BOUND FOR THE ZEROS OF POLYNOMIALS

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Abstract. In this paper, we prove a result concerning the location of the zeros of polynomials in an annulus involving binomial coefficients and \((r,s)\)-Fibonacci numbers. Our result includes not only some known results as special cases but also sharpens them more accurate.

1. Introduction and statement of results

The task of determining the roots of polynomials has been frequently investigated. Recently, Diaz-Barerro [3] has obtained bounds for zeros of polynomials in terms of binomial coefficients and Fibonacci numbers \((F_0 = 0, F_1 = 1\) and \(n \geq 2, F_n = F_{n-1} + F_{n-2}\)). In fact he has proved the following result.

THEOREM A. Let \(p(z) = \sum_{k=0}^{n} a_k z^k\), \((a_k \neq 0)\) be a non-constant complex polynomial of degree \(n\). Then all the zeros of \(p(z)\) lie in the annulus \(R = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}\), where

\[
\begin{align*}
r_1 &= \frac{3}{2} \min_{1 \leq k \leq n} \left\{ \frac{2^n F_k(n)}{F_{4n}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \\
r_2 &= \frac{2}{3} \max_{1 \leq k \leq n} \left\{ \frac{F_{4n}}{2^n F_k(n)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}.
\end{align*}
\]  

(1.1)

Let \(P_n\) denote the \(n^{th}\) Pell number (i.e. \(P_0 = 0, P_1 = 1\) and for \(n \geq 2, P_n = 2P_{n-1} + P_{n-2}\)). Diaz-Barerro and Egozcue[4] introduced bounds for the zeros of polynomials in terms of binomial coefficients and Pell numbers by proving:

THEOREM B. Let \(p(z) = \sum_{k=0}^{n} a_k z^k\), \((a_k \neq 0)\) be a non-constant complex polynomial of degree \(n\). Then all the zeros of \(p(z)\) lie in the annulus \(R = \{z \in \mathbb{C} : r_3 \leq |z| \leq r_4\}\), where

\[
\begin{align*}
r_3 &= \min_{1 \leq k \leq n} \left\{ \frac{2^k P_k(n)}{P_{2n}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \\
r_4 &= \max_{1 \leq k \leq n} \left\{ \frac{P_{2n}}{2^k P_k(n)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}.
\end{align*}
\]  

(1.2)

Bidkham and Shashahani[2] proved the following result which is better than above result when \(r_3\) is obtained for \(k = n\).


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THEOREM C. Let \( p(z) = \sum_{k=0}^{n} a_k z^k \), \( (a_k \neq 0) \) be a non-constant complex polynomial of degree \( n \) and

\[
\lambda_k = \frac{5^n P_k^{(n)}}{[(1 + \sqrt{2})^{2n} + (1 - \sqrt{2})^{2n}]P_2n}.
\]

Then all the zeros of \( p(z) \) lie in the annulus \( R = \{ z \in \mathbb{C} : r_5 \leq |z| \leq r_6 \} \), where

\[
r_5 = \frac{12}{5} \min_{1 \leq k \leq n} \left\{ \lambda_k \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \quad \text{and} \quad r_6 = \frac{5}{12} \max_{1 \leq k \leq n} \left\{ \frac{1}{\lambda_k} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}.
\]

Also Affane, Biaz and Govil [1] have obtained a lower bound for the zeros of polynomials in terms of binomial coefficients by proving:

THEOREM D. Let \( p(z) = \sum_{k=0}^{n} a_k z^k \), \( (a_k \neq 0) \) be a non-constant complex polynomial of degree \( n \). Then all the zeros of \( p(z) \) lie in the \( R = \{ z \in \mathbb{C} : r_7 \leq |z| \leq r_8 \} \), where

\[
r_7 = \min_{1 \leq k \leq n} \left\{ \frac{k}{n2^{n-1}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \quad \text{and} \quad r_8 = 1 + \delta_k,
\]

being \( \delta_k \) the unique positive root of the \( k^{th} \) degree equation

\[
Q_k(x) = x^k + \sum_{v=2}^{k} \left[ \binom{k-1}{k-v} - \sum_{j=1}^{v-1} \binom{k-j-1}{k-v} \left| \frac{a_{n-j}}{a_n} \right| \right] x^{k+1-v} - A = 0,
\]

and \( A = \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right| , \quad a_j = 0 \text{ if } j < 0. \)

In this paper, we obtain a result that not only includes the above results as special cases but also gives a method for obtaining sharper bounds for the location of the zeros of a polynomial.

Let \( \alpha \) and \( \beta \) be the roots of quadratic equation

\[
x^2 - tx - s = 0
\]

being \( t, s \) strictly positive real numbers. First we define the \((t,s)\)–Fibonacci numbers as follows:

DEFINITION 1. For any positive real numbers \( t, s \), the \((t,s)\)–Fibonacci sequence, say \( \{F_{t,s,n}\}_{n \in \mathbb{N}} \) is defined by

\[
F_{t,s,n+1} = tF_{t,s,n} + sF_{t,s,n-1} \quad \text{for} \quad n \geq 1
\]

with initial conditions

\[
F_{t,s,0} = 0, \quad F_{t,s,1} = 1.
\]
THEOREM 1. Let \( p(z) = \sum_{k=0}^{n} a_k z^k \), \( (a_k \neq 0) \) be a non-constant complex polynomial of degree \( n \). Then for \( j \geq 1 \), all its zeros lie in the annulus

\[
R = \{ z \in \mathbb{C} : \ r_9 \leq |z| \leq r_{10} \}
\]

where

\[
r_9 = \min_{1 \leq k \leq n} \left\{ \frac{\binom{n}{k} F_{t,s,k}(F_{t,s,2j})^k(sF_{t,s,2j-1})^{n-k}}{F_{t,s,2jn}} \left| \frac{a_0}{a_k} \right|^\frac{1}{k} \right\} \tag{1.6}
\]

and

\[
r_{10} = \max_{1 \leq k \leq n} \left\{ \frac{F_{t,s,2jn}}{\binom{n}{k} F_{t,s,k}(F_{t,s,2j})^k(sF_{t,s,2j-1})^{n-k}} \left| \frac{a_{n-k}}{a_n} \right|^\frac{1}{k} \right\} \tag{1.7}
\]

REMARK 1. For \( t = s = 1 \) and \( j = 2 \), Theorem 2 reduces to Theorem A. If we take \( t = 2 \), \( s = 1 \) in Theorem 2, we get the following result:

COROLLARY 1. Let \( p(z) = \sum_{k=0}^{n} a_k z^k \), \( (a_k \neq 0) \) be a non-constant complex polynomial of degree \( n \). Then for \( j \geq 1 \), all its zeros lie in the annulus

\[
R = \{ z \in \mathbb{C} : \ s_1 \leq |z| \leq s_2 \}
\]

where

\[
s_1 = \min_{1 \leq k \leq n} \left\{ \frac{\binom{n}{k} P_k(P_{2j})^k(P_{2j-1})^{n-k}}{P_{2jn}} \left| \frac{a_0}{a_k} \right|^\frac{1}{k} \right\}
\]

and

\[
s_2 = \max_{1 \leq k \leq n} \left\{ \frac{P_{2jn}}{\binom{n}{k} P_k(P_{2j})^k(P_{2j-1})^{n-k}} \left| \frac{a_{n-k}}{a_n} \right|^\frac{1}{k} \right\}.
\]

REMARK 2. If \( j = 1 \), then Corollary 1 reduces to Theorem B. If \( j = 2 \), then Corollary 1 reduces to Theorem C.

2. Lemmas

LEMA 1. Let \( r \) and \( s \) be the roots of quadratic equation \( x^2 - ax - b = 0 \), being \( a, b \) strictly positive real number. Define the two sequence \( \{ A_n \}_{n \in \mathbb{N}} \) and \( \{ B_n \}_{n \in \mathbb{N}} \) by

\[
B_n = \sum_{k=0}^{n-1} r^k s^{n-1-k} \quad \text{and} \quad A_n = cr^n + ds^n, \quad \text{where} \quad c \quad \text{and} \quad d \quad \text{are real constants. If} \quad j \geq 2, \quad \text{then}
\]

\[
\sum_{k=0}^{n} \binom{n}{k} (bB_{j-1})^{n-k}(B_j)^k A_k = A_{jn}.
\]

This lemma is due to Diaz–Barreiro and Egozcue [4].

As an special case of Lemma 1, by considering \( x^2 - tx - s = 0 \) and \( A_n = B_n = F_{t,s,n} \), we get the following lemma:
LEMMA 2. For $j \geq 1$
\[
\sum_{k=0}^{n} \binom{n}{k} (sF_{t,s,2j-1})^{n-k}(F_{t,s,2j})^{k}F_{t,s,k} = F_{t,s,2jn}.
\]
holds.

LEMMA 3. The $n^{th}$ $(t, s)$–Fibonacci number is given by
\[
F_{t,s,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{(Binet's formula)} \quad (2.1)
\]
and for $j \geq 1$
\[
F_{t,s,2jn} = (\alpha^{2j-1}n + \beta^{2j-1}n)F_{t,s,2j-1n}, \quad (2.2)
\]
where $\alpha, \beta$ are the roots of the characteristic equation
\[
r^2 = tr + s \quad \text{(2.3)}
\]
for $\alpha > \beta$.

Proof. The roots of the characteristic equation (2.3) are $\alpha = (t + \sqrt{t^2 + 4s})/2$ and $\beta = (t - \sqrt{t^2 + 4s})/2$. Since $t, s$ are positive, then
\[
\beta < 0 < \alpha \quad \text{and} \quad |\beta| < |\alpha|,
\]
\[
\alpha + \beta = t \quad \text{and} \quad \alpha\beta = -s,
\]
\[
\alpha - \beta = \sqrt{t^2 + 4s}.
\]
It is trivial that (2.1) is true for $n = 0, 1$. By induction if we suppose that (2.1) is true for the terms $F_{t,s,n-1}$ and $F_{t,s,n}$, then
\[
F_{t,s,n+1} = tF_{t,s,n} + sF_{t,s,n-1} = t\frac{\alpha^n - \beta^n}{\alpha - \beta} + s\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \quad (2.4)
\]
\[
= \frac{1}{\alpha - \beta} \{ (t\alpha + s)\alpha^{n-1} - (t\beta + s)\beta^{n-1} \},
\]
since $\alpha + \beta = t$ and $\alpha\beta = -s$, we have $t\alpha + s = \alpha^2$ and $t\beta + s = \beta^2$, therefore by (2.4) we obtain
\[
F_{t,s,n+1} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \quad (2.5)
\]
By using the above formula for $(t, s)$–Fibonacci sequence, for $a = \frac{1}{\sqrt{t^2 + 4s}}$, we have
\[
F_{t,s,n} = a(\alpha^n - \beta^n)
\]
\[
F_{t,s,2n} = (\alpha^n + \beta^n)F_{t,s,n}
\]
\[
F_{t,s,4n} = (\alpha^{2n} + \beta^{2n})F_{t,s,2n}
\]
\[
\ldots
\]
\[
F_{t,s,2jn} = (\alpha^{2j-1}n + \beta^{2j-1}n)F_{t,s,2j-1n},
\]
This completes the proof of Lemma 3. □
3. Proof of the Theorem 1

If \( a_0 = 0 \), then \( r_9 = 0 \). We suppose that \( a_0 \neq 0 \). It is clear that from the definition of \( r_9 \) in (1.6) that

\[
r^k_{9} \leq \left\{ \frac{\binom{n}{k} F_{i,s,k}(F_{i,s,2^j})^k(sF_{i,s,2^j-1})^{n-k}}{F_{i,s,2/n}} \right\} \frac{a_0}{a_k} \quad k = 1, 2, \ldots, n. \tag{3.1}
\]

Then, for \( |z| < r_9 \), where \( r_9 \) is as defined in (1.6), we get

\[
|p(z)| = |a_0 + \sum_{k=1}^{n} a_k z^k| \\
\geq |a_0| - \sum_{k=1}^{n} |a_k| |z|^k \\
> |a_0| - \sum_{k=1}^{n} |a_k| r^k_{9} \\
= |a_0| \left( 1 - \sum_{k=1}^{n} \left| \frac{a_k}{a_0} \right| r^k_{9} \right) \\
\geq |a_0| \left( 1 - \sum_{k=1}^{n} \frac{\binom{n}{k} F_{i,s,k}(F_{i,s,2^j})^k(sF_{i,s,2^j-1})^{n-k}}{F_{i,s,2/n}} \right) \quad \text{(by (3.1))} \\
= 0 \quad \text{(by Lemma 2)}.
\]

This implies \( p(z) \neq 0 \) for \( |z| < r_9 \). Next, we show that all the zeros of \( p(z) \) lie in \( |z| \leq r_{10} \), where \( r_{10} \) is defined in (1.7). From (1.7), we have

\[
\left| \frac{a_{n-k}}{a_n} \right| \leq \frac{\binom{n}{k} F_{i,s,k}(F_{i,s,2^j})^k(sF_{i,s,2^j-1})^{n-k}}{F_{i,s,2/n}} r^k_{10}, \quad (1 \leq k \leq n)
\]

or

\[
\sum_{k=1}^{n} \left| \frac{a_{n-k}}{a_n} \right| \frac{1}{r^k_{10}} \leq \sum_{k=1}^{n} \frac{\binom{n}{k} F_{i,s,k}(F_{i,s,2^j})^k(sF_{i,s,2^j-1})^{n-k}}{F_{i,s,2/n}}. \tag{3.3}
\]

On the other hand

\[
|p(z)| = |a_n z^n + \ldots + a_1 z + a_0| \\
\geq |a_n z^n| - \sum_{k=1}^{n} |a_{n-k}| |z|^{n-k} \\
= |a_n z^n| \left( 1 - \sum_{k=1}^{n} \left| \frac{a_{n-k}}{a_n} \right| \frac{1}{|z|^k} \right) \\
> |a_n z^n| \left( 1 - \sum_{k=1}^{n} \frac{a_{n-k}}{a_n} \frac{1}{|z|^k_{10}} \right). \tag{3.4}
\]

Using (3.3) and Lemma 2, we have \( |p(z)| \neq 0 \) for \( |z| > r_{10} \). Consequently all the zeros of \( p(z) \) lie in \( |z| \leq r_{10} \) and this proves second part of Theorem 1. \( \square \)
4. Quality of zero bounds

For \( k = n \) in Corollary 1, we have

\[
\|s_1\| = \left\{ \frac{P_n (P_{2j})^n}{P_{2jn}} \left| \frac{a_0}{a_n} \right| \right\}^{\frac{1}{n}} .
\]

Using Lemma 3 (inequality (2.2)), we can obtain

\[
\|s_1\| = \left\{ A_j \left| \frac{a_0}{a_n} \right| \right\}^{\frac{1}{n}},
\]

where

\[
A_j = \frac{(P_{2j})^n}{\left[ \alpha^{2j-n} + \beta^{2j-n} \right] \left[ \alpha^{2j-2n} + \beta^{2j-2n} \right] \cdots \left[ \alpha^n + \beta^n \right]},
\]

with \( \alpha = 1 + \sqrt{2} \) and \( \beta = 1 - \sqrt{2} \) and

\[
\lim_{j \to \infty} A_j = \lim_{j \to \infty} \frac{(P_{2j})^n}{\left[ \alpha^{2j-n} + \beta^{2j-n} \right] \left[ \alpha^{2j-2n} + \beta^{2j-2n} \right] \cdots \left[ \alpha^n + \beta^n \right]} \cdot \frac{\alpha^n - \beta^n}{\alpha^n - \beta^n} = \lim_{j \to \infty} \frac{(P_{2j})^n}{\left[ \alpha^{2j-n} + \beta^{2j-n} \right] \left[ \alpha^{2j-2n} + \beta^{2j-2n} \right] \cdots \left[ \alpha^n + \beta^n \right]} \cdot \frac{\alpha^n - \beta^n}{\alpha^n - \beta^n} = \lim_{j \to \infty} \frac{1}{(2\sqrt{2})^n} \cdot \frac{\alpha^n - \beta^n}{1 - \beta^{2j+1}}.
\]

Since \( |\beta| < 1 \), then we get

\[
\lim_{j \to \infty} A_j = \frac{1}{(2\sqrt{2})^n} \cdot \frac{\alpha^n - \beta^n}{1 - \beta^{2j+1}} = \frac{P_n}{(2\sqrt{2})^{n-1}}.
\]

As \( A_j \) is a monotonic increasing sequence, hence if \( s_1 \) is obtained for \( k = n \), then it not only improves Theorems B and C, but also for \( j \to \infty \), it is the best possible result and we have

\[
\|s_1\| = \left\{ \frac{P_n}{(2\sqrt{2})^{n-1}} \left| \frac{a_0}{a_n} \right| \right\}^{\frac{1}{n}}.
\]

Since \( P_n > (\sqrt{2})^{n-1} \) for \( n \geq 2 \), hence if \( r_7 \) is obtained for \( k = n \), then (4.2) is better than the one in Theorem D. We can illustrate this by the following examples.

**Examples.**

i) For the polynomial \( f_1(z) = 20z^3 + z^2 + z + 1000 \), it is found that the annulus obtained by Theorems A, B and C are respectively \( 2.656646 \leq |z| \leq 5.10873 \), \( 3.057107 \leq |z| \leq 4.43952 \) and \( 3.147073 \leq |z| \leq 4.312606 \). The lower bound by Theorem D is 2.32079 and the upper bound is implicit and finally, if we use Corollary 1 for \( j \to \infty \), it turns out to be \( 3.14980 \leq |z| \leq 4.30887 \).
ii) For the polynomial $f_2(z) = 10iz^3 - z^2 - (\sqrt{3} + i)z + 400$, it is found that the annulus obtained by Theorems A, B and C are respectively $2.466212 \leq |z| \leq 4.742524$, $2.837967 \leq |z| \leq 4.121285$ and $2.921484 \leq |z| \leq 4.003469$. The lower bound by Theorem D is 2.154435 and the upper bound is implicit and finally, if we use Corollary 1 for $j \rightarrow \infty$, it turns out to be $2.924018 \leq |z| \leq 4$.

iii) For $f_3(z) = 10z^3 - z^2 - 2z + 300$, it is found that the annulus obtained by Theorems A, B and C are respectively $2.240702 \leq |z| \leq 4.308869$, $2.578464 \leq |z| \leq 3.744436$ and $2.654344 \leq |z| \leq 3.637393$. The lower bound by Theorem D is 1.957434 and the upper bound is implicit and finally, if we use Corollary 1 for $j \rightarrow \infty$, it turns out to be $2.656646 \leq |z| \leq 3.634241$.

REFERENCES


