ON CERTAIN SUBCLASSES OF $p$–VALENT FUNCTIONS

M. K. AOUF, R. M. EL-ASHWAH, AND AHMED M. ABD-ELTAWAB

Abstract. This paper gives some inclusion relationships of certain class of $p$-valent functions which are defined by using the new linear operator $\mathbb{R}^{(\alpha,\beta)}_{\beta,p}$. Further, a property preserving integrals is considered.

1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p}z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}),$$

which are analytic and $p$-valent in the unit disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $A(1) = A$.

If $f$ and $g$ are analytic in $U$, we say that $f$ is subordinate to $g$, written symbolically as, $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w$, which (by definition) is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1 \ (z \in U)$ such that $f(z) = g(w(z)) \ (z \in U)$. In particular, if the function $g$ is univalent in $U$, we have the equivalence (see for example [7]):

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

For functions $f \in A(p)$ given by (1.1) and $g \in A(p)$ given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p}z^{k+p} \quad (p \in \mathbb{N}),$$

the Hadamard product (or convolution) of $f$ and $g$ is given by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p}b_{k+p}z^{k+p} = (g * f)(z).$$


Keywords and phrases: Hadamard product, linear operators, Jack’s lemma, inclusion relationships.
For the function \( f \in A(p) \) We introduced the operator \( \mathcal{R}^{\alpha,\gamma}_{\beta,p} : A(p) \rightarrow A(p) \) as follows:

\[
\mathcal{R}^{\alpha,\gamma}_{\beta,p}f(z) = \left( p + \alpha + \beta - \gamma \right) \frac{(\alpha - \gamma + 1)\int_0^z \left( 1 - \frac{t}{z} \right)^{\alpha-1} t^{\beta-1} f(t) dt}{p + \beta - 1} \]

\[
= \frac{\Gamma(p + \alpha + \beta - \gamma + 1)}{\Gamma(p + \beta) \Gamma(\alpha - \gamma + 1) z^\beta} \int_0^z \left( 1 - \frac{t}{z} \right)^{\alpha-\gamma} t^{\beta-1} f(t) dt
\]

\[
= z^p + \frac{\Gamma(p + \alpha + \beta - \gamma + 1)}{\Gamma(p + \beta)} \sum_{k=1}^{\infty} \frac{\Gamma(\beta + p + k)}{\Gamma(\alpha + \beta + p - k + 1)} a_{k+p} z^{k+p}
\]

From (1.2), it is easy to verify that

\[
z\left( \mathcal{R}^{\alpha+1,\gamma}_{\beta,p}f(z) \right)' = (\alpha + \beta + p - \gamma + 1) \mathcal{R}^{\alpha,\gamma}_{\beta,p}f(z) - (\alpha + \beta - \gamma + 1) \mathcal{R}^{\alpha+1,\gamma}_{\beta,p}f(z). \quad (1.3)
\]

**Remark 1.** (i) For \( \gamma = 1 \),

\[
\mathcal{R}^{\alpha,1}_{\beta,p}f(z) = Q^{\alpha}_{\beta,p}f(z)
\]

\[
= \left( p + \alpha + \beta - 1 \right) \frac{\alpha}{p + \beta - 1} \int_0^z \left( 1 - \frac{t}{z} \right)^{\alpha-1} t^{\beta-1} f(t) dt
\]

\[
= \frac{\Gamma(p + \alpha + \beta - 1)}{\Gamma(p + \beta) \Gamma(\alpha) z^\beta} \int_0^z \left( 1 - \frac{t}{z} \right)^{\alpha-1} t^{\beta-1} f(t) dt
\]

\[
= z^p + \frac{\Gamma(p + \alpha + \beta)}{\Gamma(p + \beta)} \sum_{k=1}^{\infty} \frac{\Gamma(\beta + p + k)}{\Gamma(\alpha + \beta + p + k)} a_{k+p} z^{k+p}
\]

(ii) For \( \alpha = \gamma \) and \( \beta = c \),

\[
\mathcal{R}^{\alpha,\alpha}_{\beta,c}f(z) = J_{c,p}f(z)
\]

\[
= \frac{p + c}{z^c} \int_0^z t^{c-1} f(t) dt
\]

\[
= z^p + \sum_{k=1}^{\infty} \left( \frac{c + p}{c + k + p} \right) a_{k+p} z^{k+p}
\]

where \( J_{c,p} \) is the familiar integral operator, which was defined by Saitoh et al. [9].

The operator \( J_{c,1} = J_c \) was introduced by Bernardi [1] and we note that \( J_{1,1} = J \) was introduced and studied by Libera [4] and Livingston [6].
DEFINITION 1. We say that a function \( f \in A(p) \) is in the class \( R_p(\alpha + 1, \beta, \gamma, \lambda) \), if it satisfies the following condition:

\[
\Re \left\{ \frac{z \left( \Re^{\alpha+1,\gamma} f(z) \right)'}{pz^p} \right\} > \lambda, \quad (z \in U),
\]

where \( \beta > -p, \alpha > \gamma - 2, \gamma \in \mathbb{R}, 0 \leq \lambda < 1, p \in \mathbb{N} \) and \( z \in U \).

Using (1.3) condition (1.4) can be re-written in the form

\[
\Re \left\{ (\alpha + \beta + p - \gamma + 1) \frac{\Re^{\alpha,\gamma} f(z)}{pz^p} - (\alpha + \beta - \gamma + 1) \frac{\Re^{\alpha+1,\gamma} f(z)}{pz^p} \right\} > \lambda
\]

\((\beta > -p, \alpha > \gamma - 2, \gamma \in \mathbb{R}, 0 \leq \lambda < 1, p \in \mathbb{N} \) and \( z \in U \)).

REMARK 2. (i) For \( \gamma = 1 \), we have

\[
R_p(\alpha + 1, \beta, 1, \lambda) = R_p(\alpha + 1, \beta, \lambda)
\]

\[=
\left\{ f : f \in A(p) \text{ and } \Re \left\{ \frac{z \left( \Re^{\alpha+1} f(z) \right)'}{pz^p} \right\} > \lambda, \quad (\beta > -p; \alpha > -1; 0 \leq \lambda < 1; p \in \mathbb{N} \text{ and } z \in U) \right\};
\]

(ii) For \( \gamma = \alpha + 1 \) and \( \beta = c \), we have

\[
R_p(\alpha + 1, c, \alpha + 1, \lambda) = R_p(c, \lambda)
\]

\[=
\left\{ f : f \in A(p) \text{ and } \Re \left\{ \frac{z (\Re^{c} f(z))'}{pz^p} \right\} > \lambda, \quad (c > -p; 0 \leq \lambda < 1; p \in \mathbb{N} \text{ and } z \in U) \right\};
\]

2. Basic properties of the class \( R_p(\alpha + 1, \beta, \gamma, \lambda) \)

Unless otherwise mentioned, we shall assume in the reminder of this paper that \( \beta > -p, \alpha > \gamma - 2, \gamma \in \mathbb{R}, 0 \leq \lambda < 1, \) and \( p \in \mathbb{N} \).

We begin by recalling the following result (Jack’s lemma), which we shall apply in proving our inclusion theorems below.

LEMA 1. ([2]) Let the (nonconstant) function \( w(z) \) be analytic in \( U \), with \( w(0) = 0 \). If \( |w(z)| \) attains its maximum value on the circle \( |z| = r < 1 \) at a point \( z_0 \in U \), then \( z_0 w'(z_0) = \xi w(z_0) \), where \( \xi \) is a real number and \( \xi \geq 1 \).

THEOREM 1. The following inclusion property for the class \( R_p(\alpha + 1, \beta, \gamma, \lambda) \) holds true:

\[
R_p(\alpha, \beta, \gamma, \lambda) \subset R_p(\alpha + 1, \beta, \gamma, \lambda).
\]

(2.1)
Proof. Let \( f \in R_p(\alpha, \beta, \gamma, \lambda) \) and define a regular function \( w(z) \) in \( U \) such that \( w(0) = 0, w(z) \neq -1 \) by

\[
\frac{z \left( \Re^{\alpha+1, \gamma}_{\beta, p} f(z) \right)'}{pz^p} = \frac{1 + (2\lambda - 1)w(z)}{1 + w(z)}
\]

then from (1.3) we have

\[
(\alpha + \beta + p - \gamma + 1) \Re^{\alpha, \gamma}_{\beta, p} f(z) - (\alpha + \beta - \gamma + 1) \Re^{\alpha+1, \gamma}_{\beta, p} f(z) = pz^p \frac{1 + (2\lambda - 1)w(z)}{1 + w(z)}.
\]

Differentiating (2.3) with respect to \( z \), we obtain

\[
\frac{z \left( \Re^{\alpha, \gamma}_{\beta, p} f(z) \right)'}{pz^p} = \frac{1 + (2\lambda - 1)w(z)}{1 + w(z)} - \frac{2(1 - \lambda)}{\alpha + \beta + p - \gamma + 1 (1 + w(z))^2} zw'(z).
\]

We claim that \(|w(z)| < 1\) for \( z \in U \). Otherwise there exists a point \( z_0 \in U \) such that \( \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \). Applying Jack’s lemma, we have

\[
z_0 w'(z_0) = \xi w(z_0) \quad (\xi \geq 1).
\]

From (2.4) and (2.5) we have

\[
\frac{z_0 \left( \Re^{\alpha, \gamma}_{\beta, p} f(z_0) \right)'}{pz_0^p} = \frac{1 + (2\lambda - 1)w(z_0)}{1 + w(z_0)} - \frac{2(1 - \lambda)}{\alpha + \beta + p - \gamma + 1 (1 + w(z_0))^2} \xi w(z_0).
\]

Since \( \Re \left\{ \frac{1 + (2\lambda - 1)w(z_0)}{1 + w(z_0)} \right\} = \lambda, \xi \geq 1 \), and \( \frac{\xi w(z_0)}{(1 + w(z_0))^2} \) is real and positive, we see that

\[
\Re \left\{ \frac{z_0 \left( \Re^{\alpha, \gamma}_{\beta, p} f(z_0) \right)'}{pz_0^p} \right\} < \lambda,
\]

which obviously contradicts \( f \in R_p(\alpha, \beta, \gamma, \lambda) \). Hence \(|w(z)| < 1\) for \( z \in U \) and it follows from (2.2) that \( f \in R_p(\alpha + 1, \beta, \gamma, \lambda) \). This completes the proof of Theorem 1. \( \square \)

THEOREM 2. Let \( c \) be any real number and \( c > -p \). If \( f \in R_p(\alpha + 1, \beta, \gamma, \lambda) \), then

\[
J_{c,p} \in R_p(\alpha + 1, \beta, \gamma, \lambda),
\]

where \( J_{c,p} \) is defined by (1.5).

Proof. From (1.5), we have

\[
\frac{z \left( \Re^{\alpha+1, \gamma}_{\beta, p} J_{c,p} f(z) \right)'}{pz^p} = (c + p) \Re^{\alpha+1, \gamma}_{\beta, p} f(z) - c \Re^{\alpha+1, \gamma}_{\beta, p} J_{c,p} f(z).
\]

Define a regular function \( w(z) \) in \( U \) such that \( w(0) = 0, w(z) \neq -1 \) by

\[
\frac{z \left( \Re^{\alpha+1, \gamma}_{\beta, p} J_{c,p} f(z) \right)'}{pz^p} = \frac{1 + (2\lambda - 1)w(z)}{1 + w(z)}.
\]
From (2.7) and (2.8) we have
\[
(c + p) \Re_{\beta,p}^{\alpha+1,\gamma} f(z) - c \Re_{\beta,p}^{\alpha+1,\gamma} J_{c,p} f(z) = pz^p \frac{1 + (2\lambda - 1)w(z)}{1 + w(z)}. \tag{2.9}
\]

Differentiating (2.9) with respect to \(z\), and using (2.8) we obtain
\[
\frac{z}{pz^p} \left( \Re_{\beta,p}^{\alpha+1,\gamma} f(z) \right)' = \frac{1 + (2\lambda - 1)w(z)}{1 + w(z)} - \frac{2(1 - \lambda)}{c + p} \frac{zw'(z)}{(1 + w(z))^2}. \tag{2.10}
\]

The remaining part of the proof of Theorem 2 is similar to that of Theorem 1.

**THEOREM 3.** If \(f \in A(p)\), and satisfy the condition
\[
\Re \left\{ \frac{z}{pz^p} \left( \Re_{\beta,p}^{\alpha+1,\gamma} f(z) \right)' \right\} > \lambda - \frac{(1 - \lambda)}{2(c + p)} \quad (c > -p). \tag{2.11}
\]

Then
\[
J_{c,p} \in R_p(\alpha + 1, \beta, \gamma, \lambda),
\]
where \(J_{c,p}\) is defined by (1.5).

The proof of Theorem 3 is similar to that of Theorem 2 and so we omit it.

**THEOREM 4.** Let \(f(z)\) be defined by (1.5). If \(J_{c,p} \in R_p(\alpha + 1, \beta, \gamma, \lambda)\), then \(f \in R_p(\alpha + 1, \beta, \gamma, \lambda)\) in \(|z| < \frac{c + p}{1 + \sqrt{(c + p)^2 + 1}}\), where \(J_{c,p}\) is defined by (1.5).

**Proof.** Since \(J_{c,p} \in R_p(\alpha + 1, \beta, \gamma, \lambda)\) we can write
\[
z \left( \Re_{\beta,p}^{\alpha+1,\gamma} J_{c,p} f(z) \right)' = pz^p [\lambda + (1 - \lambda)u(z)], \tag{2.12}
\]
where \(u(z) \in P\), the class of functions with positive real part in the unit disk \(U\) and normalized by \(u(0) = 1\). We can re-write (2.13) as
\[
(c + p) \Re_{\beta,p}^{\alpha+1,\gamma} f(z) - c \Re_{\beta,p}^{\alpha+1,\gamma} J_{c,p} f(z) = pz^p [\lambda + (1 - \lambda)u(z)]. \tag{2.13}
\]

Differentiating (2.13) with respect to \(z\), and using (2.7) we obtain
\[
\left( \frac{z}{pz^p} \left( \Re_{\beta,p}^{\alpha+1,\gamma} f(z) \right)' \right)' - \lambda = (1 - \lambda)^{-1} u(z) + \frac{1}{c + p} zu'(z). \tag{2.14}
\]

Using the well-known estimate (see [8]) \(|zu'(z)| \leq \frac{2r}{1 - r^2} \Re u(z), |z| = r\), (2.14) yields
\[
\Re \left\{ \left( \frac{z}{pz^p} \left( \Re_{\beta,p}^{\alpha+1,\gamma} f(z) \right)' \right)' - \lambda \right\} (1 - \lambda)^{-1} \geq \left(1 - \frac{2r}{(c + p)(1 - r^2)}\right) \Re u(z). \tag{2.15}
\]
The right-hand side of (2.15) is positive if $r < \frac{c+p}{1+\sqrt{(c+p)^2+1}}$.

The result is sharp for the function $f$ defined by $f(z) = \frac{1}{(c+p)^c} \left( z^c J_{c,p} f(z) \right)'$ where $J_{c,p}$ is given by

$$
\left( \Re^{\alpha+1, \gamma}_{\beta, p} J_{c,p} f(z) \right)' = pz^{p-1} \frac{1+(2\lambda-1)z}{1+z}.
$$

**Remark 3.** (i) Taking $\gamma = 1$ in the above results, we obtain analogues results for the subclasses which are defined in Remark 2 (i);

(ii) Taking $\gamma = \alpha + 1$ and $\beta = c$ ($c > -p$) in the above results, we obtain analogues results for the subclasses which are defined in Remark 2 (ii).

**References**


(Received May 20, 2013)