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ON CERTAIN SUBCLASSES OF p–VALENT FUNCTIONS

M. K. AOUF, R. M. EL-ASHWAH, AND AHMED M. ABD-ELTAWAB

Abstract. This paper gives some inclusion relationships of certain class of p-valent functions which are defined by using the new linear operator $\Re_{\beta,p}^{\alpha,\gamma}$. Further, a property preserving integrals is considered.

1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1,2,3,...\}), \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $A(1) = A$.

If f and g are analytic in U, we say that f is subordinate to g, written symbolically as, $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1\,(z \in U)$ such that $f(z) = g(w(z))\,(z \in U)$. In particular, if the function g is univalent in U, we have the equivalence (see for example [7]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f \in A(p)$ given by (1.1) and $g \in A(p)$ given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p} \quad (p \in \mathbb{N}),$$

the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z).$$

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For the function $f \in A(p)$ We introduced the operator $\Re^{\alpha,\gamma}_{\beta,p} : A(p) \to A(p)$ as follows:

$$\begin{aligned} \Re^{\alpha,\gamma}_{\beta,p} f(z) &= \binom{p+\alpha+\beta-\gamma}{p+\beta-1} \frac{(\alpha-\gamma+1)}{z^{\beta}} \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-\gamma} t^{\beta-1} f(t)\, dt \\ &= \frac{\Gamma(p+\alpha+\beta-\gamma+1)}{\Gamma(p+\beta)\Gamma(\alpha-\gamma+1)} \frac{1}{z^{\beta}} \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-\gamma} t^{\beta-1} f(t)\, dt \\ &= z^p + \frac{\Gamma(p+\alpha+\beta-\gamma+1)}{\Gamma(p+\beta)} \sum_{k=1}^{\infty} \left[\frac{\Gamma(\beta+p+k)}{\Gamma(\alpha+\beta+p+k-\gamma+1)}\right] a_{k+p} z^{k+p} \\ &(\beta > -p;\ \alpha > \gamma-1; \gamma \in \mathbb{R}; p \in \mathbb{N}; z \in U) \end{aligned} \tag{1.2}$$

From (1.2), it is easy to verify that

$$z\left(\Re^{\alpha+1,\gamma}_{\beta,p} f(z)\right)' = (\alpha+\beta+p-\gamma+1)\Re^{\alpha,\gamma}_{\beta,p} f(z) - (\alpha+\beta-\gamma+1)\Re^{\alpha+1,\gamma}_{\beta,p} f(z). \tag{1.3}$$

REMARK 1. (i) For $\gamma = 1$,

$$\begin{aligned} \Re^{\alpha,1}_{\beta,p} f(z) &= Q^{\alpha}_{\beta,p} f(z) \\ &= \binom{p+\alpha+\beta-1}{p+\beta-1} \frac{\alpha}{z^{\beta}} \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\, dt \\ &= \frac{\Gamma(p+\alpha+\beta)}{\Gamma(p+\beta)\Gamma(\alpha)} \frac{1}{z^{\beta}} \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\, dt \\ &= z^p + \frac{\Gamma(p+\alpha+\beta)}{\Gamma(p+\beta)} \sum_{k=1}^{\infty} \left[\frac{\Gamma(\beta+p+k)}{\Gamma(\alpha+\beta+p+k)}\right] a_{k+p} z^{k+p} \qquad (1.4) \\ &(\beta > -p;\ \alpha > 0; p \in \mathbb{N}; z \in U), \end{aligned}$$

where the operator $Q^{\alpha}_{\beta,p}$ was introduced and studied by Liu and Owa [5] and $Q^{\alpha}_{\beta,1} = Q^{\alpha}_{\beta}$, where the operator Q^{α}_{β} was introduced and studied by Jung et al. [3];

(ii) For $\alpha = \gamma$ and $\beta = c$,

$$\begin{aligned} \Re^{\alpha,\alpha}_{\beta,p} f(z) &= J_{c,p} f(z) \\ &= \frac{p+c}{z^c} \int_0^z t^{c-1} f(t)\, dt \\ &= z^p + \sum_{k=1}^{\infty} \left(\frac{c+p}{c+k+p}\right) a_{k+p} z^{k+p} \qquad (1.5) \\ &(c > -p; p \in \mathbb{N}; z \in U) \end{aligned}$$

where $J_{c,p}$ is the familiar integral operator, which was defined by Saitoh et al. [9]. The operator $J_{c,1} = J_c$ was introduced by Bernardi [1] and we note that $J_{1,1} = J$ was introduced and studied by Libera [4] and Livingston [6].

DEFINITION 1. We say that a function $f \in A(p)$ is in the class $R_p(\alpha+1,\beta,\gamma,\lambda)$, if it satisfies the following condition:

$$\operatorname{Re}\left\{\frac{z\left(\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}f(z)\right)'}{pz^p}\right\} > \lambda, \quad (z \in U), \tag{1.6}$$

where $\beta > -p,\ \alpha > \gamma-2, \gamma \in \mathbb{R}, 0 \leqslant \lambda < 1, p \in \mathbb{N}$ and $z \in U$.

Using (1.3) condition (1.4) can be re-written in the form

$$\operatorname{Re}\left\{(\alpha+\beta+p-\gamma+1)\frac{\mathfrak{R}_{\beta,p}^{\alpha,\gamma}f(z)}{pz^p} - (\alpha+\beta-\gamma+1)\frac{\mathfrak{R}_{\beta,p}^{\alpha+1,\gamma}f(z)}{pz^p}\right\} > \lambda$$
$$(\beta > -p,\ \alpha > \gamma-2, \gamma \in \mathbb{R}, 0 \leqslant \lambda < 1, p \in \mathbb{N} \text{ and } z \in U).$$

REMARK 2. (i) For $\gamma = 1$, we have

$$\begin{aligned} R_p(\alpha+1,\beta,1,\lambda) &= R_p(\alpha+1,\beta,\lambda) \\ &= \left\{ f : f \in A(p) \text{ and } \operatorname{Re}\left\{\frac{z\left(Q_{\beta,p}^{\alpha+1}f(z)\right)'}{pz^p}\right\} > \lambda, \right. \\ &\qquad \left. (\beta > -p;\ \alpha > -1; 0 \leqslant \lambda < 1; p \in \mathbb{N} \text{ and } z \in U) \right\}; \end{aligned}$$

(ii) For $\gamma = \alpha+1$ and $\beta = c$, we have

$$\begin{aligned} R_p(\alpha+1,c,\alpha+1,\lambda) &= R_p(c,\lambda) \\ &= \left\{ f : f \in A(p) \text{ and } \operatorname{Re}\left\{\frac{z\left(J_{c,p}(z)\right)'}{pz^p}\right\} > \lambda, \right. \\ &\qquad \left. (c > -p;\ 0 \leqslant \lambda < 1; p \in \mathbb{N} \text{ and } z \in U) \right\}; \end{aligned}$$

2. Basic properties of the class $R_p(\alpha+1,\beta,\gamma,\lambda)$

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\beta > -p,\ \alpha > \gamma-2, \gamma \in \mathbb{R},\ 0 \leqslant \lambda < 1$, and $p \in \mathbb{N}$.

We begin by recalling the following result (Jack's lemma), which we shall apply in proving our inclusion theorems below.

LEMMA 1. ([2]) *Let the (nonconstant) function $w(z)$ be analytic in U, with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then $z_0 w'(z_0) = \xi w(z_0)$, where ξ is a real number and $\xi \geqslant 1$.*

THEOREM 1. *The following inclusion property for the class $R_p(\alpha+1,\beta,\gamma,\lambda)$ holds true:*

$$R_p(\alpha,\beta,\gamma,\lambda) \subset R_p(\alpha+1,\beta,\gamma,\lambda). \tag{2.1}$$

Proof. Let $f \in R_p(\alpha,\beta,\gamma,\lambda)$ and define a regular function $w(z)$ in U such that $w(0)=0, w(z) \neq -1$ by

$$\frac{z\left(\Re_{\beta,p}^{\alpha+1,\gamma} f(z)\right)'}{pz^p} = \frac{1+(2\lambda-1)\,w(z)}{1+w(z)} \tag{2.2}$$

then from (1.3) we have

$$(\alpha+\beta+p-\gamma+1)\,\Re_{\beta,p}^{\alpha,\gamma} f(z) - (\alpha+\beta-\gamma+1)\,\Re_{\beta,p}^{\alpha+1,\gamma} f(z) = pz^p \frac{1+(2\lambda-1)\,w(z)}{1+w(z)}. \tag{2.3}$$

Differentiating (2.3) with respect to z, we obtain

$$\frac{z\left(\Re_{\beta,p}^{\alpha,\gamma} f(z)\right)'}{pz^p} = \frac{1+(2\lambda-1)\,w(z)}{1+w(z)} - \frac{2\,(1-\lambda)}{\alpha+\beta+p-\gamma+1}\,\frac{zw'(z)}{(1+w(z))^2}. \tag{2.4}$$

We claim that $|w(z)|<1$ for $z\in U$. Otherwise there exists a point $z_0 \in U$ such that $\max_{|z|\leqslant|z_0|}\ |w(z)| = |w(z_0)| = 1$. Applying Jack's lemma, we have

$$z_0 w'(z_0) = \xi w(z_0) \qquad (\xi \geqslant 1). \tag{2.5}$$

From (2.4) and (2.5) we have

$$\frac{z_0\left(\Re_{\beta,p}^{\alpha,\gamma} f(z_0)\right)'}{pz_0^p} = \frac{1+(2\lambda-1)\,w(z_0)}{1+w(z_0)} - \frac{2\,(1-\lambda)}{\alpha+\beta+p-\gamma+1}\,\frac{\xi w(z_0)}{(1+w(z_0))^2}. \tag{2.6}$$

Since $\operatorname{Re}\left\{\frac{1+(2\lambda-1)w(z_0)}{1+w(z_0)}\right\} = \lambda, \xi \geqslant 1$, and $\frac{\xi w(z_0)}{(1+w(z_0))^2}$ is real and positive, we see that $\operatorname{Re}\left\{\frac{z_0\left(\Re_{\beta,p}^{\alpha,\gamma} f(z_0)\right)'}{pz_0^p}\right\} < \lambda$, which obviously contradicts $f \in R_p(\alpha,\beta,\gamma,\lambda)$. Hence $|w(z)| < 1$ for $z \in U$ and it follows from (2.2) that $f \in R_p(\alpha+1,\beta,\gamma,\lambda)$. This completes the proof of Theorem 1. □

THEOREM 2. *Let c be any real number and $c > -p$. If $f \in R_p(\alpha+1,\beta,\gamma,\lambda)$, then*

$$J_{c,p} \in R_p(\alpha+1,\beta,\gamma,\lambda)\,,$$

where $J_{c,p}$ is defined by (1.5).

Proof. From (1.5), we have

$$z\left(\Re_{\beta,p}^{\alpha+1,\gamma} J_{c,p} f(z)\right)' = (c+p)\,\Re_{\beta,p}^{\alpha+1,\gamma} f(z) - c\Re_{\beta,p}^{\alpha+1,\gamma} J_{c,p} f(z)\,. \tag{2.7}$$

Define a regular function $w(z)$ in U such that $w(0)=0, w(z)\neq -1$ by

$$\frac{z\left(\Re_{\beta,p}^{\alpha+1,\gamma} J_{c,p} f(z)\right)'}{pz^p} = \frac{1+(2\lambda-1)\,w(z)}{1+w(z)}. \tag{2.8}$$

From (2.7) and (2.8) we have

$$(c+p)\Re_{\beta,p}^{\alpha+1,\gamma} f(z) - c\Re_{\beta,p}^{\alpha+1,\gamma} J_{c,p} f(z) = pz^p \frac{1+(2\lambda-1)w(z)}{1+w(z)}. \tag{2.9}$$

Differentiating (2.9) with respect to z, and using (2.8) we obtain

$$\frac{z\left(\Re_{\beta,p}^{\alpha+1,\gamma} f(z)\right)'}{pz^p} = \frac{1+(2\lambda-1)w(z)}{1+w(z)} - \frac{2(1-\lambda)}{c+p}\frac{zw'(z)}{(1+w(z))^2}. \tag{2.10}$$

The remaining part of the proof of Theorem 2 is similar to that of Theorem 1. □

THEOREM 3. *If $f \in A(p)$, and satisfy the condition*

$$\operatorname{Re}\left\{\frac{z\left(\Re_{\beta,p}^{\alpha+1,\gamma} f(z)\right)'}{pz^p}\right\} > \lambda - \frac{(1-\lambda)}{2(c+p)} \quad (c > -p). \tag{2.11}$$

Then

$$J_{c,p} \in R_p(\alpha+1,\beta,\gamma,\lambda),$$

where $J_{c,p}$ is defined by (1.5).

The proof of Theorem 3 is similar to that of Theorem 2 and so we omit it.

THEOREM 4. *Let $f(z)$ be defined by (1.5). If $J_{c,p} \in R_p(\alpha+1,\beta,\gamma,\lambda)$, then $f \in R_p(\alpha+1,\beta,\gamma,\lambda)$ in $|z| < \frac{c+p}{1+\sqrt{(c+p)^2+1}}$, where $J_{c,p}$ is defined by (1.5).*

Proof. Since $J_{c,p} \in R_p(\alpha+1,\beta,\gamma,\lambda)$ we can write

$$z\left(\Re_{\beta,p}^{\alpha+1,\gamma} J_{c,p} f(z)\right)' = pz^p\left[\lambda + (1-\lambda)u(z)\right], \tag{2.12}$$

where $u(z) \in P$, the class of functions with positive real part in the unit disk U and normalized by $u(0) = 1$. We can re-write (2.13) as

$$(c+p)\Re_{\beta,p}^{\alpha+1,\gamma} f(z) - c\Re_{\beta,p}^{\alpha+1,\gamma} J_{c,p} f(z) = pz^p\left[\lambda + (1-\lambda)u(z)\right]. \tag{2.13}$$

Differentiating (2.13) with respect to z, and using (2.7) we obtain

$$\left(\frac{z\left(\Re_{\beta,p}^{\alpha+1,\gamma} f(z)\right)'}{pz^p} - \lambda\right)(1-\lambda)^{-1} = u(z) + \frac{1}{c+p} zu'(z). \tag{2.14}$$

Using the well-known estimate (see [8]) $|zu'(z)| \leqslant \frac{2r}{1-r^2}\operatorname{Re} u(z), |z| = r$, (2.14) yields

$$\operatorname{Re}\left\{\left(\frac{z\left(\Re_{\beta,p}^{\alpha+1,\gamma} f(z)\right)'}{pz^p} - \lambda\right)(1-\lambda)^{-1}\right\} \geqslant \left(1 - \frac{2r}{(c+p)(1-r^2)}\right)\operatorname{Re} u(z). \tag{2.15}$$

The right-hand side of (2.15) is positive if $r < \frac{c+p}{1+\sqrt{(c+p)^2+1}}$.

The result is sharp for the function f defined by $f(z) = \frac{1}{(c+p)z^{c-1}} \left(z^c J_{c,p} f(z)\right)'$ where $J_{c,p}$ is given by $\left(\Re_{\beta,p}^{\alpha+1,\gamma} J_{c,p} f(z)\right)' = pz^{p-1} \frac{1+(2\lambda-1)z}{1+z}$.

REMARK 3. (i) Taking $\gamma = 1$ in the above results, we obtain analogues results for the subclasses which are defined in Remark 2 (i);

(ii) Taking $\gamma = \alpha + 1$ and $\beta = c$ $(c > -p)$ in the above results, we obtain analogues results for the subclasses which are defined in Remark 2 (ii).

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M. K. Aouf
Department of Mathematics, Faculty of Science
Mansoura University
Mansoura 35516, Egypt
e-mail: mkaouf127@yahoo.com

R. M. El-Ashwah
Department of Mathematics, Faculty of Science
Damietta University
New Damietta 34517, Egypt
e-mail: r_elashwah@yahoo.com

Ahmed M. Abd-Eltawab
Department of Mathematics, Faculty of Science
Fayoum University
Fayoum 63514, Egypt
e-mail: ams03@fayoum.edu.eg