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A MIXED PARSEVAL–PLANCHEREL FORMULA

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Abstract. In this note, a general formula is proved. It expresses the integral on the line of the product of a function f and a periodic function g in terms of the Fourier transform of f and the Fourier coefficients of g. This allows the evaluation of some oscillatory integrals.

1. Introduction and notation

In [6] the following integral was described as "difficult":

$$\int_{-\infty}^{\infty} \frac{dx}{(\cosh a + \cos x)\cosh x} \quad \text{for } a > 0, \tag{1}$$

it was used to test the trapezoidal rule after transforming the integral using a "sinh" transformation. Also, in [5] S. Tsipelis proposed to evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{\log(\cos^2 x)}{1 + e^{2|x|}} dx. \tag{2}$$

Both integrals are of the form $\int_{\mathbb{R}} f(x)g(x)dx$ where g is a 2π-periodic function. The particular case, where f is of the form $x \mapsto 1/(x+z)$, (for some $z \in \mathbb{C} \setminus \mathbb{R}$,) was thoroughly investigated in [3] using methods that are different from those discussed here.

In this note, we prove a general formula, that allows us to express this kind of integrals in terms of the Fourier transform of f and the Fourier coefficients of g.

Before we proceed, let us recall some standard notation. The spaces $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, and $L^{2,\mathrm{loc}}(\mathbb{R})$ are, respectively, the space of integrable functions, the space of square integrable functions, and the space of locally square integrable functions on $\mathbb{R}$. The spaces $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ are equipped with the standard norms denoted $\|\cdot\|_1$ and $\|\cdot\|_2$:

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(t)|^p \, dt\right)^{1/p}, \quad \text{for } p = 1, 2.$$

We consider also $L^1(\mathbb{T})$, (resp. $L^2(\mathbb{T})$), the space of integrable, (resp. square integrable), 2π-periodic functions. The spaces $L^1(\mathbb{T})$ and $L^2(\mathbb{T})$ are equipped with the standard norms denoted $\|\cdot\|_{L^1(\mathbb{T})}$ and $\|\cdot\|_{L^2(\mathbb{T})}$ and defined as follows:

$$\|f\|_{L^p(\mathbb{T})} = \left(\frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|^p \, dt\right)^{1/p}, \quad \text{for } p = 1, 2.$$

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For a function $f \in L^1(\mathbb{R})$ we recall that its Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-i\omega t}\, dt, \quad \text{for } \omega \in \mathbb{R}.$$

And for a 2π-periodic function $g \in L^1(\mathbb{T})$ we recall that the exponential Fourier coefficient $C_n(g)$ of g is defined by

$$C_n(g) = \frac{1}{2\pi} \int_{\mathbb{T}} g(t) e^{-int}\, dt, \quad \text{for } n \in \mathbb{Z},$$

In section 2 we will prove our main results and in section 3 we will give some detailed examples and applications.

2. The main result

In this section we state and prove the main theorem.

THEOREM 1. (The mixed Parseval-Plancherel formula) *Consider a function f from $L^{2,\mathrm{loc}}(\mathbb{R})$, and a 2π-periodic function g from $L^2(\mathbb{T})$. Suppose that*

$$M(f) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} \left\| \mathbb{1}_{I_k} f \right\|_2 < +\infty, \tag{3}$$

where $\mathbb{1}_{I_k}$ is the characteristic function of the interval $I_k = [2\pi k, 2\pi(k+1)]$. Then

$$\int_{\mathbb{R}} f(x)\overline{g(x)}\, dx = \sum_{n \in \mathbb{Z}} \widehat{f}(n)\overline{C_n(g)}. \tag{4}$$

where $\widehat{f}$ is the Fourier transform of f, and $(C_n(g))_{n \in \mathbb{Z}}$ is the family of exponential Fourier coefficients of g.

Proof. First, note that $\left\| \mathbb{1}_{I_k} f \right\|_1 \leqslant \sqrt{2\pi} \left\| \mathbb{1}_{I_k} f \right\|_2$ for every $k \in \mathbb{Z}$. It follows that

$$\int_{\mathbb{R}} |f(x)|\, dx = \sum_{k \in \mathbb{Z}} \left\| \mathbb{1}_{I_k} f \right\|_1 \leqslant \sqrt{2\pi} M(f) < +\infty.$$

Thus, f belongs to $L^1(\mathbb{R})$, and we can consider its Fourier transform. Similarly,

$$\int_{\mathbb{R}} |f(x)g(x)|\, dx = \sum_{k \in \mathbb{Z}} \int_{I_k} |f(x)g(x)|\, dx \leqslant \sqrt{2\pi} M(f) \|g\|_{L^2(T)} < +\infty,$$

thus fg belongs also to $L^1(\mathbb{R})$.

Now, let us consider the the family $(f_k)_{k \in \mathbb{Z}}$ defined by $f_k(x) = f(x + 2\pi k)$. Clearly $\| \mathbb{1}_m f_k \|_2 = \| \mathbb{1}_{m+k} f \|_2$. Thus

$$\sum_{k \in \mathbb{Z}} \| \mathbb{1}_m f_k \|_2 = M(f) < +\infty$$

and the series $\sum_{k\in\mathbb{Z}} \mathbb{I}_m f_k$ is normally convergent in $L^2(\mathbb{R})$ for every $m \in \mathbb{Z}$. This proves that the formula $F = \sum_{k\in\mathbb{Z}} f_k$ defines a function F that belongs to $L^{2,\mathrm{loc}}(\mathbb{R})$. Moreover, this function is clearly 2π-periodic, and $\|F\|_{L^2(\mathbb{T})} \leqslant \frac{1}{\sqrt{2\pi}} M(f)$. Now, the classical Parseval's formula, (see [2, Chap. I, §5.] or [4, Chap. 5, §3.],) implies that

$$\frac{1}{2\pi}\int_{\mathbb{T}} F(x)\overline{g(x)}\,dx = \sum_{n\in\mathbb{Z}} C_n(F)\overline{C_n(g)}. \tag{5}$$

Using the fact that $\sum_{k=-n}^{n-1} \mathbb{I}_{I_0} f_k$ converges to $\mathbb{I}_{I_0}F$ in $L^2(\mathbb{R})$, and that $\mathbb{I}_{I_0} g \in L^2(\mathbb{R})$, we conclude that

$$\begin{aligned}\int_0^{2\pi} F(x)\overline{g(x)}\,dx &= \lim_{n\to\infty}\sum_{k=-n}^{n-1}\int_0^{2\pi} f_k(x)\overline{g(x)}\,dx\\ &= \lim_{n\to\infty}\sum_{k=-n}^{n-1}\int_{2\pi k}^{2\pi(k+1)} f(x)\overline{g(x)}\,dx\\ &= \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx \end{aligned} \tag{6}$$

where, for the last equality, we used the fact that $fg \in L^1(\mathbb{R})$.

Similarly,

$$\begin{aligned}2\pi C_n(F) = \int_0^{2\pi} F(x)e^{-int}\,dx &= \lim_{n\to\infty}\sum_{k=-n}^{n-1}\int_0^{2\pi} f_k(x)e^{-int}\,dx\\ &= \lim_{n\to\infty}\sum_{k=-n}^{n-1}\int_{2\pi k}^{2\pi(k+1)} f(x)e^{-int}\,dx\\ &= \int_{\mathbb{R}} f(x)e^{-int}\,dx = \widehat{f}(n) \end{aligned} \tag{7}$$

where we used again the fact that $f \in L^1(\mathbb{R})$ for the last equality. Replacing (6) and (7) in (5), the desired formula follows. □

The next corollary is straightforward.

COROLLARY 1. *Consider a function f from $L^{2,\mathrm{loc}}(\mathbb{R})$, and a T-periodic, square integrable function g. Suppose that*

$$M_T(f) \overset{\mathrm{def}}{=} \sum_{k\in\mathbb{Z}} \left\| \mathbb{I}_{[kT,(k+1)T]} f \right\|_2 < +\infty, \tag{8}$$

Then

$$\int_{\mathbb{R}} f(x)\overline{g(x)}\,dx = \sum_{n\in\mathbb{Z}} \widehat{f}\left(\frac{2\pi n}{T}\right)\overline{C_n(g)}. \tag{9}$$

where $\widehat{f}$ is the Fourier transform of f, and $(C_n(g))_{n\in\mathbb{Z}}$ is the family of exponential Fourier coefficients of g.

3. Examples

EXAMPLE 1. For positive real numbers a and b, let g and f be the functions defined by

$$g(x)=\frac{1}{\cosh a+\cos x},\qquad f(x)=\frac{1}{\cosh(bx)},$$

It is known [1, Chap.I, §9] that $\widehat{f}(\omega)=\frac{\pi}{b}f\left(\frac{\pi}{2b}\omega\right)$. Moreover, it is easy to note that for every $k\in\mathbb{Z}$ we have $\left\|\mathbb{1}_{I_k}f\right\|_2\leqslant Be^{-2\pi b|k|}$ for some absolute constant B.

Furthermore, it is easy to check that

$$g(x)=\frac{1}{\sinh a}\sum_{n\in\mathbb{Z}}(-1)^n e^{-|n|a}e^{inx},$$

that is

$$C_n(g)=\frac{(-1)^n e^{-|n|a}}{\sinh a},\quad \text{for } n\in\mathbb{Z}.$$

Applying Theorem 1, we obtain

$$\int_{-\infty}^{\infty}\frac{dx}{(\cosh a+\cos x)\cosh(bx)}=\frac{\pi}{b\sinh a}+\frac{2\pi}{b\sinh a}\sum_{n=1}^{\infty}\frac{(-1)^n e^{-na}}{\cosh(\pi n/(2b))}$$

In particular, for $b=1$, we obtain the following expression of the integral (1) as a rapidly convergent series:

$$\int_{-\infty}^{\infty}\frac{dx}{(\cosh a+\cos x)\cosh x}=\frac{\pi}{\sinh a}+\frac{2\pi}{\sinh a}\sum_{n=1}^{\infty}\frac{(-1)^n e^{-na}}{\cosh(\pi n/2)}.$$

This is a simpler alternative series expansion to the one given in [6].

EXAMPLE 2. In our second example, let g and f be the functions defined by

$$g(x)=\log(\cos^2 x),\qquad f(x)=\frac{1}{1+e^{2|x|}}.$$

It is easy to note that for every $k\in\mathbb{Z}$ we have $\left\|\mathbb{1}_{I_k}f\right\|_2\leqslant Be^{-2\pi|k|}$ for some constant B. Moreover,

$$\begin{aligned}\widehat{f}(\omega)&=2\int_0^{\infty}\frac{e^{-2x}}{1+e^{-2x}}\cos(\omega x)\,dx\\&=2\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^{\infty}e^{-2kx}\cos(\omega x)\,dx\\&=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{4k}{4k^2+\omega^2}.\end{aligned}$$

Further, since

$$g(x)=2\log\left|1+e^{2ix}\right|-2\log 2=2\Re\,\mathrm{Log}(1+e^{2ix})-2\log 2$$

with Log being the principal branch of the logarithm, we conclude that for every $n \in \mathbb{Z}$ we have

$$C_{2n+1}(g) = 0, \quad \text{and} \quad C_{2n}(g) = \begin{cases} (-1)^{n-1}/|n| & \text{if } n \neq 0, \\ -2\log 2 & \text{if } n = 0. \end{cases}$$

Using Theorem 1, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\log(\cos^2 x)}{1+e^{2|x|}}\,dx &= \sum_{n\in\mathbb{Z}} \widehat{f}(2n)\overline{C_{2n}(g)} \\ &= -2\log 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} + 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{k+n}\frac{k}{(k^2+n^2)n}\right) \\ &= -2\log^2 2 + 2J \end{aligned} \tag{10}$$

with

$$J = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{k+n}\frac{k}{(k^2+n^2)n}\right) \tag{11}$$

Now, this double series is not absolutely convergent, so we must be careful. First, exchanging the roles of k and n we have

$$J = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{k+n}n}{p(n^2+k^2)}\right)$$

Now, using the properties of convergent alternating series we have

$$\sum_{n=1}^{\infty}\frac{(-1)^{k+n}n}{k(n^2+k^2)} = \sum_{n=1}^{q-1}\frac{(-1)^{k+n}n}{k(n^2+k^2)} + R_q(k),$$

with

$$R_q(k) = \frac{(-1)^k}{k}\sum_{n=q}^{\infty}\frac{(-1)^n n}{n^2+k^2} \quad \text{and} \quad \left|R_q(k)\right| \leqslant \frac{1}{k}\cdot\frac{q}{k^2+q^2}$$

Thus

$$J = \sum_{n=1}^{q-1}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+n}n}{k(n^2+k^2)}\right) + \varepsilon_q \tag{12}$$

with $\varepsilon_q = \sum_{k=1}^{\infty} R_q(k)$ and

$$\varepsilon_q \leqslant \sum_{k=1}^{\infty}\frac{q}{k(k^2+q^2)}.$$

Now, since

- $\dfrac{q}{k(k^2+q^2)} \leqslant \dfrac{1}{2k^2}$ for every q,
- the series $\sum_{k=1}^{\infty}\dfrac{1}{2k^2}$ is convergent,

- and $\lim_{q\to\infty} \frac{q}{k(k^2+q^2)} = 0$ for every k,

we conclude that $\lim_{q\to\infty} \varepsilon_q = 0$. So, letting q tend to $+\infty$ in (12) we get

$$J = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+n} n}{k(n^2+k^2)} \right) \tag{13}$$

Taking the sum of the two expressions (11) and (13) of J we obtain

$$2J = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+n}}{n^2+k^2} \left(\frac{n}{k} + \frac{k}{n} \right) \right) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+n}}{nk} \right) = (-\log 2)^2 = \log^2 2.$$

Replacing back in (10) we obtain

$$\int_{-\infty}^{\infty} \frac{\log(\cos^2 x)}{1+e^{2|x|}} \, dx = -\log^2 2.$$

EXAMPLE 3. For positive real numbers a and b, let g and f be the functions defined by

$$g(x) = \frac{1}{\cosh a - \cos x}, \qquad f(x) = e^{-x^2/(4b)},$$

It is known [1, Chap.I, §4] that $\widehat{f}(\omega) = 2\sqrt{\pi b} f(2b\omega)$. Moreover,

$$C_n(g) = \frac{e^{-|n|a}}{\sinh a}, \quad \text{for } n \in \mathbb{Z}.$$

Hence

$$\int_{\mathbb{R}} \frac{e^{-x^2/(4b)}}{\cosh a - \cos x} \, dx = \frac{2\sqrt{\pi b}}{\sinh a} \left(1 + 2 \sum_{n=1}^{\infty} e^{-an-bn^2} \right)$$

In particular, for $b = a$ we get

$$\int_{\mathbb{R}} \frac{e^{-x^2/(4a)}}{\cosh a - \cos x} \, dx = \frac{2\sqrt{\pi a}}{\sinh a} \left(1 + 2 \sum_{n=1}^{\infty} e^{-an(n+1)} \right) = \frac{2\sqrt{\pi a}}{\sinh a} (e^{a/4} \vartheta_2(0, e^{-a}) - 1),$$

where $\vartheta_2(u,q)$ is one of the well-known Jacobi Theta functions [7, Chap. XXI].

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