ZEROS OF CERTAIN POLYNOMIALS AND ANALYTIC FUNCTIONS WITH RESTRICTED COEFFICIENTS

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Abstract. In this paper, we consider the problem of finding the number of zeros of a special class of polynomial functions and functions analytic in a prescribed region by subjecting the real and imaginary parts of its coefficients to certain restrictions.

1. Introduction and statement of results

If \( P(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) such that
\[
a_n \geq a_{n-1} \geq a_{n-2} \geq \ldots \geq a_1 \geq a_0 > 0,
\]
then \( P(z) \) has all zeros in the closed unit disc. This famous result is known as Eneström-Kakeya theorem (see [5, 6, 8]). In the literature [1, 3, 9] there exist extensions and generalizations of Eneström-Kakeya theorem. By using Schwartz lemma, Aziz and Mohammad [1] generalized Eneström-Kakeya theorem in a different way and proved the following:

**Theorem A.** If \( P(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) with positive and real coefficients. If \( t_1 > t_2 \geq 0 \) can be found such that
\[
ar_{r}t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \quad r = 1, 2, \ldots, n + 1, \quad a_{-1} = a_{n+1} = 0
\]
then all the zeros of \( P(z) \) lie in \( |z| \leq t_1 \).

Regarding the number of zeros in \( |z| \leq \frac{1}{2} \) of the polynomial \( P(z) = \sum_{j=0}^{n} a_j z^j \), Mohammad [7] proved the following.

**Theorem B.** If \( P(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) such that
\[
a_n \geq a_{n-1} \geq a_{n-2} \geq \ldots \geq a_1 \geq a_0 > 0,
\]
then the number of zeros of \( P(z) \) in \( |z| \leq \frac{1}{2} \) does not exceed
\[
1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}.
\]

Dewan [2] generalized Theorem A to polynomials with complex coefficients and proved the following result:


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Theorem C. If \( P(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) with complex coefficients if \( \text{Re} \, a_j = \alpha_j, \text{Im} \, a_j = \beta_j, \, j = 0, 1, 2, \ldots, n \) and
\[
\alpha_n > \alpha_{n-1} > \alpha_{n-2} \geq \ldots \geq \alpha_1 > \alpha_0 > 0,
\]
then the number of zeros of \( P(z) \) in \( |z| \leq \frac{1}{2} \) does not exceed
\[
1 + \frac{1}{\log 2} \log \frac{\alpha_n + \sum_{j=0}^{n} |\beta_j|}{|a_0|}.
\]

Upadhye [11] gave a generalization of Theorem C for the region \( |z| \leq \delta, \, 0 < \delta < 1 \). In fact, she proved the following:

Theorem D. If \( P(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) with complex coefficients if \( \text{Re} \, a_j = \alpha_j, \text{Im} \, a_j = \beta_j, \, j = 0, 1, 2, \ldots, n \) and for some \( k \geq 1 \)
\[
k \alpha_n > \alpha_{n-1} > \alpha_{n-2} \geq \ldots \geq \alpha_1 > \alpha_0 > 0,
\]
then the number of zeros of \( P(z) \) in \( \frac{|a_0|}{M_1} \leq |z| \leq \delta, \, 0 < \delta < 1 \) does not exceed
\[
1 + \frac{1}{\log \frac{1}{\delta}} \log \frac{k(|\alpha_n| + |\alpha_n| + |\alpha_0| - \alpha_0 + 2 \sum_{j=0}^{n} |\beta_j|)}{|a_0|}
\]
where
\[
M_1 = k(|\alpha_n| + |\alpha_n| + |\beta_0| - \alpha_0 + 2 \sum_{j=1}^{n} |\beta_j|).
\]

In this paper, we relax the restriction on the coefficients of polynomial and prove the more general result from which the other results follows by fairly uniform procedure.

Theorem 1. If \( P(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) with complex coefficients such that
\[
|a_n| \leq |a_{n-1}| \leq \ldots \leq |a_{k+1}| \leq \lambda |a_k| \geq |a_{k-1}| \geq \ldots \geq |a_1| \geq |a_0|,
\]
where \( 0 \leq k \leq n \) and for some real \( \beta \),
\[
|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, 2, \ldots, n,
\]
then the number of zeros of \( P(z) \) in \( |z| \leq \frac{1}{2} \) does not exceed
\[
\frac{2\lambda |a_k| \cos \alpha + 2(\lambda - 1)|a_k| \sin \alpha + |a_n|(\sin \alpha - \cos \alpha + 1) + 2 \sin \alpha \sum_{j=0}^{n-1} |aj| + 2|1 - \lambda||a_k|}{\log 2 \log \frac{2\lambda |a_k| \cos \alpha + 2(\lambda - 1)|a_k| \sin \alpha + |a_n|(\sin \alpha - \cos \alpha + 1) + 2 \sin \alpha \sum_{j=0}^{n-1} |aj| + 2|1 - \lambda||a_k|}{|a_0|}}.
\]

With \( \lambda = 1 \) in Theorem 1, we have
**COROLLARY 1.** If \( P(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) with complex coefficients such that
\[
|a_n| \leq |a_{n-1}| \leq \ldots \leq |a_{k+1}| \leq |a_k| \geq |a_{k-1}| \geq \ldots \geq |a_1| \geq |a_0|,
\]
where \( 0 \leq k \leq n \) and for some real \( \beta \),
\[
|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, 2, \ldots
\]
then the number of zeros of \( P(z) \) in \(|z| \leq \frac{1}{2}\) does not exceed
\[
\frac{1}{\log 2} \log \frac{2|a_k| \cos \alpha + |a_n| (\sin \alpha - \cos \alpha + 1) + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.
\]

**REMARK 1.** For \( \lambda = 1, k = n \) and \( \alpha = \beta = 0 \) and assume all the coefficients to be positive in Theorem 1, we obtain Theorem B.

**THEOREM 2.** If \( P(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) with complex coefficients such that
\[
|a_n| \leq |a_{n-1}| \leq \ldots \leq |a_{k+1}| \leq \lambda |a_k| \geq |a_{k-1}| \geq \ldots \geq |a_1| \geq |a_0|,
\]
where \( 0 \leq k \leq n - 1 \) and for some real \( \beta \),
\[
|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, 2, \ldots
\]
then all the zeros of \( P(z) \) lie in
\[
\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{a_n} \left\{ 2\lambda |a_k| \cos \alpha + 2|\lambda - 1||a_k| \sin \alpha - |a_{n-1}|(\sin \alpha + \cos \alpha) 
+ 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| - |a_0|(\sin \alpha + \cos \alpha - 1) + 2|1 - \lambda||a_k| \right\}.
\]

We now turn to study the zeros of certain related analytic functions and in this direction we prove the following.

**THEOREM 3.** Let \( f(z) = \sum_{j=0}^{\infty} a_j z^j \) (not identically zero) be analytic in \(|z| \leq 1\). If \( \text{Re } a_j = \alpha_j \) and \( \text{Im } a_j = \beta_j, \quad j = 0, 1, 2, \ldots, n \) and for some finite \( k \),
\[
0 < \alpha_0 < \alpha_1 < \ldots < \lambda \alpha_k \geq \alpha_{k+1} \geq \ldots
\]
with \( \lambda \geq 1 \), then the number of zeros of \( f(z) \) in \(|z| \leq \frac{1}{2}\) does not exceed
\[
1 + \frac{1}{\log 2} \left\{ \frac{\lambda \alpha_k + (\lambda - 1)|\alpha_k| + \sum_{j=0}^{\infty} |\beta_j|}{|a_0|} \right\}.
\]

With \( \lambda = 1 \) in Theorem 3, we have
COROLLARY 2. If \( f(z) = \sum_{j=0}^{\infty} a_j z^j \) be analytic in \( |z| \leq 1 \). If \( \text{Re} \ a_j = \alpha_j \) and \( \text{Im} \ a_j = \beta_j, \ j = 0, 1, 2, \ldots, n \) and for some finite \( k \)

\[
0 < \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_k \geq \alpha_{k+1} \geq \ldots
\]

then the number of zeros of \( f(z) \) in \( |z| \leq \frac{1}{2} \) does not exceed

\[
1 + \frac{1}{\log 2} \left\{ \frac{\alpha_k + \sum_{j=0}^{\infty} |\beta_j|}{|a_0|} \right\}.
\]

2. Lemma

For the proof of some these results we need the following lemma which is due to Govil and Rahman [4].

LEMMA. For any two complex numbers \( b_0 \) and \( b_1 \) such that \( |b_0| \geq |b_1| \) and

\[
|\arg \ b_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \ j = 0, 1
\]

for some real \( \beta \), then

\[
|b_0 - b_1| \leq (|b_0| - |b_1|) \cos \alpha + (|b_0| + |b_1|) \sin \alpha.
\]

3. Proof of theorems

Proof of Theorem 1. Consider the polynomial

\[
F(z) = (1-z)P(z)
\]

\[
= (1-z) \left(a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0\right)
\]

\[
= -a_n z^{n+1} + (a_n-a_{n-1}) z^n + \ldots + (a_k-a_{k-1}) z^k + \ldots + (a_2-a_1) z^2 + (a_1-a_0) z + a_0.
\]

Therefore for \( |z| = 1 \) we have

\[
|F(z)| \leq |a_n| + |a_n-a_{n-1}| + \ldots + |a_{k+1}-a_k| + |a_k-a_{k-1}| + \ldots + |a_1-a_0| + |a_0|
\]

\[
= |a_n| + |a_n-a_{n-1}| + \ldots + |a_{k+1} - \lambda a_k + \lambda a_k - a_k| + |a_k - \lambda a_k + \lambda a_k - a_{k-1}|
\]

\[
+ \ldots + |a_1 - a_0| + |a_0|
\]

\[
\leq |a_n| + |a_n-a_{n-1}| + \ldots + |a_{k+1} - \lambda a_k| + |\lambda a_k - a_{k-1}| + 2|1-\lambda| |a_k|
\]

\[
+ \ldots + |a_1 - a_0| + |a_0|.
\]
Using above lemma, we have

\[ |F(z)| \leq |a_n| + 2|1 - \lambda||a_k| + (|a_{n-1}| - |a_n|) \cos \alpha + (|a_{n-1}| + |a_n|) \sin \alpha \]
\[ + (|a_{n-2}| - |a_{n-1}|) \cos \alpha \ldots \]
\[ + (|a_{n-2}| + |a_{n-1}|) \sin \alpha + (\lambda |a_k| - |a_{k+1}|) \cos \alpha + (\lambda |a_k| + |a_{k+1}|) \sin \alpha \]
\[ + (\lambda |a_k| - |a_{k-1}|) \cos \alpha + (\lambda |a_k| + |a_{k-1}|) \sin \alpha + \ldots + (|a_2| - |a_1|) \cos \alpha \]
\[ + (|a_2| + |a_1|) \sin \alpha + (|a_1| - |a_0|) \cos \alpha + (|a_1| + |a_0|) \sin \alpha + |a_0| \]
\[ = |a_n| + 2|1 - \lambda||a_k| + |a_n| \sin \alpha - |a_n| \cos \alpha + 2|a_{n-1}| \sin \alpha + \ldots + 2|a_{k+1}| \sin \alpha \]
\[ + 2\lambda |a_k| \sin \alpha + 2\lambda |a_k| \cos \alpha + \ldots + 2|a_1| \sin \alpha + |a_0| \sin \alpha - |a_0| \cos \alpha + |a_0| \]
\[ = |a_n|(\sin \alpha - \cos \alpha + 1) + 2\lambda |a_k| \cos \alpha + 2|\lambda - 1||a_k| \sin \alpha + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| \]
\[ - |a_0|(\cos \alpha + \sin \alpha - 1) + 2|1 - \lambda||a_k| \]
\[ \leq |a_n|(\sin \alpha - \cos \alpha + 1) + 2\lambda |a_k| \cos \alpha + 2|\lambda - 1||a_k| \sin \alpha \]
\[ + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| + 2|1 - \lambda||a_k| \]
\[ = M. \]

Thus \( |F(z)| \leq M \) for \( |z| = 1 \). Also for \( |F(0)| = |a_0| \neq 0 \).

Now it is known (see, [10, pp. 171]) that if \( f(z) \) is analytic, \( f(0) \neq \frac{1}{2} \) and \( |f(z)| \leq M \) in \( |z| \leq 1 \), then the number of zeros of \( f(z) \) in \( |z| \leq \frac{1}{2} \), does not exceed

\[ \frac{1}{\log 2} \log \frac{M}{|f(0)|}. \]

Thus, the number of zeros of \( F(z) \) in \( |z| \leq \frac{1}{2} \) does not exceed

\[ \frac{1}{\log 2} \log \frac{2\lambda |a_k| \cos \alpha + 2|\lambda - 1||a_k| \sin \alpha + |a_n|(\sin \alpha - \cos \alpha + 1) + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| + 2|1 - \lambda||a_k|}{|a_0|}. \]

As the number of zeros of \( P(z) \) in \( |z| \leq \frac{1}{2} \) is also equal to the number of zeros \( F(z) \) the theorem follows. \[ \square \]

**Proof of Theorem 2.** Consider the polynomial

\[ F(z) = (1 - z)P(z) \]
\[ = -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots + (a_k - a_{k-1})z^k + \ldots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0 \]

Let \( |z| > 1 \), then \( 0 \leq k \leq n - 1 \), we have

\[ |F(z)| \geq |a_n z^{n+1} - (a_n - a_{n-1})z^n| - |(a_{n-1} - a_{n-2})z^{n-1} + \ldots + (a_{k+1} - a_k)z^{k+1} \]
\[ + (a_k - a_{k-1})z^k + \ldots + (a_1 - a_0)z + a_0 | \]
\[
\begin{align*}
|a_n z^{n+1} - (a_n - a_{n-1}) z^n| & = |(a_n - a_{n-1}) z^n| + |(a_n - a_{n-2}) z^{n-1} + \ldots + (a_{k+1} - \lambda a_k + \lambda a_k - a_k) z^{k+1} + (a_k + \lambda a_k - a_k - \lambda a_k) z^k + \ldots + (a_1 - a_0) z + a_0| \\
& \geq |a_n z^{n+1} + a_{n-1} z^n - a_n z^n| - \left\{ |a_n - a_{n-2}| |z|^{n-1} + \ldots + |a_{k+1} - \lambda a_k| |z|^{k+1} + |\lambda - 1| |a_k||z|^{k+1} + |\lambda a_k - a_k - \lambda a_k||z|^k + \ldots + |a_1 - a_0| |z| + |a_0| \right\} \\
& \geq |z^n| |a_n z + a_{n-1} - a_n| - \left\{ |a_n - a_{n-2}| \frac{|z|^{n-1}}{|z|} + \ldots + |a_{k+1} - \lambda a_k| \frac{|z|^{k+1}}{|z|^{n-k}} + |\lambda - 1| |a_k| \frac{|z|^{k+1}}{|z|^{n-k}} + |\lambda a_k - a_k - \lambda a_k| \frac{|z|^k}{|z|^{n-k}} + \ldots + |a_1 - a_0| |z|^{n-1} + |a_0| \frac{|z|^n}{|z|^n} \right\}.
\end{align*}
\]

Using lemma and making use of the fact that \( \frac{1}{|z|} < 1, \ j = 1, 2, \ldots, n \), we have

\[
F(z) \geq |z^n| |a_n z + a_{n-1} - a_n| - \left\{ (|a_n - a_{n-2}| + |a_{n-2}| + |a_{n-1}|) \sin \alpha + \ldots + (|\lambda a_k - a_{k+1}| + |\lambda a_k + a_k + 1|) \cos \alpha + (|\lambda a_k| + |a_k - a_{k-1}|) \sin \alpha + (|\lambda a_k - a_k - 1|) \sin \alpha + \ldots + (|a_1 - a_0|) \cos \alpha + (|a_1 - a_0|) \sin \alpha \\
+ |a_0| + 2|1 - \lambda||a_k| \right\} \\
\geq |z^n| |a_n z + a_{n-1} - a_n| - \left\{ 2\lambda |a_k| \cos \alpha + 2|\lambda - 1||a_k| \sin \alpha - |a_{n-1}| (\sin \alpha + \cos \alpha) + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| - |a_0| (\sin \alpha + \cos \alpha - 1) + 2|1 - \lambda||a_k| \right\} > 0,
\]

if

\[
|a_n z + a_{n-1} - a_n| > 2\lambda |a_k| \cos \alpha + 2|\lambda - 1||a_k| \sin \alpha - |a_{n-1}| (\sin \alpha + \cos \alpha) + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| - |a_0| (\sin \alpha + \cos \alpha - 1) + 2|1 - \lambda||a_k|.
\]

Hence \( F(z) \) does not vanish in

\[
\left| z + \frac{a_{n-1}}{a_n} - 1 \right| < \frac{1}{a_n} \left\{ 2\lambda |a_k| \cos \alpha + 2|\lambda - 1||a_k| \sin \alpha - |a_{n-1}| (\sin \alpha + \cos \alpha) + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| - |a_0| (\sin \alpha + \cos \alpha - 1) + 2|1 - \lambda||a_k| \right\}.
\]

Therefore those zeros of \( F(z) \) whose modulus is greater than one lie in

\[
\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{a_n} \left\{ 2\lambda |a_k| \cos \alpha + 2|\lambda - 1||a_k| \sin \alpha - |a_{n-1}| (\sin \alpha + \cos \alpha) + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| - |a_0| (\sin \alpha + \cos \alpha - 1) + 2|1 - \lambda||a_k| \right\}.
\]
But those zeros of $F(z)$ whose modulus is less than or equal to one already (1). Hence we conclude that all the zeros of $F(z)$ and those of $p(z)$ lie (1). This completes the proof. □

**Proof of Theorem 3.** Consider the function

$$F(z) = (z - 1)f(z) = (z - 1)(a_o + a_1z + a_2z^2 + a_3z^3 + ...$$

$$= -a_o + \sum_{j=1}^{\infty} (a_{j-1} - a_j)z^j.$$  

Hence for $|z| = 1$

$$|F(z)| \leq |a_0| + \sum_{j=1}^{\infty} |a_{j-1} - a_j|$$

$$\leq (|a_0| + |\beta_0|) + \sum_{j=1}^{k} |\alpha_j - \alpha_{j-1}| + \sum_{j=k+1}^{\infty} |\alpha_{j-1} - \alpha_j| + \sum_{j=1}^{\infty} |\beta_{j-1} - \beta_j|$$

$$= (|a_0| + |\beta_0|) + \sum_{j=1}^{k} |\alpha_j - \alpha_{j-1}| + \sum_{j=k+1}^{\infty} |\alpha_{j-1} - \alpha_j| + \sum_{j=1}^{\infty} |\beta_{j-1} - \beta_j|$$

$$\leq |a_0| + |\beta_0| + |\alpha_1 - \alpha_0| + |\alpha_2 - \alpha_1| + ... + |\alpha_{k-1} - \alpha_k| + |\alpha_k - \alpha_{k-1}|$$

$$+ |\alpha_k - \alpha_{k+1}| + |\alpha_{k+1} - \alpha_{k+2}| + |\alpha_{k+2} - \alpha_{k+3}| + ... + \sum_{j=1}^{\infty} (|\beta_{j-1}| + |\beta_j|)$$

$$\leq a_0 + |\beta_0| + \alpha_1 - \alpha_0 + \alpha_2 - \alpha_1 + ... + \alpha_{k-1} - \alpha_k + |\alpha_k - \alpha_{k-1}|$$

$$+ |\alpha_k - \alpha_{k+1}| + |\alpha_{k+1} - \alpha_{k+2}| + |\alpha_{k+2} - \alpha_{k+3}| + ... + \sum_{j=1}^{\infty} (|\beta_{j-1}| + |\beta_j|)$$

$$= \alpha_{k-1} + |\alpha_k - \alpha_{k-1}| + |\alpha_k - \alpha_{k+1}| + \alpha_{k+1} + 2 \sum_{j=0}^{\infty} |\beta_j|$$

$$= \alpha_{k-1} + |\alpha_k - \lambda \alpha_{k-1}| + |\alpha_k + \lambda \alpha_{k-1}| + \alpha_{k+1} + 2 \sum_{j=0}^{\infty} |\beta_j|$$

$$= \alpha_{k-1} + |\lambda \alpha_k - \alpha_{k-1}| + (\lambda - 1)|\alpha_k| + |\lambda \alpha_k - \alpha_{k+1}| + (\lambda - 1)|\alpha_k| + \alpha_{k+1} + 2 \sum_{j=0}^{\infty} |\beta_j|$$

$$= \alpha_{k-1} + |\lambda \alpha_k - \alpha_{k-1}| + (\lambda - 1)|\alpha_k| + \lambda \alpha_k - \alpha_{k+1} + (\lambda - 1)|\alpha_k| + \alpha_{k+1} + 2 \sum_{j=0}^{\infty} |\beta_j|$$

$$= 2\lambda \alpha_k + 2(\lambda - 1)|\alpha_k| + 2 \sum_{j=0}^{\infty} |\beta_j|$$

which implies that

$$\frac{|F(z)|}{|F(0)|} \leq \frac{2\lambda \alpha_k + 2(\lambda - 1)|\alpha_k| + 2 \sum_{j=0}^{\infty} |\beta_j|}{|a_0|}.$$
Now it is known (see, [10, pp. 171]) that if $g(z)$ is analytic, $g(0) \neq \frac{1}{2}$ and $|g(z)| \leq M$ in $|z| \leq 1$, then the number of zeros of $g(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|f(0)|}.$$ 

Thus, the number of zeros of $F(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{2\lambda \alpha_k + 2(\lambda - 1)|\alpha_k| + 2 \sum_{j=0}^{\infty} |\beta_j|}{|a_0|},$$

or equivalently

$$1 + \frac{1}{\log 2} \left\{ \frac{\lambda \alpha_k + (\lambda - 1)|\alpha_k| + \sum_{j=0}^{\infty} |\beta_j|}{|a_0|} \right\}.$$ 

As the number of zeros of $f(z)$ in $|z| \leq \frac{1}{2}$ is also equal to the number of zeros $F(z)$, the theorem follows. $\Box$

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