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POLYNOMIAL PROBLEMS OF THE CASAS–ALVERO TYPE

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Abstract. We establish necessary and sufficient conditions for an arbitrary polynomial of degree n, especially with only real roots, to be trivial, i.e. to have the form $a(x-b)^n$. To do this, we derive new properties of polynomials and their roots. In particular, it concerns new bounds and genetic sum representations of the Abel-Goncharov interpolation polynomials. Moreover, we prove the Sz.-Nagy type identities, the Laguerre and Obreshkov-Chebotarev type inequalities for roots of polynomials and their derivatives. As applications these results are associated with the known problem, conjectured by Casas-Alvero in 2001, which says, that any complex univariate polynomial, having a common root with each of its non-constant derivative must be a power of a linear polynomial. We investigate particular cases of the problem, when the conjecture holds true or, possibly, is false.

1. Introduction and preliminary results

It is well known from elementary calculus that an arbitrary polynomial f with complex coefficients (complex polynomial) of degree $n \in \mathbb{N}$

$$f(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n, \; a_0 \neq 0, \tag{1}$$

having a root $\lambda \in \mathbb{C}$ of multiplicity μ, $1 \leqslant \mu \leqslant n$, shares it with each of its derivatives up to order $\mu - 1$, but $f^{(\mu)}(\lambda) \neq 0$. When λ is a unique root of f, it has the form $f(z) = a(z-\lambda)^n$, $\mu = n$ and λ is the same root of each derivative of f up to order $n-1$. We will call such a polynomial a trivial polynomial. Obviously, as it follows from the fundamental theorem of algebra, f has at least two distinct roots, i.e. a polynomial of degree n is non-trivial, if and only if its maximum multiplicity of roots r does not exceed $n-1$.

In 2001 Casas-Alvero [1] conjectured that an arbitrary polynomial f degree $n \geqslant 1$ with complex coefficients is of the form $f(z) = a(z-b)^n, a, b \in \mathbb{C}$, if and only if f shares a root with each of its derivatives $f^{(1)}, f^{(2)}, \ldots, f^{(n-1)}$.

We will call a possible non-trivial polynomial, which has a common root with each of its non-constant derivatives a CA-polynomial. The conjecture says that there exist no CA-polynomials. The problem is still open. However, it is proved for small degrees, for infinitely many degrees, for instance, for all powers n, when n is a prime (see in [2], [3], [4]). We observe that such a kind of CA-polynomial of degree $n \geqslant 2$ cannot have all distinct roots since at least one root is common with its first derivative. Therefore it has a multiplicity at least 2 and a maximum of possible distinct roots is $n-1$.

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Our main goal here is to derive necessary and sufficient conditions for an arbitrary polynomial (1) to be trivial. For example, solving a simple differential equation of the first order, we easily prove that a polynomial is trivial, if and only if it is divisible by its first derivative. In the sequel we establish other criteria, which will guarantee that an arbitrary polynomial has a unique joint root.

Without loss of generality one can assume in the sequel that f is a monic polynomial of degree n, i.e. $a_0 = 1$ in (1). Generally, it has k distinct roots λ_j of multiplicities $r_j,\ j = 1,\ldots,k,\ 1 \leqslant k \leqslant n$ such that

$$r_1 + r_2 + \ldots + r_k = n. \tag{2}$$

By r we will denote the maximum of multiplicities (2), $r = \max_{1\leqslant j\leqslant k}(r_j),\ r_0 = \min_{1\leqslant j\leqslant k}(r_j)$ and by $\xi_\nu^{(m)},\ \nu = 1,\ldots,n-m$ the zeros of the m-th derivative $f^{(m)}$, $m = 1,\ldots,n-1$. For further needs we specify zeros of the $n-1$st and $n-2$nd derivatives, denoting them by $\xi_1^{(n-1)} = z_{n-1}$ and $\xi_2^{(n-2)} = z_{n-2}$, respectively. It is easy to find another zero of the $n-2$-nd derivative, which is equal to $\xi_1^{(n-2)} = 2z_{n-1} - z_{n-2}$. When zeros $z_{n-1},\ z_{n-2}$ are real we write, correspondingly, $x_{n-1},\ x_{n-2}$. The value z_{n-1} is called the centroid. It is a center of gravity of roots and by Gauss-Lucas theorem it is contained in the convex hull of all non-constant polynomial derivatives (see details in [5]).

The paper is structured as follows: In Section 2 we study properties of the Abel-Goncharov interpolation polynomials, including integral and series representations and upper bounds. Section 3 deals with the Sz.-Nagy type identities and Obreshkov-Chebotarev type inequalities for roots of polynomials and their derivatives. As applications new criteria are found for an arbitrary polynomial with only real roots to be trivial. Section 4 is devoted to the Laguerre type inequalities for polynomials with only real roots to localize their zeros. The final Section 5 contains applications of these results towards solution of the Casas-Alvero conjecture and its particular cases.

2. Abel-Goncharov polynomials, their upper bounds and integral and genetic sum's representations

We begin, choosing a sequence of complex numbers (repeated terms are permitted) $z_0, z_1, z_2, \ldots, z_{n-1}, n \in \mathbb{N}$, where $z_0 \in \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, $z_m \in \{\xi_1^{(m)}, \xi_2^{(m)}, \ldots, \xi_{n-m}^{(m)}\}$, $m = 1, 2, \ldots, n-1$, satisfying conditions $f^{(m)}(z_m) = 0,\ m = 0, 1, \ldots,\ n-1$ and, clearly $f^{(n)}(z) = n!$. Then we represent $f(z)$ in the form

$$f(z) = z^n + P_{n-1}(z), \tag{3}$$

where $P_{n-1}(z)$ is a polynomial of degree at most $n-1$. To determine $P_{n-1}(z)$ we differentiate the latter equality m times, and we calculate the corresponding derivatives in z_m to obtain

$$P_{n-1}^{(m)}(z_m) = -\frac{n!}{(n-m)!} z_m^{n-m}, \quad m = 0, 1, \ldots, n-1. \tag{4}$$

But this is the known Abel-Goncharov interpolation problem (see [6]) and the polynomial $P_{n-1}(z)$ can be uniquely determined via the linear system (4) of n equations with n unknowns and triangular matrix with non-zero determinant. So, following [6], we derive

$$P_{n-1}(z) = -\sum_{k=0}^{n-1} \frac{n!}{(n-k)!} z_k^{n-k} G_k(z), \tag{5}$$

where $G_k(z), k = 0, 1, \ldots, n-1$ is the system of the Abel-Goncharov polynomials [6], [7], [8]. On the other hand it is known that

$$G_n(z) = z^n - \sum_{k=0}^{n-1} \frac{n!}{(n-k)!} z_k^{n-k} G_k(z).$$

Thus comparing with (3), we find that

$$G_n(z) \equiv G_n(z, z_0, z_1, z_2, \ldots, z_{n-1}) = f(z),$$

and

$$G_n(\lambda_j, z_0, z_1, z_2, \ldots, z_{n-1}) = f(\lambda_j) = 0, \quad j = 1, 2, \ldots, k.$$

Plainly, one can relate possible CA-polynomials with the corresponding Abel-Goncharov polynomials, fixing a sequence $\{z_m\}_0^{n-1}$ such that

$$z_m \in \{\lambda_1, \lambda_2, \ldots, \lambda_k\}, \quad m = 0, 1, \ldots, n-1.$$

Further, It is known [6] that the Abel-Goncharov polynomial can be represented as a multiple integral in the complex plane

$$G_n(z) = n! \int_{z_0}^{z} \int_{z_1}^{s_1} \cdots \int_{z_{n-1}}^{s_{n-1}} ds_n \ldots ds_1. \tag{6}$$

Moreover, making simple changes of variables in (6), it can be verified that $G_n(z)$ is a homogeneous function of degree n (cf. [7]). Therefore

$$G_n(\alpha z) = G_n(\alpha z, \alpha z_0, \alpha z_1, \ldots, \alpha z_{n-1}) = \alpha^n G_n(z), \quad \alpha \neq 0. \tag{7}$$

The following Goncharov upper bound holds for G_n (see [9], [6], [7], [11])

$$|G_n(z)| \leqslant \left(|z - z_0| + \sum_{s=0}^{n-2} |z_{s+1} - z_s| \right)^n. \tag{8}$$

Let us represent the Abel-Goncharov polynomials $G_n(z)$ in a different way. To do this, we will use the following representation of the Gauss hypergeometric function given by relation (2.2.6.1) in [12], namely

$$\int_a^b (z-a)^{\alpha-1}(b-z)^{\beta-1}(z+c)^{\gamma} dz$$
$$= (b-a)^{\alpha+\beta-1}(a+c)^{\gamma} B(\alpha,\beta)\, {}_2F_1\left(\alpha, -\gamma; \alpha+\beta; \frac{a-b}{a+c}\right), \tag{9}$$

where α,β,γ are positive integers, $a,b,c \in \mathbb{C}$ and $B(\alpha,\beta)$ is the Euler beta-function. So, our goal will be a representation of the Abel-Goncharov polynomials in terms of the so-called genetic sums considered, for instance, in [10]. Moreover, this will lead us to a sharper upper bound for these polynomials, improving the Goncharov bound (8). Indeed, $G_1(z) = z - z_0$. When $n \geqslant 2$, we use the multiple integral representation (6), and appealing to the representation (9), we obtain recursively

$$G_n(z) = n!\int_{z_0}^{z}\int_{z_1}^{s_1}\dots\int_{z_{n-2}}^{s_{n-2}}(s_{n-1}-z_{n-1})ds_{n-1}\dots ds_1$$

$$= n!(z_{n-2}-z_{n-1})\int_{z_0}^{z}\int_{z_1}^{s_1}\dots\int_{z_{n-3}}^{s_{n-3}}(s_{n-2}-z_{n-2})\,{}_2F_1\left(1,-1;\,2;\,\frac{z_{n-2}-s_{n-2}}{z_{n-2}-z_{n-1}}\right)ds_{n-2}\dots ds_1$$

$$= n!\sum_{j_1=0}^{1}\frac{(-1)_{j_1}(-1)^{j_1}}{(2)_{j_1}}(z_{n-2}-z_{n-1})^{1-j_1}\int_{z_0}^{z}\int_{z_1}^{s_1}\dots\int_{z_{n-3}}^{s_{n-3}}(s_{n-2}-z_{n-2})^{1+j_1}ds_{n-2}\dots ds_1.$$

Hence, employing properties of the Pochhammer symbol and repeating this process, we find

$$G_n(z) = n!\sum_{j_1=0}^{1}\frac{(z_{n-2}-z_{n-1})^{1-j_1}}{(2)_{j_1}(1-j_1)!}\int_{z_0}^{z}\int_{z_1}^{s_1}\dots\int_{z_{n-3}}^{s_{n-3}}(s_{n-2}-z_{n-2})^{1+j_1}ds_{n-2}\dots ds_1$$

$$= n!\sum_{j_1=0}^{1}\sum_{j_2=0}^{1+j_1}\frac{(z_{n-2}-z_{n-1})^{1-j_1}(z_{n-3}-z_{n-2})^{1+j_1-j_2}}{(2)_{j_2}(1-j_1)!(1+j_1-j_2)!}\times$$

$$\times\int_{z_0}^{z}\int_{z_1}^{s_1}\dots\int_{z_{n-4}}^{s_{n-4}}(s_{n-3}-z_{n-3})^{1+j_2}ds_{n-3}\dots ds_1.$$

Continuing to calculate iterated integrals with the use of (9), we arrive finally at the following genetic sum representation of the Abel-Goncharov polynomials ($j_0 = j_n = 0,\ z_{-1} \equiv z$)

$$G_n(z) = n!\sum_{j_1=0}^{1}\sum_{j_2=0}^{1+j_1}\dots\sum_{j_{n-1}=0}^{1+j_{n-2}}\prod_{s=0}^{n-1}\frac{(z_{n-2-s}-z_{n-1-s})^{1+j_s-j_{s+1}}}{(1+j_s-j_{s+1})!}. \tag{10}$$

Analogously, we derive the genetic sum representation for the m-th derivative $G_n^{(m)}(z)$, namely ($j_0 = 0$)

$$G_n^{(m)}(z) = n!\sum_{j_1=0}^{1}\sum_{j_2=0}^{1+j_1}\dots\sum_{j_{n-1-m}=0}^{1+j_{n-2-m}}\frac{(z-z_m)^{1+j_{n-1-m}}}{(1+j_{n-1-m})!}\prod_{s=0}^{n-2-m}\frac{(z_{n-2-s}-z_{n-1-s})^{1+j_s-j_{s+1}}}{(1+j_s-j_{s+1})!}, \tag{11}$$

where $m = 0,1,\dots,n-1$.

Meanwhile, the Taylor expansions of $G_n^{(m)}(z)$ in the neighborhood of points z_m give the formulas

$$G_n^{(m)}(z) = \frac{n!}{(n-m)!}(z-z_m)^{n-m} + \frac{G_n^{(n-1)}(z_m)}{(n-m-1)!}(z-z_m)^{n-m-1} + \dots + G_n^{(1+m)}(z_m)(z-z_m), \tag{12}$$

where $m = 0, 1, \ldots, n-1$. Thus comparing coefficients in front of $(z - z_m)^s$, $s = 1, \ldots, n-m-1$ in (11) and (12), we find the values of derivatives $G_n^{(s+m)}(z_m)$ in terms of $z_m, z_{m+1}, \ldots, z_{n-1}$. Precisely, we obtain ($j_0 = 0$)

$$G_n^{(s+m)}(z_m) = n! \sum_{j_1=0}^{1} \sum_{j_2=0}^{1+j_1} \cdots \sum_{j_{n-2-m}=0}^{1+j_{n-3-m}} \frac{(z_m - z_{m+1})^{2+j_{n-2-m}-s}}{(2+j_{n-2-m}-s)!} \times \\ \times \prod_{l=0}^{n-3-m} \frac{(z_{n-2-l} - z_{n-1-l})^{1+j_l-j_{l+1}}}{(1+j_l-j_{l+1})!}, \quad (13)$$

where $s = 1, 2, \ldots, n-m$, $m = 0, 1, \ldots, n-1$.

Finally, in this section, we will establish a sharper upper bound for the Abel-Goncharov polynomials. We have

THEOREM 1. *Let* $z, z_0, z_1, z_2, \ldots, z_{n-1} \in \mathbb{C}$, $n \geqslant 1$. *The following upper bound holds for the Abel-Goncharov polynomials*

$$|G_n(z, z_0, z_1, z_2, \ldots, z_{n-1})| \\ \leqslant \sum_{k_0=0}^{1} \sum_{k_1=0}^{2-k_0} \cdots \sum_{k_{n-2}=0}^{n-1-k_0-k_1-\ldots-k_{n-3}} \left(\frac{n!}{k_0! k_1! \ldots k_{n-2}! \, (n-k_0-k_1-\ldots-k_{n-2})!} \right) \times \\ \times \prod_{s=0}^{n-1} |z_{n-2-s} - z_{n-1-s}|^{k_s}, \quad (14)$$

where $z_{-1} \equiv z$ *and*

$$\left(\frac{n!}{l_0! l_1! \ldots l_m!} \right) = \frac{n!}{l_0! l_1! \ldots l_m!}, \quad l_0 + l_1 + \ldots + l_m = n$$

are multinomial coefficients. This bound is sharper than the Goncharov upper bound (8).

Proof. In fact, making simple substitutions $k_s = 1 + j_s - j_{s+1}$, $s = 0, 1, \ldots, n-1$, $j_0 = j_n = 0$ and writing identity (10) for the Abel-Goncharov polynomials (6), we estimate their absolute value, coming out immediately with inequality (14). Furthermore, appealing to the multinomial theorem, we estimate the right-hand side of (14) in the following way

$$\sum_{k_0=0}^{1} \sum_{k_1=0}^{2-k_0} \cdots \sum_{k_{n-2}=0}^{n-1-k_0-k_1-\ldots-k_{n-3}} \left(\frac{n!}{k_0! k_1! \ldots k_{n-2} \, (n-k_0-k_1-\ldots-k_{n-2})!} \right) \times \\ \times \prod_{s=0}^{n-1} |z_{n-2-s} - z_{n-1-s}|^{k_s} \\ \leqslant \sum_{l_0+l_1+\ldots+l_{n-1}=n} \left(\frac{n!}{l_0! l_1! \ldots l_{n-1}!} \right) \prod_{s=0}^{n-1} |z_{n-2-s} - z_{n-1-s}|^{l_s}$$

$$= \left(\sum_{m=0}^{n-1} |z_{m-1} - z_m| \right)^n ,$$

where the summation now is taken over all combinations of nonnegative integer indices l_0 through l_{n-1} such that the sum of all l_j is n. Thus it yields (8) and completes the proof.

3. Sz.-Nagy type identities for roots of polynomials and their derivatives

In this section we prove Sz.-Nagy type identities [5] for zeros of monic polynomials with complex coefficients and their derivatives. All notations of roots and their multiplicities given in Section 1 are involved.

We begin with

LEMMA 1. *Let f be a monic polynomial of degree $n \geqslant 2$ with complex coefficients, $m = 0,1,\ldots,n-2$ and $z \in \mathbb{C}$. Then the following Sz.-Nagy type identities, which relate the roots of f and its m-th derivative, hold*

$$z_{n-1} - z = \frac{1}{n} \sum_{j=1}^{k} r_j(\lambda_j - z) = \frac{1}{n-m} \sum_{j=1}^{n-m} (\xi_j^{(m)} - z), \tag{15}$$

$$(z_{n-1} - z_{n-2})^2 = \frac{1}{n(n-1)} \left[\sum_{j=1}^{k} r_j(\lambda_j - z)^2 - n(z_{n-1} - z)^2 \right]$$
$$= \frac{1}{(n-m)(n-m-1)} \left[\sum_{j=1}^{n-m} (\xi_j^{(m)} - z)^2 - (n-m)(z_{n-1} - z)^2 \right], \tag{16}$$

$$(z_{n-1} - z_{n-2})^2 = \frac{1}{n^2(n-1)} \sum_{1 \leqslant j < s \leqslant k} r_j r_s (\lambda_j - \lambda_s)^2$$
$$= \frac{1}{(n-m)^2(n-m-1)} \sum_{1 \leqslant j < s \leqslant n-m} (\xi_j^{(m)} - \xi_s^{(m)})^2. \tag{17}$$

Proof. In fact, the first Viéte formula (see [5]) says that the coefficient a_1 $(a_0 = 1)$ in (1) is equal to

$$-a_1 = r_1\lambda_1 + r_2\lambda_2 + \ldots + r_k\lambda_k.$$

On the other hand, differentiating (1) $n-1$ times, we find $z_{n-1} = -a_1/n$. Thus in view of (2) we prove the first identity in (15). The second identity can be done similarly, using that a centroid is differentiation invariant, see, for instance, [5]. In order to establish the first identity in (16), we call formula (11) to find

$$\frac{f^{(n-2)}(z)}{(n-2)!} = \frac{n(n-1)}{2} (z - z_{n-2})(z + z_{n-2} - 2z_{n-1}). \tag{18}$$

Moreover, as a consequence of the second Viéte formula, the coefficient a_2 in (1), which equals

$$a_2 = \frac{f^{(n-2)}(z)}{(n-2)!} - \frac{n(n-1)}{2} z^2 + n(n-1) z_{n-1} z \tag{19}$$

can be expressed as follows

$$a_2 = \frac{1}{2}\left(\sum_{j=1}^{k} r_j\lambda_j\right)^2 - \frac{1}{2}\sum_{j=1}^{k} r_j\lambda_j^2. \tag{20}$$

Hence letting $z = z_{n-2}$ in (18), and taking into account (15) with $z = 0$, we deduce

$$2a_2 = n^2 z_{n-1}^2 - \sum_{j=1}^{k} r_j\lambda_j^2 = 2n(n-1)z_{n-1}z_{n-2} - n(n-1)z_{n-2}^2.$$

Therefore, using again (15) and (2), we easily come up with the first identity in (16). The second one can be proven in the same manner, involving roots of derivatives. Finally, we prove the first identity in (17). Concerning the second identity, see Lemma 6.1.5 in [5]. Indeed, calling the first identity in (16), letting $z = z_{n-1}$ and employing (15), we derive

$$n^2(n-1)(z_{n-1}-z_{n-2})^2 = n\sum_{j=1}^{k} r_j\lambda_j^2 + \left(\sum_{s=1}^{k} r_s\lambda_s\right)^2 - 2\left(\sum_{s=1}^{k} r_s\lambda_s\right)\left(\sum_{j=1}^{k} r_j\lambda_j\right)$$

$$= n\sum_{j=1}^{k} r_j\lambda_j^2 - \sum_{s=1}^{k} r_j^2\lambda_j^2 - 2\sum_{1\leqslant j<s\leqslant k} r_j r_s\lambda_j\lambda_s = \sum_{1\leqslant j<s\leqslant k} r_j r_s(\lambda_j-\lambda_s)^2. \quad \square$$

The following result gives an identity, which is associated with zeros of a monic polynomial and common zeros of its derivatives. Precisely, we have

LEMMA 2. *Let f be a monic polynomial of exact degree $n \geqslant 2$, having k distinct roots of multiplicities* (2)*. Let $z_{n-1} = \lambda_1$ be a common root of f of multiplicity r_1 with the unique root of its $n-1$st derivative. Let also $z_m = \xi_{n-m}^{(m)} = \lambda_{k_m}$ be a common root of f of multiplicity r_{k_m} and its m-th derivative, $m \in \{1,2,\dots,n-2\}$. Then, involving other roots of $f^{(m)}$, the following identity holds*

$$\left[\frac{n-m-2}{(n-m)^2} + \frac{r_{k_m}+r_1-n}{n(n-1)}\right]\sum_{s=1}^{n-m-1}(z_m-\xi_s^{(m)})^2 + \frac{n-m-2}{(n-m)^2}\sum_{1\leqslant s<t\leqslant n-m-1}(\xi_s^{(m)}-\xi_t^{(m)})^2$$

$$= \frac{(n-m)^2 r_{k_m} - (n-r_1)(n-m+2)}{n(n-1)}(z_m-z_{n-1})^2$$

$$+ \frac{2}{n(n-1)}\sum_{j\neq 1,k_m} r_j \sum_{1\leqslant s<t\leqslant n-m-1}(\lambda_j-\xi_s^{(m)})(\lambda_j-\xi_t^{(m)}). \tag{21}$$

Proof. We begin, appealing to (15) and letting $z = 0$. We get

$$\sum_{s=1}^{n-m}\xi_s^{(m)} = (n-m)z_{n-1}, \quad \xi_{n-m}^{(m)} = z_m. \tag{22}$$

Hence via identities (17) with $z = z_m$ we write the chain of equalities

$$
\begin{aligned}
\sum_{1\leqslant s<t\leqslant n-m} (\xi_s^{(m)} - \xi_t^{(m)})^2 &= \frac{(n-m-1)(n-m)^2}{n(n-1)} r_{k_m}(z_m - z_{n-1})^2 \\
&\quad + \frac{(n-m-1)(n-m)^2}{n(n-1)} \sum_{j\neq 1,k_m} r_j(\lambda_j - z_{n-1})^2 \\
&= \frac{(n-m-1)(n-m)^2}{n(n-1)} r_{k_m}(z_m - z_{n-1})^2 \\
&\quad + \frac{n-m-1}{n(n-1)} \sum_{j\neq 1,k_m} r_j \left(\lambda_j - z_m + \sum_{s=1}^{n-m-1} (\lambda_j - \xi_s^{(m)})\right)^2 \\
&= \frac{(n-m-1)(n-m)^2}{n(n-1)} r_{k_m}(z_m - z_{n-1})^2 \\
&\quad + \frac{n-m-1}{n(n-1)} \left[\sum_{j\neq 1,k_m} r_j(\lambda_j - z_m)^2 + \sum_{j\neq 1,k_m} r_j \left(\sum_{s=1}^{n-m-1} (\lambda_j - \xi_s^{(m)}) \right)^2 \right. \\
&\quad \left. + 2 \sum_{j\neq 1,k_m} r_j \sum_{s=1}^{n-m-1} (\lambda_j - z_m)(\lambda_j - \xi_s^{(m)}) \right] \\
&= \frac{(n-m-1)(n-m)^2}{n(n-1)} r_{k_m}(z_m - z_{n-1})^2 \\
&\quad + \frac{n-m-1}{n(n-1)} \left[(2(n-m)-1) \sum_{j\neq 1,k_m} r_j(\lambda_j - z_m)^2 \right. \\
&\quad \left. + \sum_{j\neq 1,k_m} r_j \left(\sum_{s=1}^{n-m-1} (\lambda_j - \xi_s^{(m)}) \right)^2 + 2 \sum_{j\neq 1,k_m} r_j \sum_{s=1}^{n-m-1} (\lambda_j - z_m)(z_m - \xi_s^{(m)}) \right] \\
&= \frac{(n-m-1)(n-m)^2}{n(n-1)} r_{k_m}(z_m - z_{n-1})^2 \\
&\quad + \frac{n-m-1}{n(n-1)} \left[(2(n-m)-1) \sum_{j\neq 1,k_m} r_j(\lambda_j - z_m)^2 \right. \\
&\quad \left. + \sum_{j\neq 1,k_m} r_j \left(\sum_{s=1}^{n-m-1} (\lambda_j - \xi_s^{(m)}) \right)^2 - 2(n-m)(n-r_1)(z_m - z_{n-1})^2 \right] \\
&= \frac{(n-m-1)}{n(n-1)} \left((n-m)^2 r_{k_m} - n + r_1 \right) (z_m - z_{n-1})^2 \\
&\quad + (n-m-1)(3(n-m)-2)(z_{n-1} - z_{n-2})^2 \\
&\quad + \frac{(n-m-1)(n-r_1)}{n(n-1)} \sum_{s=1}^{n-m-1} (z_{n-1} - \xi_s^{(m)})^2
\end{aligned}
$$

$$-\frac{r_{k_m}(n-m-1)}{n(n-1)}\sum_{s=1}^{n-m-1}(z_m-\xi_s^{(m)})^2$$
$$+2\,\frac{n-m-1}{n(n-1)}\sum_{j\neq 1,k_m} r_j \sum_{1\leqslant s<t\leqslant n-m-1}(\lambda_j-\xi_s^{(m)})(\lambda_j-\xi_t^{(m)}).$$

Applying again (17), (22), we split the right-hand side of the latter identity in (17) in two parts, selecting the root z_m. Thus in the same manner after straightforward calculations it becomes

$$\left[\frac{n-m-2}{(n-m)^2}+\frac{r_{k_m}+r_1-n}{n(n-1)}\right]\sum_{s=1}^{n-m-1}(z_m-\xi_s^{(m)})^2+\frac{n-m-2}{(n-m)^2}\sum_{1\leqslant s<t\leqslant n-m-1}(\xi_s^{(m)}-\xi_t^{(m)})^2$$
$$=\frac{(n-m)^2 r_{k_m}-(n-r_1)(n-m+2)}{n(n-1)}(z_m-z_{n-1})^2$$
$$+\frac{2}{n(n-1)}\sum_{j\neq 1,k_m} r_j \sum_{1\leqslant s<t\leqslant n-m-1}(\lambda_j-\xi_s^{(m)})(\lambda_j-\xi_t^{(m)}),$$

completing the proof of Lemma 2. □

REMARK 1. It is easy to verify identity (21) for the least case $m=n-2$, when the double sums are empty and $\xi_1^{(n-2)}=2z_{n-1}-z_{n-2}$ (see above).

COROLLARY 1. *A polynomial with only real roots of degree $n\geqslant 2$ is trivial, if and only if its $n-2$nd derivative has a double root.*

Proof. Indeed, necessity is obvious. To prove sufficiency we see that since the $n-2$nd derivative has a double real root x_{n-2}, it is equal to the root x_{n-1} of the $n-1$st derivative. Therefore letting in (16) $z=x_{n-1}$, we find that its left-hand side becomes zero and, correspondingly, all squares in the right-hand side are zeros. This gives the conclusion that all roots are equal to x_{n-1}. □

COROLLARY 2. *Let f be an arbitrary polynomial of degree $n\geqslant 3$ with at least two distinct roots, whose $n-2$nd derivative has a double root. Then it contains at least one complex root.*

Proof. In fact, if all roots are real it is trivial via Corollary 1. □

Evidently, each derivative up to $f^{(r-1)}$ of a polynomial f with only real roots, where r is the maximal multiplicity, shares a root with f. Moreover, owing to the Rolle theorem all roots of $f^{(m)}$, $m=r,r+1,\dots,n-1$ are simple, we have that a possible common root with f is simple too (we note, that a number of common roots does not exceed $k-2$, because minimal and maximal roots cannot be zeros of $f^{(m)}$, $m\geqslant r$). This circumstance gives an immediate

COROLLARY 3. *There exists no non-trivial polynomial with only real roots, having two distinct zeros and sharing a root with at least one of its derivatives, whose order exceeds $r-1$, $r=\max_{1\leqslant j\leqslant k}(r_j)$.*

Proof. Indeed, in the case of existence of such a polynomial, these two distinct roots cannot be within zeros of any derivative $f^{(m)}$, $m > r$ owing to the Rolle theorem. Moreover, if any of the two roots are in common with roots of $f^{(r)}$, its multiplicity is greater than r, which is impossible.

We extend Corollary 3 to three distinct real roots. Precisely, it leads to

COROLLARY 4. *There exists no non-trivial polynomial f of degree $n \geqslant 3$ with only real roots, having three distinct zeros and sharing a root with its $n-2$nd and $n-1$st derivatives.*

Proof. Assume such a polynomial exists and let's denote its roots $\lambda_1 = x_{n-1}$, $\lambda_2 = x_{n-2}$ and λ_3 of multiplicities r_1, r_2, r_3, respectively. Hence employing identities (16), we write for this case

$$(n^2 - n - r_2)(x_{n-1} - x_{n-2})^2 = r_3(\lambda_3 - x_{n-1})^2.$$

In the meantime, squaring both sides of the first identity in (15) for this case after simple modifications , we obtain

$$r_2^2(x_{n-1} - x_{n-2})^2 = r_3^2(\lambda_3 - x_{n-1})^2.$$

Hence, comparing with the previous equality, we come out with the relation

$$(n^2 - n - r_2)r_3 = r_2^2.$$

But $n = r_1 + r_2 + r_3$, $r_j \geqslant 1, j = 1,2,3$. Consequently,

$$r_2^2 \geqslant n(n-1) - r_2 > (n-1)^2 - r_2 \geqslant (r_1 + r_2)^2 - r_2 \geqslant r_2^2 + r_2 + r_1^2 > r_2^2,$$

which is impossible. □

REMARK 2. If we omit the condition for f to have a common root with the $n-2$nd derivative in Corollary 4, it becomes false. In fact, this circumstance can be shown by the counterexample $f(x) = x^3 - x$.

The following result deals with the case of 4 distinct roots. We have,

COROLLARY 5. *There exists no non-trivial polynomial f of degree $n \geqslant 4$ with only real roots, having four distinct zeros and sharing a root with its $n-2$nd and $n-1$st derivatives.*

Proof. Similarly to the previous corollary, we assume the existence of such a polynomial and call its roots $\lambda_1 = x_{n-1}$, $\lambda_2 = x_{n-2}$ and λ_3, λ_4 of multiplicities r_j, $j = 1,2,3,4$, respectively. Hence the first identity in (16) yields

$$(n^2 - n - r_2)(x_{n-1} - x_{n-2})^2 = r_3(\lambda_3 - x_{n-1})^2 + r_4(\lambda_4 - x_{n-1})^2. \tag{23}$$

Meanwhile, using the first identity in (15) for this case, we derive in a similar manner

$$r_2^2(x_{n-1} - x_{n-2})^2 = r_3^2(\lambda_3 - x_{n-1})^2 + r_4^2(\lambda_4 - x_{n-1})^2 + 2r_3r_4(\lambda_3 - x_{n-1})(\lambda_4 - x_{n-1}).$$

Thus, after straightforward calculations, we come out with the quadratic equation

$$Ay^2+By+C=0$$

in the variable $y=(\lambda_3-x_{n-1})/(\lambda_4-x_{n-1})$ with coefficients $A=r_3r_2^2-r_3^2(n^2-n-r_2)$, $B=-2r_3r_4(n^2-n-r_2)$, $C=r_4r_2^2-r_4^2(n^2-n-r_2)$. But, it is easy to verify that $B^2-4AC>0$. Therefore the quadratic equation has two distinct real roots. Writing $\lambda_3-x_{n-1}=y(\lambda_4-x_{n-1})$ and substituting into (23), we obtain

$$(n^2-n-r_2)(x_{n-1}-x_{n-2})^2=(r_3y^2+r_4)(\lambda_4-x_{n-1})^2.$$

At the same time, since $y\neq 0$, we have $\lambda_4-x_{n-1}=y^{-1}(\lambda_3-x_{n-1})$ and

$$y^2(n^2-n-r_2)(x_{n-1}-x_{n-2})^2=(r_3y^2+r_4)(\lambda_3-x_{n-1})^2.$$

Hence,

$$\lambda_4=x_{n-1}\pm\sqrt{\frac{n^2-n-r_2}{r_3y^2+r_4}}\,|x_{n-1}-x_{n-2}|,$$

$$\lambda_3=x_{n-1}\pm|y|\sqrt{\frac{n^2-n-r_2}{r_3y^2+r_4}}\,|x_{n-1}-x_{n-2}|.$$

Consequently,

$$\lambda_4-\lambda_3=\sqrt{\frac{n^2-n-r_2}{r_3y^2+r_4}}\,|x_{n-1}-x_{n-2}|(1-|y|)=-\sqrt{\frac{n^2-n-r_2}{r_3y^2+r_4}}\,|x_{n-1}-x_{n-2}|(1+|y|)$$

$$=\sqrt{\frac{n^2-n-r_2}{r_3y^2+r_4}}\,|x_{n-1}-x_{n-2}|(1+|y|)=\sqrt{\frac{n^2-n-r_2}{r_3y^2+r_4}}\,|x_{n-1}-x_{n-2}|(|y|-1),$$

which is possible only in the case $x_{n-1}=x_{n-2},\ \lambda_3=\lambda_4$. Thus we get a contradiction with Corollary 1 and complete the proof. □

In the same manner we prove

COROLLARY 6. *There exists no non-trivial polynomial f of degree $n\geqslant 5$ with only real roots, having five distinct zeros and sharing roots with its $n-2$nd and $n-1$st derivatives.*

Proof. Assuming its existence, it has the roots $\lambda_1=x_{n-1}$, $\lambda_2=x_{n-2}$, $\lambda_3=2x_{n-1}-x_{n-2}$, λ_4 and λ_5 of multiplicities $r_j,\ j=1,2,3,4,5$, respectively. Hence

$$(n^2-n-r_2-r_3)(x_{n-1}-x_{n-2})^2=r_4(\lambda_4-x_{n-1})^2+r_5(\lambda_5-x_{n-1})^2.$$

Therefore using similar ideas as in the proof of Corollary 5, we come out again to the contradiction. □

For an arbitrary number of distinct zeros we establish the following

COROLLARY 7. *There exists no non-trivial polynomial f of degree n with only real roots, having $k \geqslant 2$ distinct zeros of multiplicities* (2) $r_j,\ j = 1,\dots,k$ *and among them all roots of $f^{(m)}$ for some m, satisfying the relations*

$$r \leqslant m < \frac{1}{2}\left(1 - \frac{1}{r_0}\right)(n-1), \tag{24}$$

where $r,\ r_0$ are maximum and minimum multiplicities of roots of f.

Proof. In fact, as a consequence of (16) we have the identity

$$\frac{(n-m)(n-m-1)}{n(n-1)} \sum_{j=1}^{k} r_j(\lambda_j - x_{n-1})^2 = \sum_{j=1}^{n-m} (\xi_j^{(m)} - x_{n-1})^2 \tag{25}$$

for some m, satisfying condition (24). Hence, since $m \geqslant r$, it has $n - m \leqslant k - 2$ and $\xi_j^{(m)} = \lambda_{m_j},\ m_j \in \{1,\dots,k\},\ j = 1,\dots,n-m$ are simple roots of $f^{(m)}$. Thus we find

$$\sum_{j=1}^{n-m}\left[r_{m_j}\frac{(n-m)(n-m-1)}{n(n-1)} - 1\right](\lambda_{m_j} - x_{n-1})^2 + \frac{(n-m)(n-m-1)}{n(n-1)} \sum_{j=n-m+1}^{k} r_{m_j}(\lambda_{m_j} - x_{n-1})^2 = 0.$$

But, owing to condition (24)

$$r_{m_j}\frac{(n-m)(n-m-1)}{n(n-1)} - 1 \geqslant r_0\frac{(n-m)(n-m-1)}{n(n-1)} - 1 \geqslant 0,\ j = 1,\dots,n-m.$$

Indeed, we have from the latter inequality

$$m \leqslant n - \frac{1}{2} - \sqrt{\frac{n^2-n}{r_0} + \frac{1}{4}}$$

and, in turn,

$$n - \frac{1}{2} - \sqrt{\frac{n^2-n}{r_0} + \frac{1}{4}} = \frac{2(1-r_0^{-1})(n^2-n)}{2n - 1 + \sqrt{4(n^2-n)r_0^{-1} + 1}} \geqslant \frac{(1-r_0^{-1})(n^2-n)}{2n-1} > \frac{1}{2}\left(1 - \frac{1}{r_0}\right)(n-1).$$

Therefore $\lambda_j = x_{n-1},\ j = 1,\dots,k$ and this contradicts to the fact that all roots are distinct. □

Finally, in this section, we will employ identities (17) to prove an analog of the Obreshkov- Chebotarev theorem for multiple roots (see [5], Theorem 6.4.3), involving estimates for smallest and largest of distances between consecutive zeros of polynomials and their derivatives. Namely, it has

THEOREM 2. *Let f be a polynomial of degree $n>2$ with only real zeros. Denote the largest and the smallest of the distances between consecutive zeros of f by Δ and δ, respectively. Denoting the corresponding quantities associated with $f^{(m)}$, $m=1,2,\ldots,\ n-2$ by $\Delta^{(m)}$ and $\delta^{(m)}$, the following inequalities hold*

$$\delta^{(m)} \leqslant \Delta\, \frac{rk}{n} \sqrt{\frac{k^2-1}{(n-m+1)(n-1)}}, \tag{26}$$

$$\delta\, \frac{r_0 k}{n} \sqrt{\frac{k^2-1}{(n-m+1)(n-1)}} \leqslant \Delta^{(m)}, \tag{27}$$

$$\delta\, \frac{r_0 k}{2n} \sqrt{\frac{k^2-1}{3(n-1)}} \leqslant |x_{n-1}-x_{n-2}| \leqslant \Delta\, \frac{rk}{2n} \sqrt{\frac{k^2-1}{3(n-1)}}, \tag{28}$$

where r_0, r are minimum and maximum multiplicities of roots of f, respectively, and $k \geqslant 2$ is a number of distinct roots.

Proof. Following similar ideas as in the proof of Theorem 6.4.3 in [5], we assume distinct roots of f in the increasing order and roots of its m-th derivative in the non-decreasing order, and taking the second identity in (17), we deduce

$$\frac{[\delta^{(m)}]^2}{(n-m)^2(n-m-1)} \sum_{1\leqslant j<s\leqslant n-m} (s-j)^2 \leqslant \frac{[\Delta r]^2}{n^2(n-1)} \sum_{1\leqslant j<s\leqslant k} (s-j)^2.$$

Hence, in view of the value of the sum

$$\sum_{1\leqslant j<s\leqslant q} (s-t)^2 = \frac{1}{12} q^2(q^2-1),$$

after simple manipulations we arrive at the inequality (26). In the same manner (cf. [5]) we establish inequalities (27), (28), employing Sz.-Nagy type identities (17). □

4. Laguerre type inequalities

In 1880 Laguerre proved his famous theorem for polynomials with only real roots, which provides their localization with upper and lower bounds (see details in [5]). Precisely, we have the following Laguerre inequalities

$$x_{n-1} - (n-1)\,|x_{n-1}-x_{n-2}| \leqslant w_j \leqslant x_{n-1} + (n-1)\,|x_{n-1}-x_{n-2}|\,,\ j=1,\ldots,n,$$

where w_j are roots of the polynomial f of degree n and x_{n-1}, x_{n-2} are roots of $f^{(n-1)}$, $f^{(n-2)}$, respectively. First we prove an analog of the Laguerre inequalities for multiple roots.

LEMMA 3. *Let f be a polynomial with only real roots of degree $n \in \mathbb{N}$, having k distinct roots $\lambda_j,\ j = 1,\ \dots, k$ of multiplicities* (2) *and $x_{n-1},\ x_{n-2}$ be roots of $f^{(n-1)},\ f^{(n-2)}$, respectively. Then the following Laguerre type inequalities hold*

$$x_{n-1} - \sqrt{\frac{(n-r_j)(n-m-1)}{r_j - m}}\,|x_{n-1} - x_{n-2}| \leqslant \lambda_j \leqslant x_{n-1} + \sqrt{\frac{(n-r_j)(n-m-1)}{r_j - m}}\,|x_{n-1} - x_{n-2}|, \quad (29)$$

where $j = 1, \dots, k,\ m = 0, 1, \dots, r_j - 1$.

Proof. In fact, appealing to the Sz.-Nagy type identities (15), (16) and the Cauchy -Schwarz inequality, we find

$$(x_{n-1} - x_{n-2})^2 = \frac{1}{(n-m)(n-m-1)}\left[\sum_{s=1}^{n-m}(\xi_s^{(m)} - \lambda_j)^2 - (n-m)(x_{n-1} - \lambda_j)^2\right]$$

$$\geqslant \frac{1}{(n-m)(n-m-1)}\left[\frac{1}{n-r_j}\left(\sum_{s=1}^{n-m}(\xi_s^{(m)} - \lambda_j)\right)^2 - (n-m)(x_{n-1} - \lambda_j)^2\right]$$

$$= \frac{r_j - m}{(n-r_j)(n-m-1)}\,(x_{n-1} - \lambda_j)^2,\ m = 0, 1, \dots, r_j - 1,$$

which yields (29). □

As a corollary we improve the Laguerre inequality (28) for multiple roots.

COROLLARY 8. *Let f be a polynomial with only real roots of degree $n \in \mathbb{N}$. Then the multiple zero λ_j of multiplicity $r_j \geqslant 1,\ j = 1, \dots, k$ lies in the interval*

$$\left[x_{n-1} - \sqrt{\left(\frac{n}{r_j} - 1\right)(n-1)}\,|x_{n-1} - x_{n-2}|,\quad x_{n-1} + \sqrt{\left(\frac{n}{r_j} - 1\right)(n-1)}\,|x_{n-1} - x_{n-2}|\right]. \quad (30)$$

Proof. Indeed, the fraction $\frac{(n-r_j)(n-m-1)}{r_j - m}$ attains its minimum value, letting $m = 0$ in (29). □

REMARK 3. When all roots are simple, the latter interval coincides with the one generated by (28).

A localization of roots of the m-th derivative $f^{(m)},\ m = 0, 1, \dots, n-2$ is given by

LEMMA 4. *Roots of the m-th derivative $f^{(m)},\ m = 0, 1, \dots, n-2$ satisfy the following Laguerre type inequalities*

$$x_{n-1} - (n-m-1)\,|x_{n-1} - x_{n-2}| \leqslant \xi_\nu^{(m)} \leqslant x_{n-1} + (n-m-1)\,|x_{n-1} - x_{n-2}|, \quad (31)$$

where $\nu = 1,\dots,n-m$.

Proof. Similarly to the proof of Lemma 3, we employ the Sz.-Nagy type identities (15), (16) and the Cauchy -Schwarz inequality to deduce

$$(x_{n-1}-x_{n-2})^2 = \frac{1}{(n-m)(n-m-1)}\left[\sum_{s=1}^{n-m}(\xi_s^{(m)}-\xi_\nu^{(m)})^2-(n-m)(x_{n-1}-\xi_\nu^{(m)})^2\right]$$

$$\geqslant \frac{1}{(n-m)(n-m-1)}\left[\frac{1}{n-m-1}\left(\sum_{s=1}^{n-m}(\xi_s^{(m)}-\xi_\nu^{(m)})\right)^2-(n-m)(x_{n-1}-\xi_\nu^{(m)})^2\right]$$

$$= \frac{1}{(n-m-1)^2}\left(x_{n-1}-\xi_\nu^{(m)}\right)^2,\ m=0,1,\dots,n-2.$$

Thus we come up with (31) and complete the proof. □

When the root $x_{n-1}=\lambda_1$ be in common with f of multiplicity r_1, we have

LEMMA 5. *Let f be a polynomial with only real roots of degree $n \geqslant 2$ and $x_{n-1} = \lambda_1$ be a common zero with f of multiplicity r_1, having $k \geqslant 2$ distinct roots λ_j of multiplicities r_j, $j=1,\dots,k$. Then the following Laguerre type inequalities hold*

$$x_{n-1}-\sqrt{\left(\frac{1}{r_s}-\frac{1}{n-r_1}\right)(n^2-n)}\,|x_{n-1}-x_{n-2}| \leqslant \lambda_s \leqslant x_{n-1}$$
$$+\sqrt{\left(\frac{1}{r_s}-\frac{1}{n-r_1}\right)(n^2-n)}\,|x_{n-1}-x_{n-2}|, \tag{32}$$

where $s=2,\dots,k$.

Proof. In the same manner we involve the first Sz.-Nagy type identity in (15) with $z=\lambda_s$, which can be written in the form

$$(n-r_1)(x_{n-1}-\lambda_s) = \sum_{j=2}^{k} r_j(\lambda_j-\lambda_s).$$

Hence squaring both sides of the latter equality and appealing to the Cauchy -Schwarz inequality, we derive by virtue of (16)

$$(n-r_1)^2(x_{n-1}-\lambda_s)^2 = \left(\sum_{j=2}^{k} r_j(\lambda_j-\lambda_s)\right)^2$$
$$\leqslant (n-r_1-r_s)\sum_{j=2}^{k} r_j(\lambda_j-\lambda_s)^2$$
$$= (n-r_1-r_s)\left[(n^2-n)(x_{n-1}-x_{n-2})^2+(n-r_1)(x_{n-1}-\lambda_s)^2\right].$$

Thus after simple calculations we easily arrive at (32). □

REMARK 4. Inequalities (27) are sharper than the corresponding relations, generated by interval (30).

The following result gives a Laguerre type localization for common roots of a possible CA-polynomial with only real roots and its m-th derivative.

LEMMA 6. *Let f be a CA-polynomial of degree $n \geqslant 2$ with only real distinct zeros of multiplicities* (2)*, including common roots $x_{n-1} = \lambda_1$ of its $n-1$st derivative and x_m of its m-th derivative, $m = r, r+1, \ldots, n-2$, where $r = \max_{1 \leqslant j \leqslant k}(r_j)$. Then the following Laguerre type inequality holds*

$$\frac{n-r_1-r_{k_m}}{(n-r_1)^2}\left(n^2 - r_1 + (n-r_1)(n-m)(n-m-2)\right)(x_{n-1}-x_{n-2})^2 \geqslant (x_{n-1}-x_m)^2, \tag{33}$$

where x_{n-2} is a root of $f^{(n-2)}$ and r_{k_m} is the multiplicity of x_m as a root of f.

Proof. Appealing again to Sz.-Nagy type identities (15), (16) with $z = x_m$, inequality (31) and the Cauchy-Schwarz inequality, we find

$$(x_{n-1}-x_{n-2})^2 = \frac{1}{n(n-1)}\left[\sum_{j=2}^{k} r_j(\lambda_j - x_m)^2 - (n-r_1)(x_{n-1}-x_m)^2\right]$$

$$\geqslant \frac{1}{n(n-1)}\left[\sum_{j=2}^{k} r_j(\lambda_j - x_m)^2 - (n-r_1)(n-m-1)^2(x_{n-1}-x_{n-2})^2\right]$$

$$\geqslant \frac{1}{n(n-1)}\left[\frac{1}{n-r_1-r_{j_m}}\left(\sum_{j=2}^{k} r_j(\lambda_j - x_m)\right)^2 - (n-r_1)(n-m-1)^2(x_{n-1}-x_{n-2})^2\right]$$

$$= \frac{n-r_1}{n(n-1)}\left[\frac{n-r_1}{n-r_1-r_{j_m}}(x_{n-1}-x_m)^2 - (n-m-1)^2(x_{n-1}-x_{n-2})^2\right].$$

Hence, making straightforward calculations, we derive (33), completing the proof of Lemma 6. □

Let us denote by d, $d^{(m)}$, $D, D^{(m)}$ the following values

$$d = \min_{2 \leqslant j \leqslant k}|\lambda_j - x_{n-1}|, \quad d^{(m)} = \min_{1 \leqslant j \leqslant n-m}|\xi_j^{(m)} - x_{n-1}|, \tag{34}$$

$$D = \max_{2 \leqslant j \leqslant k}|\lambda_j - x_{n-1}|, \quad D^{(m)} = \max_{1 \leqslant j \leqslant n-m}|\xi_j^{(m)} - x_{n-1}|, \tag{35}$$

and by

$$\mathrm{span}(f) = \lambda^* - \lambda_*,$$

where

$$\lambda^* = \max_{1 \leqslant j \leqslant k}(\lambda_j), \quad \lambda_* = \min_{1 \leqslant j \leqslant k}(\lambda_j)$$

are roots of f with multiplicities r^*, r_*, respectively. Then $D^{(m+1)} \leqslant D^{(m)} \leqslant D$ and (cf. [5]) $\mathrm{span}(f^{(m+1)}) \leqslant \mathrm{span}(f^{(m)}) \leqslant \mathrm{span}(f)$, where $\mathrm{span}(f^{(m)})$ is the span of the m-th derivative. Moreover, the strict inequalities $D^{(m)} < D$, $\mathrm{span}(f^{(m)}) < \mathrm{span}(f)$ hold when m is sufficiently large.

LEMMA 7. *Let $x_{n-1}=\lambda_1,\ x_{n-2}=\lambda_2$ be common roots of f with its $n-1$st, $n-2$nd derivatives, respectively, of multiplicities r_1,r_2 as roots of f, and the maximum distance D (see (35)) be attained at the root $\lambda_{s_0},\ s_0\in\{3,\dots,k\},\ k\geqslant 3$ of f of multiplicity r_{s_0}. Then the following inequalities hold*

$$\sqrt{\frac{n^2-n-r_2}{n-r_1-r_2}}\ |x_{n-1}-x_{n-2}|\leqslant D\leqslant\sqrt{\frac{n^2-n-r_2}{r_{s_0}}}\,|x_{n-1}-x_{n-2}|, \tag{36}$$

$$\frac{1}{2}\sqrt{\frac{r_{s_0}}{3(n-r_1)}\left(5+\frac{r_2}{n^2-n-r_2}\right)}\ \mathrm{span}(f)\leqslant D$$
$$\leqslant\sqrt{\frac{1}{n-r_1}\left[n-r_1-\frac{r_{s_0}}{4}\left(5+\frac{r_2}{n^2-n-r_2}\right)\right]}\ \mathrm{span}(f). \tag{37}$$

Proof. In order to establish (36), we employ identities (16) and under condition of the lemma we write

$$(n^2-n-r_2)(x_{n-1}-x_{n-2})^2=\sum_{j=3}^{k}r_j(\lambda_j-x_{n-1})^2\leqslant(n-r_1-r_2)D^2.$$

Since $n>r_1+r_2$ and $x_{n-2}\neq\lambda_{s_0}$ (otherwise f is trivial, because equalities $x_{n-2}=\lambda_{s_0}=\lambda^*$ or $x_{n-2}=\lambda_{s_0}=\lambda_*$ mean that the maximum multiplicity $r>n-2$, and we appeal to Corollary 3), we come up with the lower bound (36) for D. The lower bound comes immediately from the estimate

$$(n^2-n-r_2)(x_{n-1}-x_{n-2})^2=\sum_{j=3}^{k}r_j(\lambda_j-x_{n-1})^2\geqslant r_{s_0}D^2.$$

Now, since $2D\geqslant\mathrm{span}(f)$, we find from (36)

$$\mathrm{span}(f)\leqslant 2\sqrt{\frac{n^2-n-r_2}{r_{s_0}}}\ |x_{n-1}-x_{n-2}|.$$

Hence, since $D=\max\left(|\lambda^*-x_{n-1}|,\ |\lambda_*-x_{n-1}|\right)$, the $n-2$nd derivative has roots x_{n-2} and $2x_{n-1}-x_{n-2}$ and $\mathrm{span}(f)=D+\Lambda$, where $\Lambda=\min\left(|\lambda^*-x_{n-1}|,\ |\lambda_*-x_{n-1}|\right)$, we appeal to the first identity in (16), letting $z=\lambda_{s_0}$ and writing it in the form

$$(n-r_1)(x_{n-1}-\lambda_{s_0})^2=\sum_{j=2}^{k}r_j(\lambda_j-\lambda_{s_0})^2-n(n-1)(x_{n-1}-x_{n-2})^2.$$

Therefore,

$$(n-r_1)D^2\leqslant\left[n-r_1-\frac{5}{4}r_{s_0}-\frac{r_{s_0}r_2}{4(n^2-n-r_2)}\right][\mathrm{span}(f)]^2$$

and we establish the upper bound (37) for D. On the other hand $\text{span}(f) = D + \Lambda$. So,

$$D^2 \leqslant \left(1 - \frac{r_{s_0}}{4(n-r_1)}\left(5 + \frac{r_2}{n^2-n-r_2}\right)\right)\left(D^2 + \Lambda^2 + 2D\Lambda\right)$$

and we easily come out with the lower bound (37) for D, completing the proof of Lemma 7. □

LEMMA 8. *Let $x_{n-1} = \lambda_1, x_{n-2} = \lambda_2$ be common roots of f with its $n-1$st, $n-2$nd derivatives of multiplicities $r_1, r_2,\ r_1 + r_2 < n$, respectively. Then we have the following lower bound for span(f)*

$$\text{span}(f) \geqslant \sqrt{\frac{n^2-r_1}{n-r_1-r_2}}\,|x_{n-1} - x_{n-2}|. \tag{38}$$

Proof. Indeed, identities (16) with $z = x_{n-2}$ yield

$$(n^2 - r_1)(x_{n-1} - x_{n-2})^2 = \sum_{j=3}^{k} r_j(\lambda_j - x_{n-2})^2$$

and we derive

$$(n^2 - r_1)(x_{n-1} - x_{n-2})^2 \leqslant (n - r_1 - r_2)[\text{span}(f)]^2,$$

which implies (38). □

Next, we establish an analog of Lemma 5 for roots of derivatives. Precisely, we have

LEMMA 9. *Let $x_{n-1},\ x_{n-2}$ be roots of the $n-1$st, $n-2$nd derivatives of f, respectively. Then*

$$D^{(m)} \geqslant \sqrt{n-m-1}\,|x_{n-1} - x_{n-2}|, \tag{39}$$

where $m \in \{r, r+1, \dots, n-2\}$, $r = \max_{1\leqslant j\leqslant k}(r_j)$. Besides, if x_{n-1} is a root of $f^{(m)}$, then we have a stronger inequality

$$D^{(m)} \geqslant \sqrt{n-m}\,|x_{n-1} - x_{n-2}|. \tag{40}$$

Moreover,

$$2\,D^{(m)} \geqslant \text{span}(f^{(m)}) \geqslant \frac{n-m}{n-m-1}\,D^{(m)}. \tag{41}$$

and if x_{n-1} is a root of $f^{(m)}$, it becomes

$$2\,D^{(m)} \geqslant \text{span}(f^{(m)}) \geqslant \sqrt{\frac{(n-m)(n-m-1)+1}{(n-m-1)(n-m-2)}}\,D^{(m)}, \tag{42}$$

where $m \in \{r, r+1, \dots, n-3\}$.

Proof. In fact, since (see (16))

$$(n-m)(n-m-1)(x_{n-1} - x_{n-2})^2 = \sum_{j=1}^{n-m}(\xi_j^{(m)} - x_{n-1})^2 \leqslant (n-m)\left[D^{(m)}\right]^2,$$

we get (39). Analogously, we immediately come out with (40), when x_{n-1} is a root of $f^{(m)}$, because one element of the sum of squares is zero. In order to prove (41), we appeal again to (16), letting $z = \xi_{s_0}^{(m)}$, $s_0 \in \{1,2,\ldots,n-m\}$, $m \in \{r,r+1,\ldots,n-2\}$, $r = \max_{1\leqslant j\leqslant k}(r_j)$, which is a root of $f^{(m)}$, where the maximum $D^{(m)}$ is attained. Hence owing to Laguerre type inequality (31)

$$(n-m)\left[D^{(m)}\right]^2 \leqslant (n-m-1)[\mathrm{span}(f^{(m)})]^2 - \frac{n-m}{n-m-1}\left[D^{(m)}\right]^2,$$

which leads to the lower bound for $\mathrm{span}(f^{(m)})$ in (41). The upper bound is straightforward since x_{n-1} belongs to the smallest interval containing roots of $f^{(m)}$. In the same manner we establish (42), since in this case

$$(n-m-1)\left[D^{(m)}\right]^2 \leqslant (n-m-2)[\mathrm{span}(f^{(m)})]^2 - \frac{n-m}{n-m-1}\left[D^{(m)}\right]^2. \quad \square$$

REMARK 5. The case $m = n-2$ gives equalities in (39), (41). Letting the same value of m in (40), we easily get a contradiction, which means that the only trivial polynomial is within polynomials with only real roots, whose derivatives $f^{(n-2)}$, $f^{(n-1)}$ have a common root (see Corollary 1).

5. Applications to the Casas-Alvero conjecture

In this final section we will discuss properties of possible CA-polynomials, which share roots with each of their non-constant derivatives. We will investigate particular cases of the Casas-Alvero conjecture, especially for polynomials with only real roots, showing when it holds true or, possibly, is false.

We begin with

PROPOSITION 1. *The Casas-Alvero conjecture holds true, if and only if it is true for common roots $\{z_\nu\}_0^{n-1}$ lying in the unit circle.*

Proof. The necessity is trivial. Let's prove the sufficiency. Let the conjecture be true for common roots $\{z_\nu\}_0^{n-1}$ of a complex polynomial f and its non-constant derivatives, which lie in the unit circle. Associating with f an Abel-Goncharov polynomial G_n (6), one can choose an arbitrary $\alpha > 0$ such that $|z_\nu| < \alpha^{-1}$, $\nu = 0,1,\ldots,n-1$. Hence owing to (7)

$$f(\alpha z_\nu) = G_n(\alpha z_0, \alpha z_\nu, \alpha z_1, \ldots, \alpha z_{n-1}) = \alpha^n G_n(z_\nu) = \alpha^n f(z_\nu) = 0,\ \nu = 0,1,\ldots,n-1,$$

and

$$f_n^{(\nu)}(\alpha z) = n!\frac{d^\nu}{dz^\nu}\int_{\alpha z_0}^{\alpha z}\int_{\alpha z_1}^{s_1}\cdots\int_{\alpha z_{n-1}}^{s_{n-1}} ds_n \ldots ds_1 = n!\alpha^\nu \int_{\alpha z_\nu}^{\alpha z}\int_{\alpha z_{\nu+1}}^{s_{\nu+1}}\cdots\int_{\alpha z_{n-1}}^{s_{n-1}} ds_n \ldots ds_{\nu+1},$$

we find $f_n^{(\nu)}(\alpha z_\nu) = 0$. Hence αz_ν, $\nu = 0,1,\ldots,n-1$ are common roots of ν-th derivatives $f^{(\nu)}$ and f, lying in the unit circle. Consequently, since via assumption the Casas-Alvero conjecture is true when common roots are inside the unit circle, we have that f is trivial and $z_0 = z_1 = \ldots = z_{n-1} = a$ is a unique joint root of f of the multiplicity n. Proposition 1 is proved. $\square$

The following lemma will be useful in the sequel.

LEMMA 10. *Let f be a CA-polynomial with only real roots of degree $n \geqslant 2$ and $\{x_\nu\}_0^{n-1}$ be a sequence of common roots of f and the corresponding derivatives $f^{(\nu)}$. Let $f^{(s+\nu)}(x_\nu) \geqslant 0$, $s = 1,2,\ldots,n-\nu-1$ and $\nu = 0,1,\ldots,n-1$. Then x_ν is a maximal root of the derivative $f^{(\nu)}$.*

Proof. In fact, the proof is an immediate consequence of the expansion (12), where we let $G_n(x) = f(x)$. Indeed, $f^{(\nu)}(x_\nu) = 0, \nu = 0,1,\ldots,n-1$ and when $x > x_\nu$ we have from (12) $f^{(\nu)}(x) > 0, \nu = 0,1,\ldots,n-1$. So, this means that there are no roots, which are bigger than x_ν. This completes the proof of Lemma 10. $\square$

PROPOSITION 2. *Under conditions of Lemma 10 the Casas-Alvero conjecture holds true for polynomials with only real roots.*

Proof. We will show that under conditions of Lemma 10 there exists no CA-polynomial f with only real roots. Indeed, assuming its existence, we find via conditions of the lemma that the root x_0 is a maximal zero of $f(x)$. This means that $x_0 \geqslant x_1$. On the other hand, the classical theorem of Rolle states that between zeros x_0, x_1 in the case $x_0 > x_1$ there exists at least one zero of the derivative $f^{(1)}(x)$, say $\xi_1^{(1)}$, which is bigger than x_1. But this is impossible because x_1 is a maximal zero of the first derivative. Thus $x_0 = x_1 \geqslant x_2$. Then between x_1 and x_2 in the case $x_1 > x_2$ there exists a zero $\xi_2^{(1)}$ of the first derivative such that $x_1 > \xi_2^{(1)} > x_2$. Hence between x_1 and $\xi_2^{(1)}$ there exists at least one zero of the second derivative, which is bigger than x_2. But this is impossible, since x_2 is a maximal zero of $f^{(2)}(x)$. Therefore $x_0 = x_1 = x_2$. Continuing this process we observe that the sequence $\{x_\nu\}_0^{n-1}$ is stationary and f has a unique joint root, which contradicts the definition of the CA-polynomial. $\square$

COROLLARY 9. *There exists no CA-polynomial f with only real roots, having a non-increasing sequence $\{x_\nu\}_0^{n-1}$ of roots in common with f and its non-constant derivatives.*

Proof. Obviously, via (13) $f^{(s+\nu)}(x_\nu) \geqslant 0$, $s = 1,2,\ldots,n-\nu-1$ and conditions of Lemma 10 are satisfied. $\square$

COROLLARY 10. *There exists no CA-polynomial f with only real roots, such that each x_ν in the sequence $\{x_\nu\}_0^{n-1}$ is a maximal root of the derivative $f^{(\nu)}(x)$, $\nu = 0,1,\ldots,n-1$.*

Proof. The proof is similar to the proof of Proposition 2. $\square$

An immediate consequence of Corollaries 3,4,5 is

COROLLARY 11. *The CA-polynomial, if any, with only real roots has at least 5 distinct zeros.*

Let us denote by $l(m)$ the number of distinct roots of the m-th derivative $f^{(m)}$, $m = 0,1,\ldots,n-2$, which are in common with f and different from $\lambda_1 = x_{n-1}$, which is a common root with $f^{(n-1)}$, i.e. the m-th derivative $f^{(m)}$ has $l(m)$ common roots with f

$$\lambda_{j_1},\ldots,\lambda_{j_{l(m)}} \subseteq \{\lambda_2,\lambda_3,\ldots,\ \lambda_k\}$$

of multiplicities

$$r_{j_1},\ldots,r_{j_{l(m)}} \subseteq \{r_2,r_3,\ldots,\ r_k\}.$$

For instance, $l(0) = k-1$, $l(1) = k-1-s$, where s is a number of simple roots of f. So, we see that $n-m \geqslant l(m) \geqslant 0$ and since f is a CA-polynomial, $l(m) = 0$ if and only if $x_{n-1} = \lambda_1$ is the only common root of f with $f^{(m)}$.

LEMMA 11. *There exists no CA-polynomial with only real roots, having the property $l(m) = l(m+1) = 0$ for some $m \in \{r, r+1,\ldots,n-2\}$, where $r = \max_{1\leqslant j\leqslant k}(r_j)$.*

Proof. In fact, as we saw above, since all roots are real, it follows that all roots of $f^{(m)}$, $m \geqslant r$ are simple, which contradicts equalities $l(m) = l(m+1) = 0$. Indeed, the latter equalities yield that x_{n-1} is a multiple root of $f^{(m)}$. Therefore $r \geqslant r_1 > m+1 \geqslant r+1$, which is impossible. □

Further, as in Lemma 7 we involve the root λ_{s_0} of multiplicity r_{s_0}, and $D = |\lambda_{s_0} - x_{n-1}|$ (see (35)). Thus $\lambda_{s_0} = \lambda_*$ or $\lambda_{s_0} = \lambda^*$ and, correspondingly, $r_{s_0} = r_*$ or $r_{s_0} = r^*$. Hence, calling Sz.-Nagy identities (15), we let $z = x_{n-1}$ and assume without loss of generality that $\lambda_{s_0} = \lambda^*$. Then we obtain for $m \geqslant r$

$$r_*(x_{n-1}-\lambda_*) = r^*D + \sum_{j=2,\ r_j\neq r_*,\ r^*}^{k} r_j(\lambda_j - x_{n-1}) \geqslant r^*D - D^{(m)}\sum_{s=1}^{l(m)} r_{j_s} - D^{(m+1)}\sum_{s=1}^{l(m+1)} r_{l_s}$$

$$-\left(n - r_1 - r^* - r_* - \sum_{s=1}^{l(m)} r_{j_s} - \sum_{s=1}^{l(m+1)} r_{l_s}\right)D.$$

But $x_{n-1} - \lambda_* = \operatorname{span}(f) - D$. Therefore,

$$r_*\operatorname{span}(f) + (n - r_1 - 2(r^*+r_*))\,D \geqslant (D - D^{(m)})\sum_{s=1}^{l(m)} r_{j_s} + (D - D^{(m+1)})\sum_{s=1}^{l(m+1)} r_{l_s}.$$

The right-hand side of the latter inequality is, obviously, greater or equal to

$$r_0\,(l(m)+l(m+1))\,(D - D^{(m)}),$$

where $1 \leqslant r_0 = \min_{1\leqslant j\leqslant k}(r_j)$. Moreover, since $\operatorname{span}(f) \leqslant 2D$, the left-hand side does not exceed $(n-r_1)D - r^*\operatorname{span}(f)$. Thus we come up with the inequality

$$r_0\,(l(m)+l(m+1))\,(D - D^{(m)}) \leqslant (n-r_1)\,D - r^*\operatorname{span}(f)$$

or since $D - D^{(m)} > 0 \quad (m \geqslant r)$, it becomes

$$l(m) + l(m+1) \leqslant \frac{(n - r_1)D - r^* \text{span}(f)}{r_0(D - D^{(m)})}. \tag{43}$$

Meanwhile, appealing to (16), we get similarly

$$n(n-1)(x_{n-1} - x_{n-2})^2 = r^* D^2 + r_*(\lambda_* - x_{n-1})^2 + \sum_{j=2,\ r_j \neq r_*,\ r^*}^{k} r_j(\lambda_j - x_{n-1})^2$$

$$\leqslant r^* D^2 + r_*\left(\text{span}(f) - D\right)^2 + \left[D^{(m)}\right]^2 \sum_{s=1}^{l(m)} r_{j_s} + \left[D^{(m+1)}\right]^2 \sum_{s=1}^{l(m+1)} r_{l_s}$$

$$+ \left(n - r_1 - r^* - r_* - \sum_{s=1}^{l(m)} r_{j_s} - \sum_{s=1}^{l(m+1)} r_{l_s}\right) D^2.$$

Therefore, analogously to (43), we arrive at the inequality

$$l(m)+l(m+1) \leqslant \frac{(n-r_1)D^2 + r_*\left[\text{span}(f)\right]^2 - n(n-1)(x_{n-1} - x_{n-2})^2 - 2Dr_*\,\text{span}(f)}{r_0\left(D^2 - \left[D^{(m)}\right]^2\right)}.$$

PROPOSITION 3. *There exists no CA- polynomial with only real roots of degree n such that*

$$\mathit{span}(f) > (r^*)^{-1}\left[(n - r_1 - r_0)D + r_0 D^{(m)}\right],\ m \geqslant r. \tag{44}$$

Proof. Under condition (44), the right-hand side of (43) is less than one. Thus $l(m) = l(m+1) = 0$ and Lemma 11 completes the proof. □

Let $m = n-2$. Then since $l(n-1) = 0$, inequality (43) becomes

$$l(n-2) \leqslant \frac{(n - r_1)D - r^* \text{span}(f)}{r_0(D - |x_{n-1} - x_{n-2}|)}. \tag{45}$$

PROPOSITION 4. *There exists no CA- polynomial with only real roots of degree n such that*

$$D < \left[r^* \sqrt{\frac{n^2 - r_1}{n - r_1 - r_2}} - r_0\right] \frac{|x_{n-1} - x_{n-2}|}{n - r_1 - r_0}. \tag{46}$$

Proof. Indeed, employing the lower bound (38) for $\text{span}(f)$, we find that under condition (46) the right-hand side of (45) is strictly less than one. Consequently, $l(n-2) = 0$ and owing to Corollary 1 f is trivial. If the maximum of multiplicities $r > n-2$, f has at most 2 distinct zeros and it is trivial via Corollary 3. □

Finally, we prove

PROPOSITION 5. *Let CA-polynomial with only real roots exist. Then it has the property*

$$\frac{d}{D} \leqslant \sqrt{\frac{2(n-m-1)}{2(k-1)-1}}, \tag{47}$$

where d, D *are defined by* (34), (35)*, respectively, and* m, $m+1$ *belong to the interval* $\left[r, \frac{1}{2}\left(1-\frac{1}{r_0}\right)(n-1)\right)$.

Proof. Since m, $m+1$ are chosen from the interval $\left[r, \frac{1}{2}\left(1-\frac{1}{r_0}\right)(n-1)\right)$, condition (24) holds for these values. Hence assuming the existence of the CA-polynomial, we return to the Sz.-Nagy type identity (25) to have the estimate

$$0 \geqslant l(m)\left(r_0\frac{(n-m)(n-m-1)}{n(n-1)}-1\right)d^2+(k-1-l(m))d^2-(n-m-l(m))D^2$$

$$\geqslant (k-1)d^2-(n-m)D^2+l(m)(D^2-d^2).$$

Writing the same inequality for $m+1$

$$0 \geqslant (k-1)d^2-(n-m-1)D^2+l(m+1)(D^2-d^2)$$

and adding two inequalities, we find

$$0 \geqslant 2(k-1)d^2-(2(n-m)-1)D^2+(l(m)+l(m+1))(D^2-d^2),$$

which means

$$l(m)+l(m+1) \leqslant \frac{(2(n-m)-1)D^2-2(k-1)d^2}{D^2-d^2}.$$

So, for the existence of the CA-polynomial it is necessary that the right-hand side of the latter inequality is greater than or equal to 1. Thus we come up with condition (47) and complete the proof. □

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