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ZEROS OF CERTAIN POLYNOMIALS AND ANALYTIC FUNCTIONS WITH RESTRICTED COEFFICIENTS

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Abstract. In this paper, we consider the problem of finding the number of zeros of a special class of polynomial functions and functions analytic in a prescribed region by subjecting the real and imaginary parts of its coefficients to certain restrictions.

1. Introduction and statement of results

If $P(z)=\sum\limits_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that

$$a_n \geqslant a_{n-1} \geqslant a_{n-2} \geqslant \ldots \geqslant a_1 \geqslant a_0 > 0,$$

then $P(z)$ has all zeros in the closed unit disc. This famous result is known as Eneström-Kakeya theorem (see [5, 6, 8]). In the literature [1, 3, 9] there exist extensions and generalizations of Eneström-Kakeya theorem. By using Schwartz lemma, Aziz and Mohammad [1] generalized Eneström-Kakeya theorem in a different way and proved the following:

THEOREM A. *If $P(z)=\sum\limits_{j=0}^{n} a_j z^j$ is a polynomial of degree n with positive and real coefficients. If $t_1 > t_2 \geqslant 0$ can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geqslant 0, \quad r = 1,2,\ldots,n+1, \quad a_{-1} = a_{n+1} = 0$$

then all the zeros of $P(z)$ lie in $|z| \leqslant t_1$.

Regarding the number of zeros in $|z| \leqslant \frac{1}{2}$ of the polynomial $P(z)=\sum\limits_{j=0}^{n} a_j z^j$, Mohammed [7] proved the following.

THEOREM B. *If $P(z)=\sum\limits_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that*

$$a_n \geqslant a_{n-1} \geqslant a_{n-2} \geqslant \ldots \geqslant a_1 \geqslant a_0 > 0,$$

then the number of zeros of $P(z)$ in $|z| \leqslant \frac{1}{2}$ does not exceed

$$1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}.$$

Dewan [2] generalized Theorem A to polynomials with complex coefficients and proved the following result:

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THEOREM C. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients if Re* $a_j = \alpha_j$, *Im* $a_j = \beta_j$, $j = 0,1,2,\dots n$ *and*

$$\alpha_n \geqslant \alpha_{n-1} \geqslant \alpha_{n-2} \geqslant \dots \geqslant \alpha_1 \geqslant \alpha_0 > 0,$$

then the number of zeros of $P(z)$ *in* $|z| \leqslant \frac{1}{2}$ *does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{\alpha_n + \sum_{j=0}^{n} |\beta_j|}{|a_0|}.$$

Upadhye [11] gave a generalization of Theorem C for the region $|z| \leqslant \delta,\ 0 < \delta < 1$. In fact, she proved the following:

THEOREM D. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients if Re* $a_j = \alpha_j$, *Im* $a_j = \beta_j$, $j = 0,1,2,\dots,n$ *and for some* $k \geqslant 1$

$$k\alpha_n \geqslant \alpha_{n-1} \geqslant \alpha_{n-2} \geqslant \dots \geqslant \alpha_1 \geqslant \alpha_0 > 0,$$

then the number of zeros of $P(z)$ *in in* $\frac{|a_0|}{M_1} \leqslant |z| \leqslant \delta,\ 0 < \delta < 1$ *does not exceed*

$$1 + \frac{1}{\log \frac{1}{\delta}} \log \frac{k(|\alpha_n| + \alpha_n) + |\alpha_0| - \alpha_0 + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|}$$

where

$$M_1 = k(|\alpha_n| + \alpha_n) + |\beta_0| - \alpha_0 + 2\sum_{j=1}^{n} |\beta_j|.$$

In this paper, we relax the restriction on the coefficients of polynomial and prove the more general result from which the other results follows by fairly uniform procedure.

THEOREM 1. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients such that*

$$|a_n| \leqslant |a_{n-1}| \leqslant \dots \leqslant |a_{k+1}| \leqslant \lambda |a_k| \geqslant |a_{k-1}| \geqslant \dots \geqslant |a_1| \geqslant |a_0|,$$

where $0 \leqslant k \leqslant n$ *and for some real* β,

$$|\arg a_j - \beta| \leqslant \alpha \leqslant \frac{\pi}{2}, \quad j = 0,1,2,\dots,n,$$

then the number of zeros of $P(z)$ *in* $|z| \leqslant \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{2\lambda |a_k| \cos\alpha + 2|\lambda - 1||a_k| \sin\alpha + |a_n|(\sin\alpha - \cos\alpha + 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| + 2|1-\lambda||a_k|}{|a_0|}.$$

With $\lambda = 1$ in Theorem 1, we have

COROLLARY 1. *If* $P(z)=\sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients such that*

$$|a_n| \leqslant |a_{n-1}| \leqslant \ldots \leqslant |a_{k+1}| \leqslant |a_k| \geqslant |a_{k-1}| \geqslant \ldots \geqslant |a_1| \geqslant |a_0|,$$

where $0 \leqslant k \leqslant n$ *and for some real* β,

$$|\arg a_j - \beta| \leqslant \alpha \leqslant \frac{\pi}{2}, \quad j=0,1,2,...$$

then the number of zeros of $P(z)$ *in* $|z| \leqslant \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2}\log\frac{2|a_k|\cos\alpha+|a_n|(\sin\alpha-\cos\alpha+1)+2\sin\alpha\sum_{j=0}^{n-1}|a_j|}{|a_0|}.$$

REMARK 1. For $\lambda=1$, $k=n$ and $\alpha=\beta=0$ and assume all the coefficients to be positive in Theorem 1, we obtain Theorem B.

THEOREM 2. *If* $P(z)=\sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients such that*

$$|a_n| \leqslant |a_{n-1}| \leqslant \ldots \leqslant |a_{k+1}| \leqslant \lambda|a_k| \geqslant |a_{k-1}| \geqslant ... \geqslant |a_1| \geqslant |a_0|,$$

where $0 \leqslant k \leqslant n-1$ *and for some real* β,

$$|\arg a_j - \beta| \leqslant \alpha \leqslant \frac{\pi}{2}, \quad j=0,1,2,\ldots,$$

then all the zeros of $P(z)$ *lie in*

$$\left|z+\frac{a_{n-1}}{a_n}-1\right| \leqslant \frac{1}{a_n}\Big\{2\lambda|a_k|\cos\alpha+2|\lambda-1||a_k|\sin\alpha-|a_{n-1}|(\sin\alpha+\cos\alpha)$$
$$+2\sin\alpha\sum_{j=0}^{n-1}|a_j|-|a_0|(\sin\alpha+\cos\alpha-1)+2|1-\lambda||a_k|\Big\}.$$

We now turn to study the zeros of certain related analytic functions and in this direction we prove the following.

THEOREM 3. *Let* $f(z)=\sum_{j=0}^{\infty} a_j z^j$ *(not identically zero) be analytic in* $|z| \leqslant 1$. *If* *Re* $a_j=\alpha_j$ *and Im* $a_j=\beta_j$, $j=0,1,2,\ldots,n$ *and for some finite* k,

$$0<\alpha_0 \leqslant \alpha_1 \leqslant ... \leqslant \lambda\alpha_k \geqslant \alpha_{k+1} \geqslant \ldots$$

with $\lambda \geqslant 1$, *then the number of zeros of* $f(z)$ *in* $|z| \leqslant \frac{1}{2}$ *does not exceed*

$$1+\frac{1}{\log 2}\left\{\frac{\lambda\alpha_k+(\lambda-1)|\alpha_k|+\sum_{j=0}^{\infty}|\beta_j|}{|a_0|}\right\}.$$

With $\lambda=1$ in Theorem 3, we have

COROLLARY 2. *If* $f(z) = \sum_{j=0}^{\infty} a_j z^j$ *be analytic in* $|z| \leqslant 1$. *If Re* $a_j = \alpha_j$ *and Im* $a_j = \beta_j$, $j = 0,1,2,\ldots,n$ *and for some finite* k,

$$0 < \alpha_0 \leqslant \alpha_1 \leqslant \ldots \leqslant \alpha_k \geqslant \alpha_{k+1} \geqslant \ldots$$

then the number of zeros of $f(z)$ *in* $|z| \leqslant \frac{1}{2}$ *does not exceed*

$$1 + \frac{1}{\log 2}\left\{\frac{\alpha_k + \sum_{j=0}^{\infty} |\beta_j|}{|a_0|}\right\}.$$

2. Lemma

For the proof of some these results we need the following lemma which is due to Govil and Rahman [4].

LEMMA. *For any two complex numbers* b_0 *and* b_1 *such that* $|b_0| \geqslant |b_1|$ *and*

$$|\arg\ b_j - \beta| \leqslant \alpha \leqslant \frac{\pi}{2}, \quad j = 0,1$$

for some real β, *then*

$$|b_0 - b_1| \leqslant (|b_0| - |b_1|)\cos\alpha + (|b_0| + |b_1|)\sin\alpha.$$

3. Proof of theorems

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= (1-z)\left(a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0\right) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots + (a_k - a_{k-1})z^k + \ldots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0. \end{aligned}$$

Therefore for $|z| = 1$ we have

$$\begin{aligned} |F(z)| &\leqslant |a_n| + |a_n - a_{n-1}| + \ldots + |a_{k+1} - a_k| + |a_k - a_{k-1}| + \ldots + |a_1 - a_0| + |a_0| \\ &= |a_n| + |a_n - a_{n-1}| + \ldots + |a_{k+1} - \lambda a_k + \lambda a_k - a_k| + |a_k - \lambda a_k + \lambda a_k - a_{k-1}| \\ &\quad + \ldots + |a_1 - a_0| + |a_0| \\ &\leqslant |a_n| + |a_n - a_{n-1}| + \ldots + |a_{k+1} - \lambda a_k| + |\lambda a_k - a_{k-1}| + 2|1-\lambda||a_k| \\ &\quad + \ldots + |a_1 - a_0| + |a_0|. \end{aligned}$$

Using above lemma, we have

$$\begin{aligned}
|F(z)| \leqslant\ & |a_n| + 2|1-\lambda||a_k| + (|a_{n-1}| - |a_n|)\cos\alpha + (|a_{n-1}| + |a_n|)\sin\alpha \\
& +(|a_{n-2}| - |a_{n-1}|)\cos\alpha \ldots \\
& +(|a_{n-2}| + |a_{n-1}|)\sin\alpha + (\lambda|a_k| - |a_{k+1}|)\cos\alpha + (\lambda|a_k| + |a_{k+1}|)\sin\alpha \\
& +(\lambda|a_k| - |a_{k-1}|)\cos\alpha + (\lambda|a_k| + |a_{k-1}|)\sin\alpha + \ldots + (|a_2| - |a_1|)|\cos\alpha \\
& +(|a_2| + |a_1|)\sin\alpha + (|a_1| - |a_0|)|\cos\alpha + (|a_1| + |a_0|)\sin\alpha + |a_0| \\
=\ & |a_n| + 2|1-\lambda||a_k| + |a_n|\sin\alpha - |a_n|\cos\alpha + 2|a_{n-1}|\sin\alpha + \ldots + 2|a_{k+1}|\sin\alpha \\
& +2\lambda|a_k|\sin\alpha + 2\lambda|a_k|\cos\alpha + \ldots + 2|a_1|\sin\alpha + |a_0|\sin\alpha - |a_0|\cos\alpha + |a_0| \\
=\ & |a_n|(\sin\alpha - \cos\alpha + 1) + 2\lambda|a_k|\cos\alpha + 2|\lambda - 1||a_k|\sin\alpha + 2\sin\alpha\sum_{j=0}^{n-1}|a_j| \\
& -|a_0|(\cos\alpha + \sin\alpha - 1) + 2|1-\lambda||a_k| \\
\leqslant\ & |a_n|(\sin\alpha - \cos\alpha + 1) + 2\lambda|a_k|\cos\alpha + 2|\lambda - 1||a_k|\sin\alpha \\
& +2\sin\alpha\sum_{j=0}^{n-1}|a_j| + 2|1-\lambda||a_k| \\
=\ & M.
\end{aligned}$$

Thus $|F(z)| \leqslant M$ for $|z| = 1$. Also for $|F(0)| = |a_0| \neq 0$.

Now it is known (see, [10, pp. 171]) that if $f(z)$ is analytic, $f(0) \neq \frac{1}{2}$ and $|f(z)| \leqslant M$ in $|z| \leqslant 1$, then the number of zeros of $f(z)$ in $|z| \leqslant \frac{1}{2}$, does not exceed

$$\frac{1}{\log 2}\log\frac{M}{|f(0)|}.$$

Thus, the number of zeros of $F(z)$ in $|z| \leqslant \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\log\frac{2\lambda|a_k|\cos\alpha + 2|\lambda-1||a_k|\sin\alpha + |a_n|(\sin\alpha - \cos\alpha + 1) + 2\sin\alpha\sum_{j=0}^{n-1}|a_j| + 2|1-\lambda||a_k|}{|a_0|}.$$

As the number of zeros of $P(z)$ in $|z| \leqslant \frac{1}{2}$ is also equal to the number of zeros $F(z)$ the theorem follows. □

Proof of Theorem 2. Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots + (a_k - a_{k-1})z^k + \ldots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0
\end{aligned}$$

Let $|z| > 1$, then $0 \leqslant k \leqslant n-1$, we have

$$\begin{aligned}
|F(z)| \geqslant\ & |a_n z^{n+1} - (a_n - a_{n-1})z^n| - |(a_{n-1} - a_{n-2})z^{n-1} + \ldots + (a_{k+1} - a_k)z^{k+1} \\
& +(a_k - a_{k-1})z^k + \ldots + (a_1 - a_0)z + a_0|
\end{aligned}$$

$$
\begin{aligned}
&= |a_n z^{n+1} - (a_n - a_{n-1}) z^n| - |(a_{n-1} - a_{n-2}) z^{n-1} + \ldots + (a_{k+1} - \lambda a_k + \lambda a_k - a_k) z^{k+1} \\
&\quad + (a_k + \lambda a_k - a_{k-1} - \lambda a_k) z^k + \ldots + (a_1 - a_0) z + a_0| \\
&\geqslant |a_n z^{n+1} + a_{n-1} z^n - a_n z^n| - \Big\{ |a_{n-1} - a_{n-2}||z|^{n-1} + \ldots + |a_{k+1} - \lambda a_k||z|^{k+1} \\
&\quad + |\lambda - 1||a_k||z|^{k+1} + |\lambda a_k - a_{k-1}||z|^k + |1 - \lambda||a_k||z|^k + \ldots + |a_1 - a_0||z| + |a_0| \Big\} \\
&\geqslant |z|^n |a_n z + a_{n-1} - a_n| - \Big\{ \frac{|a_{n-1} - a_{n-2}|}{|z|} + \ldots + \frac{|a_{k+1} - \lambda a_k|}{|n|^{n-k-1}} + \frac{|\lambda - 1||a_k|}{|z|^{n-k-1}} \\
&\quad + \frac{|\lambda a_k - a_{k-1}|}{|z|^{n-k}} + \frac{|1 - \lambda||a_k|}{|z|^{n-k}} + \ldots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \Big\}.
\end{aligned}
$$

Using lemma and making use of the fact that $\frac{1}{|z|^j} < 1,\ j = 1, 2, \ldots, n,$ we have

$$
\begin{aligned}
|F(z)| &\geqslant |z|^n |a_n z + a_{n-1} - a_n| - \Big\{ (|a_{n-2}| - |a_{n-1}|) \cos\alpha + (|a_{n-2}| + |a_{n-1}|) \sin\alpha + \ldots \\
&\quad + (|\lambda a_k| - |a_{k+1}|) \cos\alpha + (|\lambda a_k| + |a_{k+1}|) \sin\alpha + (|\lambda a_k| - |a_{k-1}|) \cos\alpha \\
&\quad + (|\lambda a_k| - |a_{k-1}|) \sin\alpha + \ldots + (|a_1| - |a_0|) \cos\alpha + (|a_1| - |a_0|) \sin\alpha \\
&\quad + |a_0| + 2|1 - \lambda||a_k| \Big\} \\
&\geqslant |z|^n |a_n z + a_{n-1} - a_n| - \Big\{ 2\lambda |a_k| \cos\alpha + 2|\lambda - 1||a_k| \sin\alpha - |a_{n-1}| (\sin\alpha + \cos\alpha) \\
&\quad + 2 \sin\alpha \sum_{j=0}^{n-1} |a_j| - |a_0| (\sin\alpha + \cos\alpha - 1) + 2|1 - \lambda||a_k| \Big\} > 0,
\end{aligned}
$$

if

$$
\begin{aligned}
|a_n z + a_{n-1} - a_n| > 2\lambda |a_k| \cos\alpha + 2|\lambda - 1||a_k| \sin\alpha - |a_{n-1}| (\sin\alpha + \cos\alpha) \\
+ 2 \sin\alpha \sum_{j=0}^{n-1} |a_j| - |a_0| (\sin\alpha + \cos\alpha - 1) + 2|1 - \lambda||a_k|.
\end{aligned}
$$

Hence $F(z)$ does not vanish in

$$
\begin{aligned}
|z + \frac{a_{n-1}}{a_n} - 1| > \frac{1}{a_n} \Big\{ 2\lambda |a_k| \cos\alpha + 2|\lambda - 1||a_k| \sin\alpha - |a_{n-1}| (\sin\alpha + \cos\alpha) \\
+ 2 \sin\alpha \sum_{j=0}^{n-1} |a_j| - |a_0| (\sin\alpha + \cos\alpha - 1) + 2|1 - \lambda||a_k| \Big\}.
\end{aligned}
$$

Therefore those zeros of $F(z)$ whose modulus is greater than one lie in

$$
\begin{aligned}
|z + \frac{a_{n-1}}{a_n} - 1| \leqslant \frac{1}{a_n} \Big\{ 2\lambda |a_k| \cos\alpha + 2|\lambda - 1||a_k| \sin\alpha - |a_{n-1}| (\sin\alpha + \cos\alpha) \\
+ 2 \sin\alpha \sum_{j=0}^{n-1} |a_j| - |a_0| (\sin\alpha + \cos\alpha - 1) + 2|1 - \lambda||a_k| \Big\}. \qquad (1)
\end{aligned}
$$

But those zeros of $F(z)$ whose modulus is less than or equal to one already (1). Hence we conclude that all the zeros of $F(z)$ and those of $p(z)$ lie (1). This completes the proof. □

Proof of Theorem 3. Consider the function

$$\begin{aligned} F(z) &= (z-1)f(z) = (z-1)(a_o + a_1 z + a_2 z^2 + a_3 z^3 + ... \\ &= -a_o + \sum_{j=1}^{\infty}(a_{j-1} - a_j)z^j. \end{aligned}$$

Hence for $|z| = 1$

$$\begin{aligned} |F(z)| &\leqslant |a_0| + \sum_{j=1}^{\infty} |a_{j-1} - a_j| \\ &\leqslant (|\alpha_0| + |\beta_0|) + \sum_{j=1}^{\infty} |\alpha_{j-1} - \alpha_j| + \sum_{j=1}^{\infty} |\beta_{j-1} - \beta_j| \\ &= (|\alpha_0| + |\beta_0|) + \sum_{j=1}^{k} |\alpha_j - \alpha_{j-1}| + \sum_{j=k+1}^{\infty} |\alpha_{j-1} - \alpha_j| + \sum_{j=1}^{\infty} |\beta_{j-1} - \beta_j| \\ &\leqslant |\alpha_0| + |\beta_0| + |\alpha_1 - \alpha_0| + |\alpha_2 - \alpha_1| + ... + |\alpha_{k-1} - \alpha_{k-2}| + |\alpha_k - \alpha_{k-1}| \\ &\quad + |\alpha_k - \alpha_{k+1}| + |\alpha_{k+1} - \alpha_{k+2}| + |\alpha_{k+2} - \alpha_{k+3}| + \ldots + \sum_{j=1}^{\infty}(|\beta_{j-1}| + |\beta_j|) \\ &\leqslant \alpha_0 + |\beta_0| + \alpha_1 - \alpha_0 + \alpha_2 - \alpha_1 + \ldots + \alpha_{k-1} - \alpha_{k-2} + |\alpha_k - \alpha_{k-1}| \\ &\quad + |\alpha_k - \alpha_{k+1}| + \alpha_{k+1} - \alpha_{k+2} + \alpha_{k+2} - \alpha_{k+3} + \ldots + \sum_{j=1}^{\infty}(|\beta_{j-1}| + |\beta_j|) \\ &= \alpha_{k-1} + |\alpha_k - \alpha_{k-1}| + |\alpha_k - \alpha_{k+1}| + \alpha_{k+1} + 2\sum_{j=0}^{\infty} |\beta_j| \\ &= \alpha_{k-1} + |\alpha_k - \lambda\alpha_k + \lambda\alpha_k - \alpha_{k-1}| + |\alpha_k + \lambda\alpha_k - \lambda\alpha_k - \alpha_{k+1}| + \alpha_{k+1} + 2\sum_{j=0}^{\infty} |\beta_j| \\ &= \alpha_{k-1} + |\lambda\alpha_k - \alpha_{k-1}| + (\lambda-1)|\alpha_k| + |\lambda\alpha_k - \alpha_{k+1}| + (\lambda-1)|\alpha_k| + \alpha_{k+1} + 2\sum_{j=0}^{\infty} |\beta_j| \\ &= \alpha_{k-1} + \lambda\alpha_k - \alpha_{k-1} + (\lambda-1)|\alpha_k| + \lambda\alpha_k - \alpha_{k+1} + (\lambda-1)|\alpha_k| + \alpha_{k+1} + 2\sum_{j=0}^{\infty} |\beta_j| \\ &= 2\lambda\alpha_k + 2(\lambda-1)|\alpha_k| + 2\sum_{j=0}^{\infty} |\beta_j| \end{aligned}$$

which implies that

$$\frac{|F(z)|}{|F(0)|} \leqslant \frac{2\lambda\alpha_k + 2(\lambda-1)|\alpha_k| + 2\sum_{j=0}^{\infty} |\beta_j|}{|a_0|}.$$

Now it is known (see, [10, pp. 171]) that if $g(z)$ is analytic, $g(0) \neq \frac{1}{2}$ and $|g(z)| \leqslant M$ in $|z| \leqslant 1$, then the number of zeros of $g(z)$ in $|z| \leqslant \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|f(0)|}.$$

Thus, the number of zeros of $F(z)$ in $|z| \leqslant \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\left\{\log \frac{2\lambda \alpha_k + 2(\lambda - 1)|\alpha_k| + 2\sum_{j=0}^{\infty}|\beta_j|}{|a_0|}\right\}.$$

or equivalently

$$1 + \frac{1}{\log 2}\left\{\log \frac{\lambda \alpha_k + (\lambda - 1)|\alpha_k| + \sum_{j=0}^{\infty}|\beta_j|}{|a_0|}\right\}.$$

As the number of zeros of $f(z)$ in $|z| \leqslant \frac{1}{2}$ is also equal to the number of zeros $F(z)$ the theorem follows. $\square$

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