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APPLICATIONS OF GENERAL MONOTONE SEQUENCES AND FUNCTIONS TO TRIGONOMETRIC SERIES AND INTEGRALS

PÉTER KÓRUS

Abstract. Using the concept of general monotone sequences and functions we were able to extend some of the results of L. Leindler from sine series to cosine series and to sine and cosine integrals as well.

1. Introduction: general monotone sequences

In the last few years, several results were achieved in various convergence problems of trigonometric series and integrals, concerning general monotone sequences and functions, see for example [1, 2, 3, 4, 5] and its references. General monotone sequences were first introduced and used by L. Leindler in [3] named as $\gamma-$*group bounded variation sequences*. We recall that a complex sequence $\{c_k\}_{k=1}^{\infty}$ is called a β-*general monotone sequence* or shortly $(\{c_k\},\{\beta_n\}) \in \mathrm{GMS}$, see [2, 1], if

$$\sum_{k=n}^{2n} |\Delta c_k| \leqslant C\beta_n$$

where $\Delta c_k = c_k - c_{k+1}$, $C = C(\{c_k\},\{\beta_n\})$ is a positive constant independent from n and $\beta = \{\beta_n\}_{n=1}^{\infty}$ is a positive sequence. For β-general monotone sequences, the following theorems were proved in [3].

THEOREM A *Let* $(\{b_k\},\{\beta_n\}) \in \mathrm{GMS}$. *If* $\beta_n = o(n^{-1})$ *then the sine series*

$$\sum_{k=1}^{\infty} b_k \sin kx$$

is uniformly convergent in x.

THEOREM B *Let* $(\{b_k\},\{\beta_n\}) \in \mathrm{GMS}$. *If* $\beta_n = O(n^{-1})$ *or* $\beta_n = o(n^{-1})$, *furthermore* $\{n_j\}$ *is quasi geometrically increasing, then the estimates*

$$\sum_{j=1}^{\infty} \left| \sum_{k=n_j}^{n_{j+1}-1} b_k \sin kx \right| \leqslant C = C(\{b_k\},\{n_j\})$$

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or

$$\sum_{j=m}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} b_k \sin kx\right| = o(1), \quad m\to\infty$$

hold uniformly in x, respectively.

We recall that a sequence $\{n_j\}$ is said to be quasi geometrically increasing (decreasing) if there exist natural numbers μ and $C \geqslant 1$ such that for all natural numbers j

$$n_{j+\mu} \geqslant 2n_j \quad \text{and} \quad n_j \leqslant Cn_{j+1} \qquad \left(n_{j+\mu} \leqslant \frac{1}{2}n_j \quad \text{and} \quad n_{j+1} \leqslant Cn_j\right).$$

Shortly after, in [4] it was seen that

THEOREM C [4] *If the function $f \in L^1(0,2\pi)$ with $\{b_k\}$ Fourier sine coefficients, the sequence $\{n_j\}$ is quasi geometrically increasing, and $(\{c_k\},\{\beta_n\}) \in \mathrm{GMS}$ with $\beta_n = O(n^{-1})$ or $\beta_n = o(n^{-1})$, then*

$$\sum_{j=1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} b_k c_k\right| \leqslant C(\{n_j\},\{\beta_n\})\,\|f\|_L$$

or

$$\sum_{j=m}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} b_k c_k\right| = o(1), \quad m\to\infty$$

hold, respectively.

The previous results deal with sine series and it seems to be an obvious question what happens when we have cosine series. For the uniform convergence an analogous theorem to Theorem A was proved.

THEOREM D [2] *Let $(\{a_k\},\{\beta_n\}) \in \mathrm{GMS}$. If $\beta_n = o(n^{-1})$ then the cosine series*

$$\sum_{k=1}^{\infty} a_k \cos kx$$

is uniformly convergent in x if and only if

$$\sum_{k=1}^{\infty} a_k \quad \textit{converges.} \tag{1.1}$$

We note that the root of Theorems B and C goes back to S. A. Telyakovskiĭ [7]. Throughout this paper, we denote by C a positive constant that may be different in different occurrences.

2. Main results on trigonometric series

We draw an analogous result to Theorem B.

THEOREM 1 *Let* $(\{a_k\},\{\beta_n\}) \in \mathrm{GMS}$. *If* $\beta_n = O(n^{-1})$ *or* $\beta_n = o(n^{-1})$, *furthermore* $\{n_j\}$ *is quasi geometrically increasing, then the estimates*

$$\sum_{j=1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} a_k \cos kx\right| \leqslant C = C(\{a_k\},\{n_j\}) \tag{2.1}$$

or

$$\sum_{j=m}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} a_k \cos kx\right| = o(1), \quad m \to \infty \tag{2.2}$$

hold uniformly in x, *respectively, if and only if*

$$\sum_{j=1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} a_k\right| < \infty. \tag{2.3}$$

Proof. Necessity. If either (2.1) or (2.2) is satisfied, then choosing $x = 0$ we get (2.3) immediately.
Sufficiency. We only consider the necessary modifications to the proof of [3, Theorem 2.3] (i.e. Theorem B). We can suppose that $0 \leqslant x < \pi$. For $x = 0$, (2.1) or (2.2) is implied by (2.3). In the case of $0 < x < \pi$, to prove (2.1) we assume $\beta_n = O(n^{-1})$. First, we have for $n_i \leqslant N = \lfloor {}^{\pi}/_{x} \rfloor < n_{i+1}$ that

$$\begin{aligned}
&\sum_{j=1}^{i-1}\left|\sum_{k=n_j}^{n_{j+1}-1} a_k \cos kx\right| + \left|\sum_{k=n_i}^{N} a_k \cos kx\right| \\
&= \sum_{j=1}^{i-1}\left|\sum_{k=n_j}^{n_{j+1}-1} a_k \left(1+2\sin^2\frac{kx}{2}\right)\right| + \left|\sum_{k=n_i}^{N} a_k \left(1+2\sin^2\frac{kx}{2}\right)\right| \\
&\leqslant \sum_{j=1}^{i-1}\left|\sum_{k=n_j}^{n_{j+1}-1} a_k\right| + \left|\sum_{k=n_i}^{N} a_k\right| + \sum_{k=n_1}^{N} 2|a_k|\frac{kx}{2} \\
&\leqslant C + C + C\varepsilon_{n_1} N x \leqslant C,
\end{aligned} \tag{2.4}$$

where $\varepsilon_n = \sup_{k\geqslant n} k\beta_k$, since it was seen in [3] that for $k \geqslant n$, $k|a_k| \leqslant C\varepsilon_n$, furthermore by (2.3), any $\sum_{j=1}^{i}\left|\sum_{k=n_j}^{n_{j+1}-1} a_k\right|$ and $\left|\sum_{k=n_i}^{N} a_k\right|$ sums are bounded. The second needed estimation is

$$\left|\sum_{k=N+1}^{n_{i+1}-1} a_k \cos kx\right| + \sum_{j=i+1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} a_k \cos kx\right| \leqslant C\varepsilon_N. \tag{2.5}$$

This inequality is obtained using the well-known estimation $|\sum_{k=1}^{n}\cos kx| \leqslant \frac{C}{x}$ and some calculations as in [3]. If we sum up (2.4) and (2.5) we get (2.1). To see (2.2), we need inequality (2.5) and

$$\begin{aligned}
&\sum_{j=m}^{i-1}\left|\sum_{k=n_j}^{n_{j+1}-1} a_k\cos kx\right| + \left|\sum_{k=n_i}^{N} a_k\cos kx\right| \\
&\qquad\leqslant \sum_{j=m}^{i-1}\left|\sum_{k=n_j}^{n_{j+1}-1} a_k\right| + \left|\sum_{k=n_i}^{N} a_k\right| + \sum_{k=n_m}^{N} 2|a_k|\frac{kx}{2} \\
&\qquad\leqslant \varepsilon'_m + \varepsilon''_{n_i} + C\varepsilon_{n_m}
\end{aligned}$$

where

$$\varepsilon'_m = \sum_{j=m}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} a_k\right| \quad \text{and} \quad \varepsilon''_n = \sup_{M\geqslant n}\left|\sum_{k=n}^{M} a_k\right|,$$

since $\beta_n = o(n^{-1})$ implies $\varepsilon_n = o(1)$, while (2.3) implies $\varepsilon'_m = o(1)$ and $\varepsilon''_n = o(1)$.

We note that (2.3) can not be replaced by the weaker condition (1.1) in Theorem 1, consider for example the sequence $n_j = 2^j$ and define $a_k = {}^{(-1)^j}/_{j2^j}$ for $2^j \leqslant k < 2^{j+1}$. This time (1.1) is satisfied but (2.1) or (2.2) is not (just take $x = 0$).

We can also extend Theorem C for cosine coefficients as well.

COROLLARY 1 *Theorem C remains true if (2.3) is satisfied and f has $\{a_k\}$ Fourier cosine coefficients instead of $\{b_k\}$ Fourier sine coefficients.*

This can be proved by an analogous argumentation to the proof of [4, Theorem 2.1] with cosine function in place of sine function, we leave it to the reader.

3. Background: general monotone functions

In the previous sections we drew results about trigonometric series with general monotone coefficients. In the following sections we study trigonometric integrals, i.e. the Fourier transforms of general monotone functions. From now on, we deal with *admissible* functions f and g (or h, if we do not distinguish between them) defined on $\mathbb{R}_+ = [0,\infty)$, locally of bounded variation on $(0,\infty)$, vanishing at infinity, and such that $f(t) \in L^1[0,1]$ and $tg(t) \in L^1[0,1]$. We say that a function h is β*-general monotone*, or shortly $(h,\beta) \in \mathrm{GM}$, see [5, 1], if

$$\int_u^{2u} |dh(t)| < C\beta(u)$$

holds for all $u \in (0,\infty)$, where $C = C(h,\beta)$ is a positive constant independent of u, and β is a majorant, that is, a positive function on $\mathbb{R}_+$. For β-general monotone functions, the following theorems were proved in [1].

THEOREM E *Let* $(g,\beta) \in \mathrm{GM}$. *If* $\beta(t) = o(t^{-1})$ *as* $t \to \infty$, *then the sine Fourier transform*

$$G(x) = \int_0^\infty g(t)\sin xt\,dt$$

converges uniformly on $\mathbb{R}_+$.

THEOREM F *Let* $(f,\beta) \in \mathrm{GM}$. *If* $\beta(t) = o(t^{-1})$ *as* $t \to \infty$, *then the cosine Fourier transform*

$$F(x) = \int_0^\infty f(t)\cos xt\,dt$$

converges uniformly on $\mathbb{R}_+$ *if and only if*

$$\int_0^u f(t)\,dt \quad \text{converges as } u \to \infty. \tag{3.1}$$

We note that certain general monotone classes of functions were also considered in [6] for the uniform convergence of sine integrals.

4. Main results on trigonometric integrals

We prove analogous results to Theorem B for sine and cosine integrals as well.

THEOREM 2 *Let* $(g,\beta) \in \mathrm{GM}$. *If* $\beta(t) = O(t^{-1})$ *or* $\beta(t) = o(t^{-1})$ *as* $t \to \infty$, *furthermore* $\{u_j\}$ *is quasi geometrically increasing, then the estimates*

$$\sum_{j=1}^{\infty} \left| \int_{u_j}^{u_{j+1}} g(t)\sin xt\,dt \right| \leqslant C(g,\{u_j\}) \tag{4.1}$$

or

$$\sum_{j=m}^{\infty} \left| \int_{u_j}^{u_{j+1}} g(t)\sin xt\,dt \right| = o(1), \quad m \to \infty \tag{4.2}$$

hold uniformly in $x \in \mathbb{R}_+$, *respectively.*

Before we prove Theorem 2 we need two lemmas.

LEMMA 1 [1] *If* $(h,\beta) \in \mathrm{GM}$ *with* $\beta(t) = O(t^{-1})$ *as* $t \to \infty$, *then*

$$t|h(t)| \leqslant t\int_t^\infty |dh(u)| \leqslant C \sup_{u \geqslant t/2} \left(u \int_u^{2u} |dh(v)| \right) \leqslant C\varepsilon_{t/2},$$

where C *is a positive constant independent of* t *and* $\varepsilon_t = \sup_{u \geqslant t} u\beta(u)$.

LEMMA 2 *Let* $(h,\beta)\in \mathrm{GM}$ *with* $\beta(t)=O(t^{-1})$ *as* $t\to\infty$. *If for a complex function* $d(t)$ *there exists a constant* D *such that for any* $u>0$,

$$\left|\int_0^u d(t)\,dt\right|\leqslant D,$$

then for any $U\geqslant u$,

$$\left|\int_u^U h(t)d(t)\,dt\right|\leqslant C\varepsilon_{u/2}\,u^{-1},$$

where C *is a constant independent of* u. *Consequently, if* $\varepsilon_u=o(u)$, *then* $\int_0^u h(t)d(t)\,dt$ *converges as* $u\to\infty$.

Proof. By Lemma 1, we have

$$|h(t)|\leqslant\int_t^\infty|dh(u)|\leqslant C\varepsilon_{t/2}\,t^{-1}.$$

Hence using the notation

$$D(u)=\int_0^u d(t)\,dt$$

and integrating by parts we get

$$\begin{aligned}\left|\int_u^U h(t)d(t)\,dt\right|&=\left|\big[h(t)D(t)\big]_u^U-\int_u^U D(t)\,dh(t)\right|\\&\leqslant D\left(|h(u)|+|h(U)|+\int_u^U|dh(t)|\right)\leqslant C\varepsilon_{u/2}\,u^{-1},\end{aligned}$$

which proves Lemma 2.

Proof. [Proof of Theorem 2] First we show (4.1). We can suppose that $x\in(0,\infty)$, furthermore $\beta(t)=O(t^{-1})$ as $t\to\infty$. For $u_i\leqslant U={}^{\pi}/_{x}<u_{i+1}$, we have by Lemma 1 that

$$\sum_{j=1}^{i-1}\left|\int_{u_j}^{u_{j+1}}g(t)\sin xt\,dt\right|+\left|\int_{u_i}^{U}g(t)\sin xt\,dt\right|\leqslant\int_{u_1}^{U}|g(t)|xt\,dt\leqslant C\varepsilon_{u_1}Ux\leqslant C, \tag{4.3}$$

since $\varepsilon_t=\sup_{u\geqslant t}u\beta(u)=O(1)$. On the other hand, by

$$\left|\int_0^u\sin xt\,dt\right|=\left|\big[-x^{-1}\cos xt\big]_0^u\right|\leqslant 2x^{-1},$$

we can use Lemma 2 for $h = g$ and $d(t) = \sin xt$. Hence

$$\begin{aligned} I_U &= \left| \int_U^{u_{i+1}} g(t) \sin xt \, dt \right| + \sum_{j=i+1}^{\infty} \left| \int_{u_j}^{u_{j+1}} g(t) \sin xt \, dt \right| \\ &\leqslant Cx^{-1} \varepsilon_{U/2} U^{-1} + Cx^{-1} \sum_{j=i+1}^{\infty} \varepsilon_{u_j/2} u_j^{-1} \\ &\leqslant C\varepsilon_{U/2} \left(1 + U \sum_{j=i+1}^{\infty} u_j^{-1} \right). \end{aligned}$$

Since u_j^{-1} is quasi geometrically decreasing,

$$\sum_{j=m}^{\infty} u_j^{-1} \leqslant C u_m^{-1} \quad (m = 1, 2, \ldots, C \geqslant 1),$$

see [3]. This implies

$$I_U \leqslant C\varepsilon_{U/2} \left(1 + U u_{i+1}^{-1}\right) \leqslant C\varepsilon_{U/2} \leqslant C. \tag{4.4}$$

From (4.3) and (4.4) we obtain (4.1). To see (4.2), we only need to sum from m instead of 1 in (4.3), take (4.4), and note that by $\beta(t) = o(t^{-1})$, $\varepsilon_t = o(1)$.

THEOREM 3 *Let $(f, \beta) \in \mathrm{GM}$. If $\beta(t) = O(t^{-1})$ or $\beta(t) = o(t^{-1})$ as $t \to \infty$, furthermore $\{u_j\}$ is quasi geometrically increasing, then the estimates*

$$\sum_{j=1}^{\infty} \left| \int_{u_j}^{u_{j+1}} f(t) \cos xt \, dt \right| \leqslant K(f, \{u_j\}) \tag{4.5}$$

or

$$\sum_{j=m}^{\infty} \left| \int_{u_j}^{u_{j+1}} f(t) \cos xt \, dt \right| = o(1), \quad m \to \infty \tag{4.6}$$

hold uniformly in $x \in \mathbb{R}_+$, respectively, if and only if

$$\sum_{j=1}^{\infty} \left| \int_{u_j}^{u_{j+1}} f(t) \, dt \right| < \infty. \tag{4.7}$$

Proof. Necessity. If either (4.5) or (4.6) is satisfied, then by taking $x = 0$ we obtain (4.7).

Sufficiency. First let us see (4.5). For $x = 0$, (4.5), also (4.6), is implied by (4.7). If we assume that $x \in (0, \infty)$, furthermore $\beta(t) = O(t^{-1})$ as $t \to \infty$, then for $u_i \leqslant U = {}^{\pi}/_{x} <$

u_{i+1}, we have by Lemma 1 that

$$\begin{aligned}
&\sum_{j=1}^{i-1}\left|\int_{u_j}^{u_{j+1}} f(t)\cos xt\,dt\right| + \left|\int_{u_i}^{U} f(t)\cos xt\,dt\right| \\
&\quad = \sum_{j=1}^{i-1}\left|\int_{u_j}^{u_{j+1}} f(t)\left(1+2\sin^2\frac{xt}{2}\right)dt\right| + \left|\int_{u_i}^{U} f(t)\left(1+2\sin^2\frac{xt}{2}\right)dt\right| \\
&\quad \leqslant \sum_{j=1}^{i-1}\left|\int_{u_j}^{u_{j+1}} f(t)\,dt\right| + \left|\int_{u_i}^{U} f(t)\,dt\right| + \int_{u_1}^{U} 2|f(t)|\frac{xt}{2}\,dt \\
&\quad \leqslant C + C + C\varepsilon_{u_1}Ux \leqslant C, \qquad (4.8)
\end{aligned}$$

since by (4.7) any $\sum_{j=1}^{m}\left|\int_{u_j}^{u_{j+1}} f(t)\,dt\right|$ sums and $\left|\int_{u_i}^{U} f(t)\,dt\right|$ integrals are bounded. On the other hand, by

$$\left|\int_0^u \cos xt\,dt\right| = \left|\left[x^{-1}\sin xt\right]_0^u\right| \leqslant 2x^{-1},$$

we can use Lemma 2 for $h = f$ and $d(t) = \cos xt$. Hence with a similar argumentation as in the proof of Theorem 2 we get

$$I_U = \left|\int_U^{u_{i+1}} f(t)\cos xt\,dt\right| + \sum_{j=i+1}^{\infty}\left|\int_{u_j}^{u_{j+1}} f(t)\cos xt\,dt\right| \leqslant C\varepsilon_{U/2}. \qquad (4.9)$$

Since $\varepsilon_t = O(1)$, from (4.8) and (4.9) we obtain (4.5). To see (4.6), we need to sum from m instead of 1 in (4.8), i.e.

$$\begin{aligned}
&\sum_{j=m}^{i-1}\left|\int_{u_j}^{u_{j+1}} f(t)\cos xt\,dt\right| + \left|\int_{u_i}^{U} f(t)\cos xt\,dt\right| \\
&\quad \leqslant \sum_{j=m}^{i-1}\left|\int_{u_j}^{u_{j+1}} f(t)\,dt\right| + \left|\int_{u_i}^{U} f(t)\,dt\right| + \int_{u_m}^{U} 2|f(t)|\frac{xt}{2}\,dt \\
&\quad \leqslant \varepsilon'_m + \varepsilon''_{u_i} + C\varepsilon_{u_m}
\end{aligned}$$

where

$$\varepsilon'_m = \sum_{j=m}^{\infty}\left|\int_{u_j}^{u_{j+1}} f(t)\,dt\right| \quad \text{and} \quad \varepsilon''_u = \sup_{V\geqslant u}\left|\int_u^V f(t)\,dt\right|.$$

Moreover take (4.9) and note that by $\beta(t) = o(t^{-1})$, $\varepsilon_t = o(1)$ while (4.7) implies $\varepsilon'_m = o(1)$ and $\varepsilon''_u = o(1)$.

We note that (4.7) can not be replaced by the weaker condition (3.1) in Theorem 3, a counterexample is the sequence $u_j = 2^j$ and $f(t) = {}^{(-1)^j}\!/_{j2^j}$ for $2^j \leqslant t < 2^{j+1}$. Then (3.1) is satisfied but (4.5) or (4.6) is not.

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Péter Kórus
University of Szeged
Bolyai Institute
Aradi vértanúk tere 1
H-6720 Szeged, Hungary
e-mail: korpet@math.u-szeged.hu