Journal of
Classical
Analysis
Volume 3, Number 2 (2013), 137–147
doi:10.7153/jca-03-12

ON λ–PSEUDO–STARLIKE FUNCTIONS

K. O. BABALOLA

Abstract. In this paper, we isolate some new, interesting classes of λ*-pseudo-starlike* univalent functions in the open unit disk $E = \{z \in \mathbb{C} \colon |z| < 1\}$. Some characterizations of them are obtained and examples given.

1. Introduction

We begin by letting A denote the class of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

which are analytic in E and by S the subclass of A which consist of univalent functions only. Furthermore, let $R(\beta)$ and $S^*(\beta)$ be the usual subclasses of S consisting of functions which are, respectively, of bounded turning and starlike of order β, $0 \leqslant \beta < 1$, in E. That is, functions satisfying, respectively, Re $f'(z) > \beta$ and Re $zf'(z)/f(z) > \beta$ in E. In [5], Singh studied a subclass of S denoted by $B_1(\alpha)$ consisting of functions which are a special case of Bazilevič functions (known to consist of univalent functions only). Functions in $B_1(\alpha)$ satisfy the geometric condition:

$$\text{Re}\,\frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}} > 0, \;\; z \in E$$

for nonnegative real number α and are known as Bazilevič functions of type α. The class $B_1(\alpha)$ includes the starlike and bounded turning functions as the cases $\alpha = 0$ and $\alpha = 1$ respectively. For the purpose of this paper, we say $f \in A$ is a Bazilevič function of type α, order β if and only if

$$\text{Re}\,\frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}} > \beta, \;\; z \in E$$

and we denote the class of such functions by $B_1(\alpha, \beta)$.

Next we say:

DEFINITION 1. Let $f \in A$. Suppose $0 \leqslant \beta < 1$ and $\lambda \geqslant 1$ is real. Then $f(z)$ belongs to the class $\mathscr{L}_\lambda(\beta)$ of λ*-pseudo-starlike* functions of order β in the unit disk if and only if

$$\text{Re}\,\frac{z(f'(z))^\lambda}{f(z)} > \beta, \;\; z \in E.$$

Mathematics subject classification (2010): 30C45.
Keywords and phrases: Bazilevič functions, λ-pseudo-starlike, analytic and univalent functions.

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Paper JCA-03-12

REMARK 1.

(a) Throughout this work, all powers shall mean principal determinations only.

(b) If $\lambda = 1$, we have the class of starlike functions of order β, which in this context, are 1-pseudo-starlike functions of order β.

(c) If $\beta = 0$, we simply write $\mathscr{L}_\lambda$ instead of $\mathscr{L}_\lambda(0)$.

For $\lambda = 2$ we note that functions in $\mathscr{L}_2(\beta)$ are defined by

$$\operatorname{Re} f'(z)\frac{zf'(z)}{f(z)} > \beta, \;\; z \in E$$

which is a product combination of geometric expressions for bounded turning and starlike functions - an interesting analytic presentation of univalent functions as would be shown later.

It is interesting also to note that $\mathscr{L}_\infty(\beta)$ is the singleton subclass of S containing the identity map only.

We shall prove in Section 3 that $\mathscr{L}_\lambda(\beta)$ is a subclass of $B_1(\alpha,\beta)$, thus λ-pseudo-starlike functions are Bazilevič, univalent in E. We shall also obtain some characterizations of these new classes of functions. In particular, we shall obtain their integral representations, sufficient univalence conditions and some coefficient bounds and also estimates of their Fekete-Szego functional. We shall mention examples for some special cases. We remark here that though for $\lambda > 1$, these classes of λ-pseudo-starlike functions 'clone' the analytic representation of starlike functions, it is not yet known the possibilty of any inclusion relations between them.

In the next section, we shall state the preliminary results.

2. Preliminary lemmas

In this paper we shall require the following preliminary results:

LEMMA 1. *If z is a complex number having positive real part, then for any real number t such that $t \in [0,1]$, we have $\operatorname{Re} z^t \geqslant (\operatorname{Re} z)^t$.*

Proof. Let $z = \cos\theta + i\sin\theta$. The case $\theta = 0$ holds trivially. Now suppose $\theta \neq 0$. Define

$$y(t) = (\operatorname{Re} z^t)/(\operatorname{Re} z)^t = \cos t\theta/(\cos\theta)^t.$$

Then by elementary calculus, $y(t)$ attains its maximum value at $t_0 \in [0,1]$ where t_0 is given by

$$t_0 = \frac{\arctan\left(\frac{-\log\cos\theta}{\theta}\right)}{\theta}, \;\; \theta \neq 0,$$

for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ $(\theta \neq 0)$ since $\operatorname{Re} z > 0$. Then $y(t)$ is decreasing on $t \in [t_0, 1]$ and increasing on $t \in [0, t_0]$. In particular, $y(t) = \cos t\theta/(\cos\theta)^t \geqslant y(0) = y(1) = 1$, and the conclusion follows immediately. $\square$

Now let P_β be the class of functions

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

which are analytic in the unit disk E and satisfy $Re\ p(z) > \beta$ there. The class of functions P_β is known as Caratheodory functions of order β. If $\beta = 0$, we write P simply in place of P_β.

LEMMA 2. *Let $p \in P_\beta$. If $q(z) = [p(z)]^t$, $t \in [0,1]$, then $q(0) = 1$ and Re $q(z) > \beta^t$.*

Proof. It is easy to see that $q(0) = 1$. Applying Lemma 1 we see that Re $q(z) =$ Re $[p(z)]^t > (\text{Re } p(z))^t > 0$. Thus Re $q(z) > \beta^t$ as required. □

We shall require the well known Caratheodory inequality for P, which is $|c_k| \leqslant 2$, $k = 1,2,3\cdots$, as well as the following coefficient inequalities for P.

LEMMA 3. ([1]) *Let $p \in P$. Then we have sharp inequalities*

$$\left|c_2 - \sigma\frac{c_1^2}{2}\right| \leqslant \begin{cases} 2(1-\sigma), & \text{if } \sigma \leqslant 0, \\ 2, & \text{if } 0 \leqslant \sigma \leqslant 2, \\ 2(\sigma - 1), & \text{if } \sigma \geqslant 2. \end{cases}$$

LEMMA 4. ([3]) *Let $p \in P$. Then for $n \geqslant 2$ and $s \geqslant 1$,*

$$|c_n - c_s c_{n-s}| \leqslant 2.$$

These inequalities are sharp for all n and s.

COROLLARY 1. *Let $p \in P$. Then*

$$|c_n - \sigma c_s c_{n-s}| \leqslant \begin{cases} 2(3-2\sigma), & \text{if } \sigma \leqslant 1, \\ 2, & \text{if } \sigma = 1, \\ 2(2\sigma - 1), & \text{if } \sigma \geqslant 1. \end{cases}$$

Proof. Noting that

$$\begin{aligned} |c_n - \sigma c_s c_{n-s}| &= |c_n - c_s c_{n-s} + c_s c_{n-s} - \sigma c_s c_{n-s}| \\ &\leqslant |c_n - c_s c_{n-s}| + |c_s c_{n-s} - \sigma c_s c_{n-s}| \\ &\leqslant |c_n - c_s c_{n-s}| + |c_s||c_{n-s}||1-\sigma|. \end{aligned}$$

and using the Caratheodory inequality and Lemma 4, we have the result. □

LEMMA 5. ([4]) *Let q be univalent in E and let ϕ be analytic in a domain D containing $q(E)$. If $zq'(z)\phi(q(z))$ is starlike in E, then*

$$zp'(z)\phi(p(z)) \prec zq'(z)\phi(q(z)) \Rightarrow p(z) \prec q(z)$$

and $q(z)$ is the best dominant.

The next lemma is a sharp version of a result of Frasin [2]. The lemma and its proof (based on the Lemma 5 above) were suggested by the referee to whom the present author is indebted.

LEMMA 6. *Let p be holomorphic in E with $p(0) = 1$ and suppose that*

$$Re\left(1+\frac{zp'(z)}{p(z)}\right) > \frac{3\beta-1}{2\beta}, \quad z \in E.$$

Then

$$Re\, p(z) > 2^{1-1/\beta}, \quad 1/2 \leqslant \beta < 1, \quad z \in E$$

and the constant $2^{1-1/\beta}$ is the best possible.

Proof. Using Lemma 5, define $\phi(z) = 1/z$ and $q(z) = 1/(1-z)^m$ where $m = (1-\beta)/\beta \in (0,1]$. Then we get

$$\frac{zp'(z)}{p(z)} \prec \frac{mz}{1-z} \Rightarrow p(z) \prec \frac{1}{(1-z)^m}$$

which can be rewritten as

$$1+\frac{2}{m}\frac{zp'(z)}{p(z)} \prec \frac{1+z}{1-z} \Rightarrow p(z) \prec \frac{1}{(1-z)^m}$$

which is equivalent to

$$\mathrm{Re}\left(1+\frac{zp'(z)}{p(z)}\right) > \frac{3\beta-1}{2\beta} \Rightarrow p(z) \prec \frac{1}{(1-z)^m}.$$

Now the subordination $p(z) \prec 1/(1-z)^m$ implies

$$\mathrm{Re}\, p(z) > \inf_{\theta\in[0,2\pi]} \mathrm{Re}\,\frac{1}{(1-e^{i\theta})^m} = 2^{-m} = 2^{1-1/\beta}, \quad z \in E$$

which completes the proof. $\square$

3. Some properties of pseudo-starlike functions

THEOREM 1.

$$\mathscr{L}_\lambda(\beta) \subset B_1(1-1/\lambda, \beta^{1/\lambda}).$$

Proof. Let $f \in \mathscr{L}_\lambda$, then for some $p \in P_\beta$ we have $z(f'(z))^\lambda / f(z) = p(z)$. We write $z(f'(z))^\lambda / f(z)$ as

$$\left(\frac{z^{1/\lambda} f'(z)}{f(z)^{1/\lambda}}\right)^\lambda$$

so that

$$\frac{z(f'(z))^\lambda}{f(z)} = \left(\frac{z^{1/\lambda} f'(z)}{f(z)^{1/\lambda}}\right)^\lambda = p(z)$$

which implies

$$\frac{z^{1/\lambda} f'(z)}{f(z)^{1/\lambda}} = p(z)^{1/\lambda}.$$

So, by Lemma 2, we have

$$\operatorname{Re} \frac{z^{1/\lambda} f'(z)}{f(z)^{1/\lambda}} > \beta^{1/\lambda}.$$

Taking $\alpha = 1-1/\lambda$, we have $f \in B_1(1-1/\lambda, \beta^{1/\lambda})$. □

COROLLARY 2. *All pseudo-starlike functions are Bazilevič of type* $1-1/\lambda$*, order* $\beta^{1/\lambda}$ *and univalent in* E.

THEOREM 2. *Let* $f \in \mathscr{L}_\lambda(\beta)$. *Then* $f(z)$ *has integral representation*

$$f(z) = \begin{cases} \left\{ \frac{\lambda-1}{\lambda} \int_0^z \left(\frac{p(t)}{t}\right)^{\frac{1}{\lambda}} dt \right\}^{\frac{\lambda}{\lambda-1}}, & \text{if } \lambda > 1, \\ \exp \int_0^z \frac{p(t)}{t} dt, & \text{if } \lambda = 1 \end{cases}$$

for some $p \in P_\beta$.

Proof. Since $f \in \mathscr{L}_\lambda(\beta)$, then there exists $p \in P_\beta$ such that

$$\frac{z^{1/\lambda} f'(z)}{f(z)^{1/\lambda}} = p^{1/\lambda}.$$

Then taking $\alpha = 1-1/\lambda$ we have

$$\frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}} = p^{1-\alpha}$$

so that

$$(f(z)^{\alpha})' = \alpha z^{\alpha-1} p^{1-\alpha}.$$

Hence

$$f(z) = \left\{ \int_0^z \alpha t^{\alpha-1} p(t)^{1-\alpha} dt \right\}^{\frac{1}{\alpha}}$$

which gives the representation. For $\lambda = 1$, the integral representation of starlike functions is well-known. □

COROLLARY 3. *Let $f \in \mathscr{L}_2(\beta)$. Then $f(z)$ has integral representation*

$$f(z) = \left\{ \frac{1}{2} \int_0^z \left(\frac{p(t)}{t} \right)^{\frac{1}{2}} dt \right\}^2$$

for some $p \in P_\beta$.

THEOREM 3. *If $f \in A$ satisfies*

$$Re \left\{ \lambda \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right\} > -\frac{1+\beta}{2\beta}, \quad z \in E,$$

then $f \in \mathscr{L}_\lambda(2^{1-1/\beta})$, $1/2 \leqslant \beta < 1$. The constant $2^{1-1/\beta}$ is the best possible.

Proof. For $z \in E$, define

$$p(z) = \frac{z(f'(z))^\lambda}{f(z)}.$$

Then we find that

$$\frac{zp'(z)}{p(z)} = 1 - \frac{zf'(z)}{f(z)} + \lambda \frac{zf''(z)}{f'(z)}$$

so that

$$\begin{aligned} \text{Re} \left(1 + \frac{zp'(z)}{p(z)} \right) &= \text{Re} \left\{ 2 - \frac{zf'(z)}{f(z)} + \lambda \frac{zf''(z)}{f'(z)} \right\} \\ &> \frac{3\beta - 1}{2\beta} \end{aligned}$$

yields

$$\text{Re} \left\{ \lambda \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right\} > -\frac{1+\beta}{2\beta},$$

which, by Lemma 6, implies

$$\text{Re} \frac{z(f'(z))^\lambda}{f(z)} > 2^{1-1/\beta}, \quad 1/2 \leqslant \beta < 1$$

as required. □

The following corollary is a new stalikeness condition for analytic mappings of the unit disk.

COROLLARY 4. *If $f \in A$ satisfies*

$$Re\left\{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > -\frac{3}{2},\ \ z \in E,$$

then

$$Re\,\frac{zf'(z)}{f(z)} > \frac{1}{2}.$$

That is f is starlike of order $1/2$ in E.

THEOREM 4. *Let $f \in \mathscr{L}_\lambda(\beta)$. Then*

$$|a_2| \leqslant \frac{2(1-\beta)}{2\lambda-1},$$

$$|a_3| \leqslant \begin{cases} \dfrac{2(1-\beta)(4\lambda-1)}{(2\lambda-1)^2(3\lambda-1)}, & \text{if } 1 \leqslant \lambda \leqslant \dfrac{2+\sqrt{2}}{2} \\ \dfrac{2(1-\beta)}{3\lambda-1}, & \text{if } \lambda \geqslant \dfrac{2+\sqrt{2}}{2}, \end{cases}$$

$$|a_4| \leqslant \begin{cases} \dfrac{2(1-\beta)(F_1-G_1\beta+H\beta^2)}{3(2\lambda-1)^3(3\lambda-1)(4\lambda-1)}, & \text{if } 1 \leqslant \lambda \leqslant \dfrac{11+\sqrt{73}}{12} \\ \dfrac{2(1-\beta)(F_2-G_2\beta+H\beta^2)}{3(2\lambda-1)^3(3\lambda-1)(4\lambda-1)}, & \text{if } \lambda \geqslant \dfrac{11+\sqrt{73}}{12}. \end{cases}$$

where

$$F_1 = 24\lambda^4 - 44\lambda^3 + 78\lambda^2 - 25\lambda + 3,$$
$$F_2 = 168\lambda^4 - 308\lambda^3 + 258\lambda^2 - 79\lambda + 9,$$
$$G_1 = 2(24\lambda^4 - 116\lambda^3 + 162\lambda^2 - 55\lambda + 6),$$
$$G_2 = 2(24\lambda^4 - 116\lambda^3 + 162\lambda^2 - 55\lambda + 18),$$

and

$$H = 4(24\lambda^4 - 80\lambda^3 + 84\lambda^2 - 28\lambda + 3).$$

The bounds are sharp.

Proof. For $f \in \mathscr{L}_\lambda(\beta)$, there exists $p \in P$ such that

$$\frac{z(f'(z))^\lambda}{f(z)} = \beta + (1-\beta)p(z)$$

so that $z(f'(z))^\lambda = f(z)[\beta + (1-\beta)p(z)]$ and expanding in series forms we have

$$\begin{aligned} &z + 2a_2\lambda z^2 + [3a_3\lambda + 2\lambda(\lambda-1)a_2^2]z^3 \\ &+ \left[4a_4\lambda + 6\lambda(\lambda-1)a_2a_3 + \frac{4}{3}\lambda(\lambda-1)(\lambda-2)a_2^3\right]z^4 + \cdots \\ = &\ z + [(1-\beta)c_1 + a_2]z^2 + [(1-\beta)c_2 + (1-\beta)c_1a_2 + a_3]z^3 \\ &+ [(1-\beta)c_3 + (1-\beta)c_2a_2 + (1-\beta)c_1a_3 + a_4]z^4 + \cdots \end{aligned}$$

Comparing coefficients, we have

$$(2\lambda-1)a_2=(1-\beta)c_1 \tag{3.1}$$

which, using the Caratheodory inequality $|c_1|\leqslant 2$, gives the bound on a_2. Next we have

$$3a_3\lambda+2\lambda(\lambda-1)a_2^2=(1-\beta)c_2+(1-\beta)c_1a_2+a_3$$

which yields

$$(3\lambda-1)a_3=(1-\beta)\left\{c_2-\frac{2(1-\beta)(2\lambda^2-4\lambda+1)}{4\lambda^2-4\lambda+1}\frac{c_1^2}{2}\right\}. \tag{3.2}$$

Applying Lemma 3 with $\sigma=2(1-\beta)(2\lambda^2-4\lambda+1)/(4\lambda^2-4\lambda+1)$, we have the inequalities for a_3. Next we have

$$4a_4\lambda+6\lambda(\lambda-1)a_2a_3+\frac{4}{3}\lambda(\lambda-1)(\lambda-2)a_2^3=(1-\beta)c_3+(1-\beta)c_2a_2+(1-\beta)c_1a_3+a_4$$

so that

$$\begin{aligned}(4\lambda-1)a_4=&(1-\beta)c_3-(1-\beta)^2\frac{6\lambda^2-11\lambda+2}{(2\lambda-1)(3\lambda-1)}c_1c_2\\&+(1-\beta)^3\frac{24\lambda^4-80\lambda^3+84\lambda^2-28\lambda+3}{3(2\lambda-1)^3(3\lambda-1)}c_1^3.\end{aligned}$$

Now, if $6\lambda^2-11\lambda+2$ is nonpositive, that is $\lambda\leqslant(11+\sqrt{73})/12$, we have

$$\begin{aligned}(4\lambda-1)|a_4|\leqslant&(1-\beta)|c_3|+(1-\beta)^2\frac{11\lambda-6\lambda^2-2}{(2\lambda-1)(3\lambda-1)}|c_1||c_2|\\&+(1-\beta)^3\frac{24\lambda^4-80\lambda^3+84\lambda^2-28\lambda+3}{3(2\lambda-1)^3(3\lambda-1)}|c_1|^3\end{aligned}$$

and by the Caratheodory inequalities, and careful computation, we obtain the first inquality for a_4. Otherwise, we write

$$\begin{aligned}(4\lambda-1)a_4=&(1-\beta)[c_3-(1-\beta)c_1c_2]+\frac{(1-\beta)^2(6\lambda-1)}{(2\lambda-1)(3\lambda-1)}c_1c_2\\&+(1-\beta)^3\frac{24\lambda^4-80\lambda^3+84\lambda^2-28\lambda+3}{3(2\lambda-1)^3(3\lambda-1)}c_1^3\end{aligned}$$

which by triangle inquality yields

$$\begin{aligned}(4\lambda-1)|a_4|\leqslant&(1-\beta)|c_3-(1-\beta)c_1c_2|+\frac{(1-\beta)^2(6\lambda-1)}{(2\lambda-1)(3\lambda-1)}|c_1||c_2|\\&+(1-\beta)^3\frac{24\lambda^4-80\lambda^3+84\lambda^2-28\lambda+3}{3(2\lambda-1)^3(3\lambda-1)}|c_1|^3.\end{aligned}$$

Thus taking $\sigma=1-\beta<1$ in Corollary 1 and then using the Caratheodory inequalities, we obtain the second inequality for a_4 and the proof is complete. □

COROLLARY 5. *If $f(z)$ is starlike, that is $f \in \mathscr{L}_1$, then*

$$|a_2| \leqslant 2,\ |a_3| \leqslant 3,\ |a_4| \leqslant 4.$$

The bounds are sharp and well-known.

COROLLARY 6. *Let $f \in \mathscr{L}_2$. Then*

$$|a_2| \leqslant \frac{2}{3},\quad |a_3| \leqslant \frac{2}{5},\quad |a_4| \leqslant \frac{82}{105}.$$

The bounds are sharp.

THEOREM 5. *Let $f \in \mathscr{L}_\lambda(\beta)$ and μ any real number. Then*

$$|a_3-\mu a_2^2| \leqslant \begin{cases} \dfrac{2(1-\beta)(J+K\beta)}{(2\lambda-1)^2(3\lambda-1)}, & \text{if } \mu \leqslant \dfrac{4\lambda-2\lambda^2-1}{3\lambda-1}, \\ \dfrac{2}{3\lambda-1}, & \text{if } \dfrac{4\lambda-2\lambda^2-1}{3\lambda-1} \leqslant \mu \leqslant \dfrac{2\lambda^2-\beta(2\lambda^2+4\lambda-1)}{3\lambda-1}, \\ \dfrac{2(\beta-1)(J+K\beta)}{(2\lambda-1)^2(3\lambda-1)}, & \text{if } \mu \geqslant \dfrac{2\lambda^2-\beta(2\lambda^2+4\lambda-1)}{3\lambda-1} \end{cases}$$

where

$$J = 2(2-3\mu)\lambda + 2\mu - 1$$

and

$$K = 2\lambda^2 + (3\mu-4)\lambda - \mu + 1.$$

The inequalities are sharp.

Proof. From (3.1) and (3.2) we have

$$|a_3-\mu a_2^2| = (1-\beta)\left|c_2 - \frac{2(1-\beta)[2\lambda^2-4\lambda+\mu(3\lambda-1)+1]}{(2\lambda-1)^2}\frac{c_1^2}{2}\right|.$$

The conclusion now follows by taking $\sigma = 2[2\lambda^2-4\lambda+\mu(3\lambda-1)+1]/(2\lambda-1)^2$ in Lemma 3. □

COROLLARY 7. *Let $f \in \mathscr{L}_2$ and μ any real number. Then*

$$|a_3-\mu a_2^2| \leqslant \begin{cases} \dfrac{14-20\mu}{45}, & \text{if } \mu \leqslant -\dfrac{1}{5}, \\ \dfrac{2}{5}, & \text{if } -\dfrac{1}{5} \leqslant \mu \leqslant \dfrac{8}{5}, \\ \dfrac{20\mu-14}{45}, & \text{if } \mu \geqslant \dfrac{8}{5}. \end{cases}$$

The inequalities are sharp.

In addition to the identity map $f(z) = z$, we give some nontrivial examples of pseudo-starlike functions for the special choice, $\lambda = 2$.

EXAMPLE 1. The functions $f_j(z)$ $j = 1,2,3,4$ given by

$$f_1(z) = \frac{1}{4}\left(\sqrt{z(1+z)} + \ln(\sqrt{z} + \sqrt{1+z})\right)^2 = z + \frac{z^2}{3} - \frac{z^3}{45} + \frac{z^4}{105} - \frac{4z^5}{1575} + \cdots.$$

$$f_2(z) = \frac{1}{4}\left(\sqrt{z(1-z)} - \arcsin\sqrt{1-z}\right)^2 = z - \frac{z^2}{3} - \frac{z^3}{45} - \frac{z^4}{105} - \frac{4z^5}{1575} - \cdots.$$

$$f_3(z) = \left(\ln(\sqrt{z} + \sqrt{1+z})\right)^2 = z - \frac{z^2}{3} + \frac{8z^3}{45} - \frac{4z^4}{35} + \frac{384z^5}{4725} - \cdots.$$

$$f_4(z) = \left(\arcsin\sqrt{1-z}\right)^2 = z + \frac{z^2}{3} + \frac{8z^3}{45} + \frac{4z^4}{35} + \frac{384z^5}{4725} + \cdots.$$

belong to $\mathcal{L}_2$, thus Bazilevič of type $1/2$, univalent in the open unit disk.

Proof. By careful computation we find that

$$\frac{z(f_j'(z))^2}{f_j(z)} = \begin{cases} 1+z & \text{if } j = 1, \\ 1-z & \text{if } j = 2, \\ \frac{1}{1+z} & \text{if } j = 3, \\ \frac{1}{1-z} & \text{if } j = 4. \end{cases}$$

Since the right hand side of the above equations are all functions in P, we have

$$Re\,\frac{z(f_j'(z))^2}{f_j(z)} > 0, \quad j = 1,\ 2,\ 3,\ 4$$

so that $f_j \in \mathcal{L}_2$. The proof is complete. □

Based on Corollary 3, we conclude this paper by mentioning that the following are starlike univalent functions in the unit disk:

EXAMPLE 2. The functions $f_j(z)$ $j = 5,6,7,8$ given by

$$f_5(z) = -\frac{4}{25}\left(1 + \frac{5}{2}z\right)\exp\left\{-\frac{5}{2}z\right\}.$$

$$f_6(z) = \frac{4}{25}\left(\frac{5}{2}z - 1\right)\exp\left\{\frac{5}{2}z\right\}.$$

$$f_7(z) = \exp\left\{\frac{2(3z-2)}{3(1-z)^{\frac{3}{2}}}\right\}.$$

$$f_8(z) = \exp\left\{\frac{2(3z+4)}{3(1+z)^{\frac{3}{2}}}\right\}.$$

are starlike of order $1/2$ in the open unit disk.

Acknowledgements. The author is indebted to the referee for his helpful comments and especially his suggestion of Lemma 6 which helped to sharpen Theorem 3.

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(Received August 6, 2013)

K. O. Babalola
Department of Physical Sciences
Al-Hikmah University
Ilorin
and
Department of Mathematics
University of Ilorin
Ilorin
e-mail: kobabalola@gmail.com