frac{\pi}{2}$, $0 \leqslant \alpha < p$) in Corollary 2, we have the following corollary:

COROLLARY 3. *Let the function* $f \in A(p,j)$ *and* $g \in S_p^{n,\lambda}(\alpha)$. *If* $D_p^n f$ *is majorized by* $D_p^n g$, *then*

$$\left|D_p^{n+1} f(z)\right| \leqslant \left|D_p^{n+1} g(z)\right| \quad (|z| \leqslant r_3),$$

where $r_3 = r_3(p,\alpha,\lambda)$ *is given by*

$$r_3 = \frac{\beta - \{\beta^2 - 4p\left|2(p-\alpha)\cos\lambda e^{-i\lambda} - p\right|\}^{\frac{1}{2}}}{2\left|2(p-\alpha)\cos\lambda e^{-i\lambda} - p\right|}, \tag{2.12}$$

where $\beta = \left|2(p-\alpha)\cos\lambda e^{-i\lambda} - p\right| + 2 + p$.

Putting $n = 0$ in Corollary 3, we have the following

COROLLARY 4. *Let the function* $f \in A(p,j)$ *and* $g \in S_p^{\lambda}(\alpha)$. *If* $f(z)$ *is majorized by* g, *then*

$$\left|f'(z)\right| \leqslant \left|g'(z)\right| \quad (|z| \leqslant r_3),$$

where r_3 *is given by* (2.12).

REMARK 2. (i) Putting $n = 0$ in Theorem 1 we obtain the result obtained by Altintas and Srivastava [2, Theorem 1];

(ii) Specializing the parameters n, q, A and B in Theorem 1, we obtain the majorization results for the corresponding classes defined in the introduction.

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