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BOUND FOR THE ZEROS OF POLYNOMIALS

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Abstract. In this paper, we prove a result concerning the location of the zeros of polynomials in an annulus involving binomial coefficients and $(t,s)-$Fibonacci numbers. Our result includes not only some known results as special cases but also sharpens them more accurate.

1. Introduction and statement of results

The task of determining the roots of polynomials has been frequently investigated. Recently, Diaz-Barerro [3] has obtained bounds for zeros of polynomials in terms of binomial coefficients and Fibonacci numbers ($F_0 = 0$, $F_1 = 1$ and $n \geqslant 2$, $F_n = F_{n-1} + F_{n-2}$). In fact he has proved the following result.

THEOREM A. *Let $p(z) = \sum_{k=0}^{n} a_k z^k$, $(a_k \neq 0)$ be a non-constant complex polynomial of degree n. Then all the zeros of $p(z)$ lie in the annulus $R = \{z \in \mathbb{C} : \quad r_1 \leqslant |z| \leqslant r_2\}$, where*

$$r_1 = \frac{3}{2} \min_{1 \leqslant k \leqslant n} \left\{ \frac{2^n F_k \binom{n}{k}}{F_{4n}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \quad \textit{and} \quad r_2 = \frac{2}{3} \max_{1 \leqslant k \leqslant n} \left\{ \frac{F_{4n}}{2^n F_k \binom{n}{k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}. \tag{1.1}$$

Let P_n denote the n^{th} Pell number (i.e. $P_0 = 0$, $P_1 = 1$ and for $n \geqslant 2$, $P_n = 2P_{n-1} + P_{n-2}$). Diaz-Barerro and Egozcue[4] introduced bounds for the zeros of polynomials in terms of binomial coefficients and Pell numbers by proving:

THEOREM B. *Let $p(z) = \sum_{k=0}^{n} a_k z^k$, $(a_k \neq 0)$ be a non-constant complex polynomial of degree n. Then all the zeros of $p(z)$ lie in the annulus $R = \{z \in \mathbb{C} : \quad r_3 \leqslant |z| \leqslant r_4\}$, where*

$$r_3 = \min_{1 \leqslant k \leqslant n} \left\{ \frac{2^k P_k \binom{n}{k}}{P_{2n}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \quad \textit{and} \quad r_4 = \max_{1 \leqslant k \leqslant n} \left\{ \frac{P_{2n}}{2^k P_k \binom{n}{k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}. \tag{1.2}$$

Bidkham and Shashahani [2] proved the following result which is better than above result when r_3 is obtained for $k = n$.

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THEOREM C. *Let* $p(z)=\sum_{k=0}^{n} a_k z^k$, $(a_k \neq 0)$ *be a non-constant complex polynomial of degree* n *and*

$$\lambda_k = \frac{5^n P_k \binom{n}{k}}{[(1+\sqrt{2})^{2n}+(1-\sqrt{2})^{2n}]P_{2n}}.$$

Then all the zeros of $p(z)$ *lie in the annulus* $R=\{z\in\mathbb{C}: \ r_5 \leqslant |z| \leqslant r_6\}$, *where*

$$r_5 = \frac{12}{5}\min_{1\leqslant k\leqslant n}\left\{\lambda_k\left|\frac{a_0}{a_k}\right|\right\}^{\frac{1}{k}} \quad \text{and} \quad r_6 = \frac{5}{12}\max_{1\leqslant k\leqslant n}\left\{\frac{1}{\lambda_k}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{\frac{1}{k}}. \tag{1.3}$$

Also Affane, Biaz and Govil [1] have obtained a lower bound for the zeros of polynomials in terms of binomial coefficients by proving:

THEOREM D. *Let* $p(z)=\sum_{k=0}^{n} a_k z^k$, $(a_k \neq 0)$ *be a non-constant complex polynomial of degree* n. *Then all the zeros of* $p(z)$ *lie in the* $R=\{z\in\mathbb{C}: \ r_7 \leqslant |z| \leqslant r_8\}$, *where*

$$r_7 = \min_{1\leqslant k\leqslant n}\left\{\frac{k\binom{n}{k}}{n2^{n-1}}\left|\frac{a_0}{a_k}\right|\right\}^{\frac{1}{k}} \quad \text{and} \quad r_8 = 1+\delta_k, \tag{1.4}$$

being δ_k *the unique positive root of the* k^{th} *degree equation*

$$Q_k(x) = x^k + \sum_{v=2}^{k}\left[\binom{k-1}{k-v} - \sum_{j=1}^{v-1}\binom{k-j-1}{k-v}\left|\frac{a_{n-j}}{a_n}\right|\right]x^{k+1-v} - A = 0,$$

and $A=\max_{0\leqslant j\leqslant n-1}|\frac{a_j}{a_n}|$, $a_j = 0$ *if* $j<0$.

In this paper, we obtain a result that not only includes the above results as special cases but also gives a method for obtaining sharper bounds for the location of the zeros of a polynomial.

Let α and β be the roots of quadratic equation

$$x^2 - tx - s = 0$$

being t,s strictly positive real numbers. First we define the $(t,s)-$Fibonacci numbers as follows:

DEFINITION 1. For any positive real numbers t,s, the $(t,s)-$Fibonacci sequence, say $\{F_{t,s,n}\}_{n\in\mathbb{N}}$ is defined by

$$F_{t,s,n+1} = tF_{t,s,n} + sF_{t,s,n-1} \qquad \text{for} \qquad n \geqslant 1 \tag{1.5}$$

with initial conditions

$$F_{t,s,0}=0, \qquad F_{t,s,1}=1.$$

THEOREM 1. *Let $p(z)=\sum_{k=0}^{n} a_k z^k$, $(a_k \neq 0)$ be a non-constant complex polynomial of degree n. Then for $j \geqslant 1$, all its zeros lie in the annulus*

$$R=\{z\in\mathbb{C}: \ r_9 \leqslant |z| \leqslant r_{10}\}$$

where

$$r_9 = \min_{1\leqslant k\leqslant n}\left\{\frac{\binom{n}{k}F_{t,s,k}(F_{t,s,2^j})^k(sF_{t,s,2^j-1})^{n-k}}{F_{t,s,2^jn}}\left|\frac{a_0}{a_k}\right|\right\}^{\frac{1}{k}} \tag{1.6}$$

and

$$r_{10} = \max_{1\leqslant k\leqslant n}\left\{\frac{F_{t,s,2^jn}}{\binom{n}{k}F_{t,s,k}(F_{t,s,2^j})^k(sF_{t,s,2^j-1})^{n-k}}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{\frac{1}{k}} \tag{1.7}$$

REMARK 1. For $t=s=1$ and $j=2$, Theorem 2 reduces to Theorem A. If we take $t=2$, $s=1$ in Theorem 2, we get the following result:

COROLLARY 1. *Let $p(z)=\sum_{k=0}^{n} a_k z^k$, $(a_k \neq 0)$ be a non-constant complex polynomial of degree n. Then for $j \geqslant 1$, all its zeros lie in the annulus*

$$R=\{z\in\mathbb{C}: \ s_1 \leqslant |z| \leqslant s_2\}$$

where

$$s_1 = \min_{1\leqslant k\leqslant n}\left\{\frac{\binom{n}{k}P_k(P_{2^j})^k(P_{2^j-1})^{n-k}}{P_{2^jn}}\left|\frac{a_0}{a_k}\right|\right\}^{\frac{1}{k}}$$

and

$$s_2 = \max_{1\leqslant k\leqslant n}\left\{\frac{P_{2^jn}}{\binom{n}{k}P_k(P_{2^j})^k(P_{2^j-1})^{n-k}}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{\frac{1}{k}}.$$

REMARK 2. If $j=1$, then Corollary 1 reduces to Theorem B. If $j=2$, then Corollary 1 reduces to Theorem C.

2. Lemmas

LEMMA 1. *Let r and s be the roots of quadratic equation $x^2-ax-b=0$, being a,b strictly positive real number. Define the two sequence $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ by $B_n=\sum_{k=0}^{n-1} r^k s^{n-1-k}$ and $A_n = cr^n+ds^n$, where c and d are real constants. If $j\geqslant 2$, then*

$$\sum_{k=0}^{n}\binom{n}{k}(bB_{j-1})^{n-k}(B_j)^k A_k = A_{jn}.$$

This lemma is due to Diaz–Barerro and Egozcue [4].

As an special case of Lemma 1, by considering $x^2-tx-s=0$ and $A_n=B_n=F_{t,s,n}$, we get the following lemma:

LEMMA 2. *For* $j \geqslant 1$

$$\sum_{k=0}^{n} \binom{n}{k} (sF_{t,s,2^j-1})^{n-k} (F_{t,s,2^j})^k F_{t,s,k} = F_{t,s,2^j n}.$$

holds.

LEMMA 3. *The* n^{th} $(t,s)-$*Fibonacci number is given by*

$$F_{t,s,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad (Binet's formula) \tag{2.1}$$

and for $j \geqslant 1$

$$F_{t,s,2^j n} = (\alpha^{2^{j-1}n} + \beta^{2^{j-1}n}) F_{t,s,2^{j-1}n}, \tag{2.2}$$

where $\alpha,\ \beta$ *are the roots of the characteristic equation*

$$r^2 = tr + s \tag{2.3}$$

for $\alpha > \beta$.

Proof. The roots of the characteristic equation (2.3) are $\alpha = (t + \sqrt{t^2+4s})/2$ and $\beta = (t - \sqrt{t^2+4s})/2$. Since t, s are positive, then

$$\beta < 0 < \alpha \quad \text{and} \quad |\beta| < |\alpha|,$$
$$\alpha + \beta = t \quad \text{and} \quad \alpha\beta = -s,$$
$$\alpha - \beta = \sqrt{t^2+4s}.$$

It is trivial that (2.1) is true for $n = 0,\ 1$. By induction if we suppose that (2.1) is true for the terms $F_{t,s,n-1}$ and $F_{t,s,n}$, then

$$\begin{aligned} F_{t,s,n+1} = tF_{t,s,n} + sF_{t,s,n-1} &= t\frac{\alpha^n - \beta^n}{\alpha - \beta} + s\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \\ &= \frac{1}{\alpha - \beta}\{(t\alpha + s)\alpha^{n-1} - (t\beta + s)\beta^{n-1}\}, \end{aligned} \tag{2.4}$$

since $\alpha + \beta = t$ and $\alpha\beta = -s$, we have $t\alpha + s = \alpha^2$ and $t\beta + s = \beta^2$, therefore by (2.4) we obtain

$$F_{t,s,n+1} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \tag{2.5}$$

By using the above formula for $(t,s)-$Fibonacci sequence, for $a = \frac{1}{\sqrt{t^2+4s}}$, we have

$$\begin{aligned} F_{t,s,n} &= a(\alpha^n - \beta^n) \\ F_{t,s,2n} &= (\alpha^n + \beta^n) F_{t,s,n} \\ F_{t,s,4n} &= (\alpha^{2n} + \beta^{2n}) F_{t,s,2n} \\ &\cdots \\ F_{t,s,2^j n} &= (\alpha^{2^{j-1}n} + \beta^{2^{j-1}n}) F_{t,s,2^{j-1}n}. \end{aligned} \tag{2.6}$$

This completes the proof of Lemma 3. □

3. Proof of the Theorem 1

If $a_0 = 0$, then $r_9 = 0$. We suppose that $a_0 \neq 0$. It is clear that from the definition of r_9 in (1.6) that

$$r_9^k \leqslant \left\{ \frac{\binom{n}{k} F_{t,s,k} (F_{t,s,2^j})^k (sF_{t,s,2^j-1})^{n-k}}{F_{t,s,2^j n}} \left| \frac{a_0}{a_k} \right| \right\} \qquad k = 1, 2, \ldots, n. \tag{3.1}$$

Then, for $|z| < r_9$, where r_9 is as defined in (1.6), we get

$$\begin{aligned} |p(z)| &= |a_0 + \sum_{k=1}^{n} a_k z^k| \\ &\geqslant |a_0| - \sum_{k=1}^{n} |a_k||z|^k \\ &> |a_0| - \sum_{k=1}^{n} |a_k| r_9^k \\ &= |a_0| \left(1 - \sum_{k=1}^{n} \left| \frac{a_k}{a_0} \right| r_9^k \right) \\ &\geqslant |a_0| \left(1 - \sum_{k=1}^{n} \frac{\binom{n}{k} F_{t,s,k} (F_{t,s,2^j})^k (sF_{t,s,2^j-1})^{n-k}}{F_{t,s,2^j n}} \right) \quad \text{(by (3.1))} \\ &= 0 \quad \text{(by Lemma 2)}. \end{aligned} \tag{3.2}$$

This implies $p(z) \neq 0$ for $|z| < r_9$. Next, we show that all the zeros of $p(z)$ lie in $|z| \leqslant r_{10}$, where r_{10} is defined in (1.7). From (1.7), we have

$$\left| \frac{a_{n-k}}{a_n} \right| \leqslant \frac{\binom{n}{k} F_{t,s,k} (F_{t,s,2^j})^k (sF_{t,s,2^j-1})^{n-k}}{F_{t,s,2^j n}} r_{10}^k, \qquad (1 \leqslant k \leqslant n)$$

or

$$\sum_{k=1}^{n} \left| \frac{a_{n-k}}{a_n} \right| \frac{1}{r_{10}^k} \leqslant \sum_{k=1}^{n} \frac{\binom{n}{k} F_{t,s,k} (F_{t,s,2^j})^k (sF_{t,s,2^j-1})^{n-k}}{F_{t,s,2^j n}}. \tag{3.3}$$

On the other hand

$$\begin{aligned} |p(z)| &= |a_n z^n + \ldots + a_1 z + a_0| \\ &\geqslant |a_n z^n| - \sum_{k=1}^{n} |a_{n-k}||z|^{n-k} \\ &= |a_n z^n| \left(1 - \sum_{k=1}^{n} \left| \frac{a_{n-k}}{a_n} \right| \frac{1}{|z|^k} \right) \\ &> |a_n z^n| \left(1 - \sum_{k=1}^{n} \left| \frac{a_{n-k}}{a_n} \right| \frac{1}{r_{10}^k} \right). \end{aligned} \tag{3.4}$$

Using (3.3) and Lemma 2, we have $|p(z)| > 0$ for $|z| > r_{10}$. Consequently all the zeros of $p(z)$ lie in $|z| \leqslant r_{10}$ and this proves second part of Theorem 1. $\square$

4. Quality of zero bounds

For $k = n$ in Corollary 1, we have

$$s_1 = \left\{ \frac{P_n (P_{2^j})^n}{P_{2^j n}} \left| \frac{a_0}{a_n} \right| \right\}^{\frac{1}{n}}.$$

Using Lemma 3 (inequality (2.2)), we can obtain

$$s_1 = \left\{ A_j \left| \frac{a_0}{a_n} \right| \right\}^{\frac{1}{n}},$$

where

$$A_j = \frac{(P_{2^j})^n}{[\alpha^{2^{j-1}n} + \beta^{2^{j-1}n}][\alpha^{2^{j-2}n} + \beta^{2^{j-2}n}]\ldots[\alpha^n + \beta^n]},$$

with $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ and

$$\begin{aligned}
\lim_{j \longrightarrow \infty} A_j &= \lim_{j \longrightarrow \infty} \frac{(P_{2^j})^n}{[\alpha^{2^{j-1}n} + \beta^{2^{j-1}n}][\alpha^{2^{j-2}n} + \beta^{2^{j-2}n}]\ldots[\alpha^n + \beta^n]} \cdot \frac{\alpha^n - \beta^n}{\alpha^n - \beta^n} \\
&= \lim_{j \longrightarrow \infty} \frac{(P_{2^j})^n (\alpha^n - \beta^n)}{(\alpha^{2^j n} - \beta^{2^j n})} \\
&= \lim_{j \longrightarrow \infty} \frac{1}{(2\sqrt{2})^n} \frac{(\alpha^{2^j} - \beta^{2^j})^n (\alpha^n - \beta^n)}{(\alpha^{2^j n} - \beta^{2^j n})} \cdot \frac{\beta^{2^j n}}{\beta^{2^j n}} \\
&= \lim_{j \longrightarrow \infty} \frac{1}{(2\sqrt{2})^n} \frac{(\alpha^n - \beta^n)(1 - \beta^{2^{j+1}})^n}{(1 - \beta^{2^{j+1} n})}.
\end{aligned} \tag{4.1}$$

Since $|\beta| < 1$, then we get

$$\lim_{j \longrightarrow \infty} A_j = \frac{1}{(2\sqrt{2})^n} (\alpha^n - \beta^n) = \frac{P_n}{(2\sqrt{2})^{n-1}}.$$

As A_j is a monotonic increasing sequence, hence if s_1 is obtained for $k = n$, then it not only improves Theorems B and C, but also for $j \longrightarrow \infty$, it is the best possible result and we have

$$s_1 = \left\{ \frac{P_n}{(2\sqrt{2})^{n-1}} \left| \frac{a_0}{a_n} \right| \right\}^{\frac{1}{n}}. \tag{4.2}$$

Since $P_n > (\sqrt{2})^{n-1}$ for $n \geqslant 2$, hence if r_7 is obtained for $k = n$, then (4.2) is better than the one in Theorem D. We can illustrate this by the following examples.

EXAMPLES.

i) For the polynomial $f_1(z) = 20z^3 + z^2 + z + 1000$, it is found that the annulus obtained by Theorems A, B and C are respectively $2.656646 \leqslant |z| \leqslant 5.10873$, $3.057107 \leqslant |z| \leqslant 4.43952$ and $3.147073 \leqslant |z| \leqslant 4.312606$. The lower bound by Theorem D is 2.32079 and the upper bound is implicit and finally, if we use Corollary 1 for $j \longrightarrow \infty$, it turns out to be $3.14980 \leqslant |z| \leqslant 4.30887$.

ii) For the polynomial $f_2(z) = 10iz^3 - z^2 - (\sqrt{3}+i)z + 400$, it is found that the annulus obtained by Theorems A, B and C are respectively $2.466212 \leqslant |z| \leqslant 4.742524$, $2.837967 \leqslant |z| \leqslant 4.121285$ and $2.921484 \leqslant |z| \leqslant 4.003469$. The lower bound by Theorem D is 2.154435 and the upper bound is implicit and finally, if we use Corollary 1 for $j \longrightarrow \infty$, it turns out to be $2.924018 \leqslant |z| \leqslant 4$.

iii) For $f_3(z) = 10z^3 - z^2 - 2z + 300$, it is found that the annulus obtained by Theorems A, B and C are respectively $2.240702 \leqslant |z| \leqslant 4.308869$, $2.578464 \leqslant |z| \leqslant 3.744436$ and $2.654344 \leqslant |z| \leqslant 3.637393$. The lower bound by Theorem D is 1.957434 and the upper bound is implicit and finally, if we use Corollary 1 for $j \longrightarrow \infty$, it turns out to be $2.656646 \leqslant |z| \leqslant 3.634241$.

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