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ON A FAMILY OF MULTIVARIABLE POLYNOMIALS DEFINED THROUGH RODRIGUES TYPE FORMULA

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Abstract. In this paper, we present a family of multivariable polynomials defined by Rodrigues formula and we discuss their some miscellaneous properties such as generating function and recurrence relation. We also derive various classes of multilateral generating functions for these multivariable polynomials and give some special cases of the results. Furthermore, we also show that some particular cases of the polynomials reduce to the products of Hermite and Laguerre orthogonal polynomials with one variable.

1. Introduction

The concept of generating function is one of the most surprising and useful inventions in Mathematics and other disciplines. It is a powerful tool for recurrence relations which come up a lot in electrical engineering, circuit analysis, statistics, mathematics, physics, geology and any other disciplines that use differential equations. Recently, several families of multivariable polynomials defined via generating functions have been introduced and their some properties have been studied in [2, 4, 5, 6, 9, 11, 14]. This paper deals with finding a generating function for a family of multivariable polynomials defined by Rodrigues formula and obtaining some recurrence relations for these polynomials.

We recall that the classical Hermite polynomials $H_n(x)$ of degree n are defined by the Rodrigues formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2}\right) \tag{1.1}$$

and the familiar orthogonality property is as follows

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n n! \sqrt{\pi}\delta_{m,n} \tag{1.2}$$

$$(m,n \in \mathbb{N}_0 := \mathbb{N} \uplus \{0\})$$

where $\delta_{m,n}$ being the Kronecker delta. Laguerre polynomials $L_n(x)$ of degree n are defined by the Rodrigues formula

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} \left(e^{-x} x^n\right), \tag{1.3}$$

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and these polynomials hold the following orthogonality relation

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\, dx = \delta_{m,n}; \quad m,n \in \mathbb{N}_0. \tag{1.4}$$

As a generalization of Rodrigues formulas given by (1.1) and (1.3), in [1], the authors defined a family of polynomials defined through Rodrigues formula:

$$\phi_{k+n(m-1)}(x) = e^{\varphi_m(x)} \frac{d^n}{dx^n}\left(\psi_k(x)\, e^{-\varphi_m(x)}\right) \tag{1.5}$$

where $\phi_{k+n(m-1)}(x)$ is a polynomial of degree $k+n(m-1)$, $n=0,1,2,...$ and, $\psi_k(x)$ and $\varphi_m(x)$ are polynomials respectively of degree k and m; k, $m = 0,1,2,\ldots$ In that work, for these polynomials whose special cases reduce to Hermite polynomials, some recurrence relations and a generating function were given. In [2], as a generalization of polynomials (1.5), a family of polynomials in two variables was presented and some properties of them were obtained.

Motivated essentially by these works in [1, 2], we first consider the following Rodrigues formula in order to develop a general class of polynomials with r-variables

$$\begin{aligned}\phi_{\mathbf{n}}(x_1,\ldots,x_r) &:= \phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r) \\ &= e^{\varphi_m(x_1,\ldots,x_r)} \frac{\partial^{|\mathbf{n}|}}{\partial x_1^{n_1}\ldots\partial x_r^{n_r}}\left\{\psi_k(x_1,\ldots,x_r)\, e^{-\varphi_m(x_1,\ldots,x_r)}\right\}\end{aligned} \tag{1.6}$$

$$(n_i, k, m = 0,1,2,\ldots)$$

where $\psi_k(x_1,\ldots,x_r)$ and $\varphi_m(x_1,\ldots,x_r)$ are polynomials of total degree k and m with respect to the variables $x_1,\ldots,x_r$, respectively. $\phi_{\mathbf{n}}(x_1,\ldots,x_r)$ is a polynomial of total degree $N=(m-1)|\mathbf{n}|+k$ with respect to the variables $x_1,\ldots,x_r$ and $|\mathbf{n}| = n_1+\ldots+n_r$. We then give a generating function for these polynomials by applying Cauchy's integral formula and derive various families of bilateral generating functions for them. Under some special cases, we give some recurrence relations satisfied by $\phi_{\mathbf{n}}(x_1,\ldots,x_r)$. We also show that some particular cases of these polynomials reduce to known multivariable polynomials which are products of Hermite or Laguerre orthogonal polynomials with one variable.

2. A family of generating functions for the multivariable polynomials $\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)$

With the help of the Cauchy's integral formula, we give a family of generating function for the polynomials $\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)$ as follows:

THEOREM 2.1. *Let the polynomials $\psi_k(x_1,\ldots,x_r)$ and $\varphi_m(x_1,\ldots,x_r)$ be independent of $n_1,n_2,\ldots,n_r$. A generating function for the polynomials $\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)$ is* *given by*

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$
$$= \psi_k(x_1+t_1,\ldots,x_r+t_r)\, e^{\varphi_m(x_1,\ldots,x_r)-\varphi_m(x_1+t_1,\ldots,x_r+t_r)}. \tag{2.1}$$

Proof. We begin by considering the series

$$A_{n_1,\ldots,n_r}(x_1,\ldots,x_r,t_1) := \sum_{n_1=0}^{\infty} \phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\frac{t_1^{n_1}}{n_1!}. \tag{2.2}$$

From Cauchy's integral formula we have

$$\frac{\partial^{n_1}}{\partial x_1^{n_1}}\left\{\psi_k(x_1,\ldots,x_r)\, e^{-\varphi_m(x_1,\ldots,x_r)}\right\} = \frac{n_1!}{2\pi i}\oint_{C_1^*}\frac{\psi_k(z_1,x_2,\ldots,x_r)\, e^{-\varphi_m(z_1,x_2,\ldots,x_r)}}{(z_1-x_1)^{n_1+1}}dz_1$$

where the closed contour C_1^* in the complex z-plane is a circle (centered at $z_1 = x_1$) of sufficiently small radius, which is described in the positive direction (counter-clockwise), so that we find from (1.6) and (2.2) that

$$\begin{aligned}
&A_{n_1,\ldots,n_r}(x_1,\ldots,x_r,t_1)\\
&= e^{\varphi_m(x_1,\ldots,x_r)}\sum_{n_1=0}^{\infty}\frac{\partial^{n_2+\ldots+n_r}}{\partial x_2^{n_2}\ldots\partial x_r^{n_r}}\left\{\frac{1}{2\pi i}\oint_{C_1^*}\frac{\psi_k(z_1,x_2,\ldots,x_r)\, e^{-\varphi_m(z_1,x_2,\ldots,x_r)}}{(z_1-x_1)^{n_1+1}}dz_1\right\}t_1^{n_1}\\
&= \frac{e^{\varphi_m(x_1,\ldots,x_r)}}{2\pi i}\frac{\partial^{n_2+\ldots+n_r}}{\partial x_2^{n_2}\ldots\partial x_r^{n_r}}\oint_{C_1^*}\frac{\psi_k(z_1,x_2,\ldots,x_r)\, e^{-\varphi_m(z_1,x_2,\ldots,x_r)}}{z_1-x_1}\sum_{n_1=0}^{\infty}\left(\frac{t_1}{z_1-x_1}\right)^{n_1}dz_1\\
&= \frac{e^{\varphi_m(x_1,\ldots,x_r)}}{2\pi i}\frac{\partial^{n_2+\ldots+n_r}}{\partial x_2^{n_2}\ldots\partial x_r^{n_r}}\oint_{C_1}\frac{\psi_k(z_1,x_2,\ldots,x_r)\, e^{-\varphi_m(z_1,x_2,\ldots,x_r)}}{z_1-(x_1+t_1)}dz_1, \quad \left|\frac{t_1}{z_1-x_1}\right|<1\\
&= e^{\varphi_m(x_1,\ldots,x_r)}\frac{\partial^{n_2+\ldots+n_r}}{\partial x_2^{n_2}\ldots\partial x_r^{n_r}}\left\{\psi_k(x_1+t_1,x_2,\ldots,x_r)\, e^{-\varphi_m(x_1+t_1,x_2,\ldots,x_r)}\right\}
\end{aligned}$$

where C_1 is a circle (centered at $z_1 = x_1+t_1$) with radius $\varepsilon > 0$ in the complex z-plane. Thus, we get

$$\sum_{n_1=0}^{\infty}\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\frac{t_1^{n_1}}{n_1!} \tag{2.3}$$
$$= e^{\varphi_m(x_1,\ldots,x_r)}\frac{\partial^{n_2+\ldots+n_r}}{\partial x_2^{n_2}\ldots\partial x_r^{n_r}}\left\{\psi_k(x_1+t_1,x_2,\ldots,x_r)\, e^{-\varphi_m(x_1+t_1,x_2,\ldots,x_r)}\right\}.$$

When multiplying both sides of (2.3) by $\frac{t_2^{n_2}}{n_2!}$ and then summing both sides, we obtain

$$\sum_{n_1,n_2=0}^{\infty} \phi_{n_1,\ldots,n_r}\left(x_1,\ldots,x_r\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$
$$=\sum_{n_2=0}^{\infty} e^{\varphi_m(x_1,\ldots,x_r)}\frac{\partial^{n_2+\ldots+n_r}}{\partial x_2^{n_2}\ldots\partial x_r^{n_r}}\left\{\psi_k\left(x_1+t_1,x_2,\ldots,x_r\right)e^{-\varphi_m(x_1+t_1,x_2,\ldots,x_r)}\right\}\frac{t_2^{n_2}}{n_2!}. \tag{2.4}$$

Applying the Cauchy's integral formula again to the right hand-side of (2.4), for suitable contour $C_2,$ we have

$$\sum_{n_1,n_2=0}^{\infty} \phi_{n_1,\ldots,n_r}\left(x_1,\ldots,x_r\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$$
$$=\frac{e^{\varphi_m(x_1,\ldots,x_r)}}{2\pi i}\frac{\partial^{n_3+\ldots+n_r}}{\partial x_3^{n_3}\ldots\partial x_r^{n_r}}\oint_{C_2}\frac{\psi_k\left(x_1+t_1,z_2,x_3,\ldots,x_r\right)e^{-\varphi_m\left(x_1+t_1,z_2,x_3,\ldots,x_r\right)}}{z_2-\left(x_2+t_2\right)}dz_2$$
$$=e^{\varphi_m(x_1,\ldots,x_r)}\frac{\partial^{n_3+\ldots+n_r}}{\partial x_3^{n_3}\ldots\partial x_r^{n_r}}\psi_k\left(x_1+t_1,x_2+t_2,x_3,\ldots,x_r\right)e^{-\varphi_m(x_1+t_1,x_2+t_2,x_3,\ldots,x_r)}$$

for $\left|\frac{t_2}{z_2-x_2}\right|<1$. When the above method is applied $(r-3)$ times repeatedly by means of Cauchy's integral formula, we obtain

$$\sum_{n_1,\ldots,n_{r-1}=0}^{\infty} \phi_{n_1,\ldots,n_r}\left(x_1,\ldots,x_r\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\cdots\frac{t_{r-1}^{n_{r-1}}}{n_{r-1}!} \tag{2.5}$$
$$=e^{\varphi_m(x_1,\ldots,x_r)}\frac{\partial^{n_r}}{\partial x_r^{n_r}}\left\{\psi_k\left(x_1+t_1,\ldots,x_{r-1}+t_{r-1},x_r\right)e^{-\varphi_m(x_1+t_1,\ldots,x_{r-1}+t_{r-1},x_r)}\right\}.$$

By multiplying both sides of (2.5) by $\frac{t_r^{n_r}}{n_r!}$ and then summing both sides, we have

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \phi_{n_1,\ldots,n_r}\left(x_1,\ldots,x_r\right)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\cdots\frac{t_r^{n_r}}{n_r!}$$
$$=e^{\varphi_m(x_1,\ldots,x_r)}\sum_{n_r=0}^{\infty}\frac{\partial^{n_r}}{\partial x_r^{n_r}}\left\{\psi_k\left(x_1+t_1,\ldots,x_{r-1}+t_{r-1},x_r\right)e^{-\varphi_m(x_1+t_1,\ldots,x_{r-1}+t_{r-1},x_r)}\right\}\frac{t_r^{n_r}}{n_r!},$$

in which considering Cauchy's integral formula in the right hand for suitable contour C_r (centered at $z_r=x_r+t_r$), we conclude that

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \phi_{n_1,\ldots,n_r}\left(x_1,\ldots,x_r\right)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$
$$=\frac{e^{\varphi_m(x_1,\ldots,x_r)}}{2\pi i}\oint_{C_r}\frac{\psi_k\left(x_1+t_1,\ldots,x_{r-1}+t_{r-1},z_r\right)e^{-\varphi_m(x_1+t_1,\ldots,x_{r-1}+t_{r-1},z_r)}}{z_r-\left(x_r+t_r\right)}dz_r,\quad\left|\frac{t_r}{z_r-x_r}\right|<1$$
$$=\psi_k\left(x_1+t_1,x_2+t_2,\ldots,x_r+t_r\right)e^{\varphi_m(x_1,\ldots,x_r)-\varphi_m(x_1+t_1,x_2+t_2,\ldots,x_r+t_r)}$$

which completes the proof. □

Now, we consider many general families of bilateral generating relations for the polynomials $\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)$ which are generated by (2.1) without using group-theoretic (or Lie algebraic) technique but, with the help of the similar method as given in [2, 3, 4, 5, 6, 8, 9, 10, 14].

We begin by stating the following theorem.

THEOREM 2.2. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,\ldots,y_s)$ of s complex variables $y_1,\ldots,y_s$ $(s\in\mathbb{N})$ and of complex order μ, suppose that*

$$\Lambda_{\mu,\nu}(y_1,\ldots,y_s;z):=\sum_{l=0}^{\infty}a_l\Omega_{\mu+\nu l}(y_1,\ldots,y_s)z^l \tag{2.6}$$

$$(a_l\neq 0,\quad \mu,\nu\in\mathbb{C}).$$

and

$$\begin{aligned}&\Theta_{\mathbf{n},p,\mu,\nu}(x_1,\ldots,x_r;y_1,\ldots,y_s;\varsigma)\\ &:=\sum_{l=0}^{[n_1/p]}\frac{a_l}{(n_1-pl)!}\phi_{n_1-pl,n_2,\ldots,n_r}(x_1,\ldots,x_r)\,\Omega_{\mu+\nu l}(y_1,\ldots,y_s)\varsigma^l\\ &(n_1,p\in\mathbb{N}).\end{aligned} \tag{2.7}$$

Then we have

$$\begin{aligned}&\sum_{n_1,\ldots n_r=0}^{\infty}\frac{1}{n_2!\ldots n_r!}\Theta_{\mathbf{n},p,\mu,\nu}\left(x_1,\ldots,x_r;y_1,\ldots,y_s;\frac{\eta}{t_1^p}\right)t_1^{n_1}t_2^{n_2}\ldots t_r^{n_r}\\ &=\Lambda_{\mu,\nu}(y_1,\ldots,y_s;\eta)\psi_k(x_1+t_1,\ldots,x_r+t_r)\,e^{\varphi_m(x_1,\ldots,x_r)-\varphi_m(x_1+t_1,\ldots,x_r+t_r)}\end{aligned} \tag{2.8}$$

provided that each member of (2.8) exists.

Proof. For the proof of Theorem 2.2, we find it to be convenient to denote the first member of the assertion (2.8) by S. Then, upon substituting for the polynomials

$$\Theta_{\mathbf{n},p,\mu,\nu}\left(x_1,\ldots,x_r;y_1,\ldots,y_s;\frac{\eta}{t_1^p}\right)$$

from the definition (2.7) into the left-hand side of (2.8), we obtain

$$\begin{aligned}S&=\sum_{n_1,\ldots n_r=0}^{\infty}\sum_{l=0}^{[n_1/p]}\frac{a_l}{(n_1-pl)!n_2!\ldots n_r!}\phi_{n_1-pl,n_2,\ldots,n_r}(x_1,\ldots,x_r)\\ &\quad\times\Omega_{\mu+\nu l}(y_1,\ldots,y_s)t_1^{n_1-pl}t_2^{n_2}\ldots t_r^{n_r}\eta^l\\ &=\sum_{l=0}^{\infty}a_l\Omega_{\mu+\nu l}(y_1,\ldots,y_s)\eta^l\sum_{n_1,\ldots n_r=0}^{\infty}\frac{1}{n_1!n_2!\ldots n_r!}\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)t_1^{n_1}t_2^{n_2}\ldots t_r^{n_r}\\ &=\Lambda_{\mu,\nu}(y_1,\ldots,y_s;\eta)\psi_k(x_1+t_1,\ldots,x_r+t_r)\,e^{\varphi_m(x_1,\ldots,x_r)-\varphi_m(x_1+t_1,\ldots,x_r+t_r)}\end{aligned}$$

which completes the proof of Theorem 2.2. □

3. Further Remarks and Observations

If we express the multivariable function

$$\Omega_{\mu+\nu l}(y_1,...,y_s) \quad (l \in \mathbb{N}_0, \ \ s \in \mathbb{N})$$

in terms of several relatively simpler functions of one and more variables, Theorem 2.2 can be applied to yield various families of bilateral generating functions. For instance, if we set

$$s = r \ \text{ and } \ \Omega_{\mu+\nu l}(y_1,...,y_r) = h_{\mu+\nu l}^{(\gamma_1,...,\gamma_r)}(y_1,...,y_r)$$

in Theorem 2.2, where the multivariable Lagrange-Hermite polynomials $h_n^{(\alpha_1,...,\alpha_r)}(x_1, ..., x_r)$ [4] are generated by

$$\prod_{j=1}^{r} \left\{ \left(1 - x_j t^j\right)^{-\alpha_j} \right\} = \sum_{n=0}^{\infty} h_n^{(\alpha_1,...,\alpha_r)}\left(x_1,...,x_r\right) t^n \tag{3.1}$$

where $|t| < \min\left\{ |x_1|^{-1}, |x_2|^{-1/2}, ..., |x_r|^{-1/r} \right\}$, then we obtain the following result which provides a class of bilateral generating functions for the Lagrange-Hermite multivariable polynomials and the polynomials $\phi_{n_1,....,n_r}\left(x_1,...,x_r\right)$ generated by (2.1).

COROLLARY 3.1. *If* $\Lambda_{\mu,\nu}(y_1,...,y_r;z) := \sum\limits_{l=0}^{\infty} a_l h_{\mu+\nu l}^{(\gamma_1,...,\gamma_r)}(y_1,...,y_r) z^l$ *where* $a_l \neq 0, \ \nu,\mu \in \mathbb{N}_0$*; and*

$$\begin{aligned} &\Theta_{\mathbf{n},p,\mu,\nu}\left(x_1,...,x_r;y_1,...,y_r;\varsigma\right) \\ &:= \sum_{l=0}^{[n_1/p]} \frac{a_l}{(n_1-pl)!} \phi_{n_1-pl,n_2,...,n_r}\left(x_1,...,x_r\right) h_{\mu+\nu l}^{(\gamma_1,...,\gamma_r)}(y_1,...,y_r) \varsigma^l \end{aligned}$$

where $n_1, p \in \mathbb{N}$*. Then we have*

$$\begin{aligned} &\sum_{n_1,...n_r=0}^{\infty} \frac{1}{n_2!...n_r!} \Theta_{\mathbf{n},p,\mu,\nu}\left(x_1,...,x_r;y_1,...,y_r;\frac{\eta}{t_1^p}\right) t_1^{n_1}...t_r^{n_r} \\ &= \Lambda_{\mu,\nu}(y_1,...,y_r;\eta)\, \psi_k\left(x_1+t_1,...,x_r+t_r\right) e^{\varphi_m(x_1,....,x_r)-\varphi_m(x_1+t_1,...,x_r+t_r)} \end{aligned} \tag{3.2}$$

provided that each member of (3.2) exists.

REMARK 3.2. Using the generating function (3.1) and taking $a_l = 1, \ \mu = 0, \ \nu = 1$, we have

$$\begin{aligned} &\sum_{n_1,...n_r=0}^{\infty} \sum_{l=0}^{[n_1/p]} \frac{1}{(n_1-pl)!n_2!...n_r!} \phi_{n_1-pl,n_2,...,n_r}\left(x_1,...,x_r\right) t_2^{n_2}...t_r^{n_r} \\ &\quad \times h_l^{(\gamma_1,...,\gamma_r)}(y_1,...,y_r) \eta^l t_1^{n_1-pl} \\ &= \psi_k\left(x_1+t_1,...,x_r+t_r\right) e^{\varphi_m(x_1,...,x_r)-\varphi_m(x_1+t_1,...,x_r+t_r)} \prod_{j=1}^{r} \left\{ \left(1-y_j \eta^j\right)^{-\gamma_j} \right\} \end{aligned}$$

where

$$\left(|\eta| < \min\left\{|y_1|^{-1}, |y_2|^{-1/2}, ..., |y_r|^{-1/r}\right\}\right).$$

Set

$$s = 1 \text{ and } \Omega_{\mu+\nu l}(y) = H_{\mu+\nu l}(y)$$

in Theorem 2.2, where the nth Hermite polynomial $H_n(x)$ is generated by

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \exp\left(2xt - t^2\right). \tag{3.3}$$

Then, we get the following result which provides a class of bilateral generating functions for Hermite polynomials and the polynomials $\phi_{n_1,...,n_r}(x_1,...,x_r)$.

COROLLARY 3.3. *Let* $\Lambda_{\mu,\nu}(y;z) := \sum_{l=0}^{\infty} a_l H_{\mu+\nu l}(y) z^l$ *where* $a_l \neq 0$, $\nu, \mu \in \mathbb{N}_0$; *and*

$$\begin{aligned}&\Theta_{\mathbf{n},p,\mu,\nu}(x_1,...,x_r;y;\varsigma)\\ &:= \sum_{l=0}^{[n_1/p]} \frac{a_l}{(n_1-pl)!} \phi_{n_1-pl,n_2,...,n_r}(x_1,...,x_r) H_{\mu+\nu l}(y) \varsigma^l\end{aligned}$$

where $n_1, p \in \mathbb{N}$. *Then we get*

$$\begin{aligned}&\sum_{n_1,...n_r=0}^{\infty} \frac{1}{n_2!...n_r!} \Theta_{\mathbf{n},p,\mu,\nu}\left(x_1,...,x_r;y;\frac{\eta}{t_1^p}\right) t_1^{n_1}...t_r^{n_r}\\ &= \Lambda_{\mu,\nu}(y;\eta)\psi_k(x_1+t_1,...,x_r+t_r) e^{\varphi_m(x_1,...,x_r)-\varphi_m(x_1+t_1,...,x_r+t_r)}\end{aligned} \tag{3.4}$$

provided that each member of (3.4) *exists.*

REMARK 3.4. If we take $a_l = \frac{1}{l!}$, $\mu = 0$, $\nu = 1$ and then use the generating relation (3.3) for Hermite polynomials, we have

$$\begin{aligned}&\sum_{n_1,...n_r=0}^{\infty} \sum_{l=0}^{[n_1/p]} \phi_{n_1-pl,n_2,...,n_r}(x_1,...,x_r) H_l(y) \frac{1}{l!(n_1-pl)!n_2!...n_r!} \eta^l t_1^{n_1-pl}...t_r^{n_r}\\ &= \exp\left(2y\eta - \eta^2\right)\psi_k(x_1+t_1,...,x_r+t_r) e^{\varphi_m(x_1,...,x_r)-\varphi_m(x_1+t_1,...,x_r+t_r)}.\end{aligned}$$

We conclude this section by remarking that for each suitable choice of the coefficients a_l $(l \in \mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\nu l}(y_1,...,y_s)$, $(s \in \mathbb{N})$, is expressed as an appropriate product of several simpler relatively functions, the assertion of Theorem 2.2 can be applied to yield many different families of multilateral generating functions for the polynomials $\phi_{n_1,...,n_r}$.

4. Some recurrence relations of the polynomials $\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)$

In this section, we give some recurrence relations satisfied by the polynomial set $\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)$ for some special choices of the polynomials $\psi_k(x_1,\ldots x_r)$.

Setting $\psi_k(x_1,\ldots x_r) = (a_1x_1 + a_2x_2 + \ldots + a_rx_r + a_{r+1})^k$ $(a_1,\ldots,a_{r+1} \in \mathbb{R},\ k = 0,1,\ldots)$ in (1.6), we get

$$\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r) = e^{\varphi_m(x_1,\ldots,x_r)} \frac{\partial^{|\mathbf{n}|}}{\partial x_1^{n_1}\ldots\partial x_r^{n_r}} \left\{ (a_1x_1 + \ldots + a_rx_r + a_{r+1})^k e^{-\varphi_m(x_1,\ldots,x_r)} \right\}. \tag{4.1}$$

which are generated by

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r) \frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!} \tag{4.2}$$
$$= e^{\varphi_m(x_1,\ldots,x_r) - \varphi_m(x_1+t_1,x_2+t_2,\ldots,x_r+t_r)} \left[a_1(x_1+t_1) + \ldots + a_r(x_r+t_r) + a_{r+1}\right]^k$$

from Theorem 2.1.

For brevity, we need the following notations: for $\mathbf{x} = (x_1,\ldots,x_r)$ and $\mathbf{n} = (n_1,\ldots,n_r)$, let denote

$$\frac{\partial^{|\mathbf{n}|}}{\partial x_1^{n_1}\ldots\partial x_r^{n_r}} f(x_1,\ldots,x_r) = \frac{\partial^{|\mathbf{n}|}}{\partial \mathbf{x}^{\mathbf{n}}} f(x_1,\ldots,x_r).$$

Let $e_1,\ldots,e_r$ be the standard basis of $\mathbb{R}^r$, that is, the ith coordinate of e_j is 1 if $i = j$, 0 if $i \neq j$.

Now, we can give the following results for these polynomials.

THEOREM 4.1. *Let*

$$\Omega^{(l_1,\ldots,l_r)}(x_1,\ldots,x_r;n_1,\ldots,n_r) = (a_1x_1 + \ldots + a_rx_r + a_{r+1})\,\phi_{\mathbf{n}-l_1e_1-\ldots-l_re_r}(x_1,\ldots,x_r)$$
$$+ \sum_{j=1}^{r} a_j(n_j - l_j)\,\phi_{\mathbf{n}-l_1e_1-\ldots-l_re_r-e_j}(x_1,\ldots,x_r)$$
$$(l_1,l_2,\ldots,l_r \in \mathbb{N}_0).$$

Then for the polynomials $\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)$, we have the following recurrence relations for $1 \leqslant i \leqslant r$

$$-\sum_{p=0}^{m-1} \sum_{l_1+\ldots+l_r=p} \binom{n_1}{l_1}\cdots\binom{n_r}{l_r} \Omega^{(l_1,\ldots,l_r)}(x_1,\ldots,x_r;n_1,\ldots,n_r) \frac{\partial^{p+1}}{\partial \mathbf{x}^{\mathbf{l}+e_i}} \varphi_m(x_1,\ldots,x_r)$$
$$= (a_1x_1 + \ldots + a_rx_r + a_{r+1})\,\phi_{\mathbf{n}+e_i}(x_1,\ldots,x_r)$$
$$+ \sum_{j=1}^{r} a_jn_j\phi_{\mathbf{n}+e_i-e_j}(x_1,\ldots,x_r) - a_ik\phi_{\mathbf{n}}(x_1,\ldots,x_r)$$

where $\mathbf{l} = (l_1,l_2,\ldots,l_r)$ and

$$m \geqslant 1, \quad n_i \geqslant l_i + 1; \quad i = 1,2,\ldots,r.$$

Proof. Without loss generality, we can assume $i=1$. Differentiating each member of the generating function (4.2) with respect to t_1 and then using (4.2) again, we find that

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \phi_{n_1+1,n_2,\ldots,n_r}(x_1,\ldots,x_r)\frac{t_1^{n_1}t_2^{n_2}\ldots t_r^{n_r}}{n_1!\ldots n_r!} \tag{4.3}$$
$$=\frac{a_1k}{[a_1(x_1+t_1)+\ldots+a_r(x_r+t_r)+a_{r+1}]}\sum_{n_1,\ldots,n_r=0}^{\infty}\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$
$$-\frac{\partial}{\partial t_1}\varphi_m(x_1+t_1,\ldots,x_r+t_r)\sum_{n_1,\ldots,n_r=0}^{\infty}\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}.$$

Since the Taylor series of the polynomial $\frac{\partial}{\partial t_1}\varphi_m(x_1+t_1,\ldots,x_r+t_r)$ at $(t_1,\ldots,t_r)=(0,\ldots,0)$ is

$$\frac{\partial}{\partial t_1}\varphi_m(x_1+t_1,\ldots,x_r+t_r)$$
$$=\sum_{p=0}^{m-1}\sum_{l_1+\ldots+l_r=p}\frac{1}{l_1!\ldots l_r!}\frac{\partial^{p+1}}{\partial x_1^{l_1+1}\partial x_2^{l_2}\ldots\partial x_r^{l_r}}\varphi_m(x_1,\ldots,x_r)t_1^{l_1}\ldots t_r^{l_r},$$

the equality (4.3) can be written in the form

$$(a_1x_1+\ldots+a_rx_r+a_{r+1})\sum_{n_1,\ldots,n_r=0}^{\infty}\phi_{n_1+1,n_2,\ldots,n_r}(x_1,\ldots,x_r)\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!}$$
$$+(a_1t_1+\ldots+a_rt_r)\sum_{n_1,\ldots,n_r=0}^{\infty}\phi_{n_1+1,n_2,\ldots,n_r}(x_1,\ldots,x_r)\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!}$$
$$=a_1k\sum_{n_1,\ldots,n_r=0}^{\infty}\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$
$$-(a_1x_1+\ldots+a_rx_r+a_{r+1})\sum_{n_1,\ldots,n_r=0}^{\infty}\sum_{p=0}^{m-1}\sum_{l_1+\ldots+l_r=p}\left(\frac{\partial^{p+1}}{\partial x_1^{l_1+1}\partial x_2^{l_2}\ldots\partial x_r^{l_r}}\varphi_m(x_1,\ldots,x_r)\right)$$
$$\times\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\frac{t_1^{n_1+l_1}}{n_1!l_1!}\cdots\frac{t_r^{n_r+l_r}}{n_r!l_r!}$$
$$-(a_1t_1+\ldots+a_rt_r)\sum_{n_1,\ldots,n_r=0}^{\infty}\sum_{p=0}^{m-1}\sum_{l_1+\ldots+l_r=p}\left(\frac{\partial^{p+1}}{\partial x_1^{l_1+1}\partial x_2^{l_2}\ldots\partial x_r^{l_r}}\varphi_m(x_1,\ldots,x_r)\right)$$
$$\times\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\frac{t_1^{n_1+l_1}}{n_1!l_1!}\cdots\frac{t_r^{n_r+l_r}}{n_r!l_r!}.$$

If we make necessary calculations, we find the desired recurrence relation for $i=1$. Similarly, it can be easily obtained for $i=2,3,\ldots r$. □

Other recurrence relations for the polynomials $\phi_{n_1,...,n_r}(x_1,...,x_r)$ can be obtained by differentiating the generating function (4.2) with respect to x_i, $i=1,2,...,r$ as follows:

THEOREM 4.2. *Let* $\Omega^{(l_1,...,l_r)}(x_1,...,x_r;n_1,...,n_r)$ *be given as in Theorem* 4.1. *The polynomials* $\phi_{n_1,...,n_r}(x_1,...,x_r)$ *satisfy the following recurrence relations*

$$\begin{aligned}
&-\sum_{p=0}^{m-1}\sum_{l_1+...+l_r=p}\binom{n_1}{l_1}...\binom{n_r}{l_r}\Omega^{(l_1,...,l_r)}(x_1,...,x_r;n_1,...,n_r)\frac{\partial^{p+1}}{\partial\mathbf{x}^{\mathbf{l}+e_i}}\varphi_m(x_1,...,x_r)\\
&=(a_1x_1+...+a_rx_r+a_{r+1})\left\{\frac{\partial}{\partial x_i}\phi_{\mathbf{n}}(x_1,...,x_r)-\phi_{\mathbf{n}}(x_1,...,x_r)\frac{\partial}{\partial x_i}\varphi_m(x_1,...,x_r)\right\}\\
&+\sum_{j=1}^{r}a_jn_j\left\{\frac{\partial}{\partial x_i}\phi_{\mathbf{n}-e_j}(x_1,...,x_r)-\phi_{\mathbf{n}-e_j}(x_1,...,x_r)\frac{\partial}{\partial x_i}\varphi_m(x_1,...,x_r)\right\}\\
&-a_ik\phi_{\mathbf{n}}(x_1,...,x_r)
\end{aligned}$$

where

$$m\geqslant 1,\quad n_i\geqslant l_i+1;\quad i=1,2,...,r.$$

The next results can be easily obtained from Theorem 4.1 and Theorem 4.2.

THEOREM 4.3. *The polynomials* $\phi_{n_1,...,n_r}(x_1,...,x_r)$ *hold that*

$$\begin{aligned}
&(a_1x_1+...+a_rx_r+a_{r+1})\left\{\phi_{\mathbf{n}+e_i}(x_1,...,x_r)+\phi_{\mathbf{n}}(x_1,...,x_r)\frac{\partial}{\partial x_i}\varphi_m(x_1,...,x_r)\right\}\\
&=\sum_{j=1}^{r}a_jn_j\left\{\frac{\partial}{\partial x_i}\phi_{\mathbf{n}-e_j}(x_1,...,x_r)-\phi_{\mathbf{n}-e_j}(x_1,...,x_r)\frac{\partial}{\partial x_i}\varphi_m(x_1,...,x_r)-\phi_{\mathbf{n}+e_i-e_j}(x_1,...,x_r)\right\}\\
&+(a_1x_1+...+a_rx_r+a_{r+1})\frac{\partial}{\partial x_i}\phi_{\mathbf{n}}(x_1,...,x_r).
\end{aligned}$$

For the special case $\psi_k(x_1,...,x_r)=1$ in (1.6), we have the polynomials

$$\phi_{n_1,...,n_r}(x_1,...,x_r)=e^{\varphi_m(x_1,...,x_r)}\frac{\partial^{|\mathbf{n}|}}{\partial x_1^{n_1}...\partial x_r^{n_r}}\left\{e^{-\varphi_m(x_1,...,x_r)}\right\}. \tag{4.4}$$

which are of degree $N_1=(m-1)|\mathbf{n}|$; $|\mathbf{n}|=n_1+...+n_r$. As a result of Theorems 4.1–4.3, we have following:

COROLLARY 4.4. *For the polynomials* $\phi_{n_1,...,n_r}(x_1,...,x_r)$, *we have the following recurrence relations*

$$\begin{aligned}
\phi_{\mathbf{n}+e_i}(x_1,...,x_r)=&-\sum_{p=0}^{m-1}\sum_{l_1+...+l_r=p}\binom{n_1}{l_1}...\binom{n_r}{l_r}\phi_{\mathbf{n}-l_1e_1-...-l_re_r}(x_1,...,x_r)\\
&\times\frac{\partial^{p+1}}{\partial\mathbf{x}^{\mathbf{l}+e_i}}\varphi_m(x_1,...,x_r),
\end{aligned}$$

$$\frac{\partial}{\partial x_i}\phi_{\mathbf{n}}(x_1,\dots,x_r)-\phi_{\mathbf{n}}(x_1,\dots,x_r)\frac{\partial}{\partial x_i}\varphi_m(x_1,\dots,x_r)$$
$$=-\sum_{p=0}^{m-1}\sum_{l_1+\dots+l_r=p}\binom{n_1}{l_1}\dots\binom{n_r}{l_r}\phi_{\mathbf{n}-l_1e_1-\dots-l_re_r}(x_1,\dots,x_r)\frac{\partial^{p+1}}{\partial\mathbf{x}^{\mathbf{l}+e_i}}\varphi_m(x_1,\dots,x_r),$$
$$(m\geqslant 1,\quad n_i\geqslant l_i+1;\quad i=1,2,\dots,r)$$

and

$$\phi_{\mathbf{n}+e_i}(x_1,\dots,x_r)+\phi_{\mathbf{n}}(x_1,\dots,x_r)\frac{\partial}{\partial x_i}\varphi_m(x_1,\dots,x_r)=\frac{\partial}{\partial x_i}\phi_{\mathbf{n}}(x_1,\dots,x_r)$$

for $1\leqslant i\leqslant r$.

5. Some special cases of the polynomials $\phi_{n_1,\dots,n_r}(x_1,\dots,x_r)$

Under suitable choices, the family of polynomials $\phi_{n_1,\dots,n_r}(x_1,\dots,x_r)$ reduce to several known multivariable polynomials which are products of Hermite or Laguerre orthogonal polynomials with one variable. Now, let consider some special cases.

REMARK 5.1. By getting $\psi_k(x_1,\dots,x_r)=x_1\dots x_r$ and $\varphi_m(x_1,\dots,x_r)=x_1^2+\dots+x_r^2$, the family of polynomials given by (1.6) reduces to the following polynomials

$$\begin{aligned}\phi_{n_1,\dots,n_r}(x_1,\dots,x_r)&=e^{x_1^2+\dots+x_r^2}\frac{\partial^{|\mathbf{n}|}}{\partial x_1^{n_1}\dots\partial x_r^{n_r}}\left\{(x_1\dots x_r)e^{-x_1^2-\dots-x_r^2}\right\}\\&=\left(-\frac{1}{2}e^{x_1^2}\frac{d^{n_1+1}}{dx_1^{n_1+1}}e^{-x_1^2}\right)\dots\left(-\frac{1}{2}e^{x_r^2}\frac{d^{n_r+1}}{dx_r^{n_r+1}}e^{-x_r^2}\right)\\&=\frac{(-1)^{n_1+\dots+n_r}}{2^r}H_{n_1+1}(x_1)\dots H_{n_r+1}(x_r);\quad n_1,\dots,n_r=0,1,\dots,\end{aligned}$$

which are generated by

$$\sum_{n_1,\dots,n_r=0}^{\infty}\phi_{n_1,\dots,n_r}(x_1,\dots,x_r)\frac{t_1^{n_1}}{n_1!}\dots\frac{t_r^{n_r}}{n_r!}$$
$$=(x_1+t_1)\dots(x_r+t_r)e^{-\left(t_1^2+\dots+t_r^2+2x_1t_1+\dots+2x_rt_r\right)}$$

where $H_{n_1+1}(x_1),\dots,H_{n_r+1}(x_r)$ are Hermite polynomials of degree $n_1+1,\dots,n_r+1$, respectively. These polynomials satisfy the following orthogonality relation from (1.2) (see also [7])

$$\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}e^{-(x_1^2+\dots+x_r^2)}\phi_{n_1,\dots,n_r}(x_1,\dots,x_r)\phi_{k_1,\dots,k_r}(x_1,\dots,x_r)\,dx_1\dots dx_r$$
$$=2^{n_1+\dots+n_r-r}(n_1+1)!\dots(n_r+1)!\,\pi^{\frac{r}{2}}\delta_{n_1,k_1}\dots\delta_{n_r,k_r}$$

where $\delta_{n_1,k_1},\dots,\delta_{n_r,k_r}$ are Kronecker delta.

REMARK 5.2. If we take $\varphi_m(x_1,...,x_r) = x_1^2 + ... + x_r^2 + \alpha_1 x_1 + ... + \alpha_r x_r$ in (4.4), we have

$$\begin{aligned} &\phi_{n_1,...,n_r}(x_1,...,x_r) \\ &= e^{x_1^2+...+x_r^2+\alpha_1 x_1+...+\alpha_r x_r} \frac{\partial^{|\mathbf{n}|}}{\partial x_1^{n_1}...\partial x_r^{n_r}} e^{-x_1^2-...-x_r^2-\alpha_1 x_1-...-\alpha_r x_r} \\ &= \left(e^{x_1^2+\alpha_1 x_1} \frac{\partial^{n_1}}{\partial x_1^{n_1}} e^{-x_1^2-\alpha_1 x_1} \right) ... \left(e^{x_r^2+\alpha_r x_r} \frac{\partial^{n_r}}{\partial x_r^{n_r}} e^{-x_r^2-\alpha_r x_r} \right) \\ &= \left(e^{\left(x_1+\frac{\alpha_1}{2}\right)^2} \frac{\partial^{n_1}}{\partial x_1^{n_1}} e^{-\left(x_1+\frac{\alpha_1}{2}\right)^2} \right) ... \left(e^{\left(x_r+\frac{\alpha_r}{2}\right)^2} \frac{\partial^{n_r}}{\partial x_r^{n_r}} e^{-\left(x_r+\frac{\alpha_r}{2}\right)^2} \right) \\ &= (-1)^{n_1+...+n_r} H_{n_1}\left(x_1+\frac{\alpha_1}{2}\right) ... H_{n_r}\left(x_r+\frac{\alpha_r}{2}\right); \quad n_1,...,n_r = 0,1,... \end{aligned} \tag{5.1}$$

where $H_{n_1}\left(x_1+\frac{\alpha_1}{2}\right),...,H_{n_r}\left(x_r+\frac{\alpha_r}{2}\right)$ are Hermite polynomials of degree $n_1,...,n_r$, respectively.

From Theorem 2.1, these polynomials defined by (5.1) are generated by

$$\sum_{n_1,...,n_r=0}^{\infty} \phi_{n_1,...,n_r}(x_1,...,x_r) \frac{t_1^{n_1}}{n_1!} ... \frac{t_r^{n_r}}{n_r!} = e^{-\left(t_1^2+...+t_r^2+(2x_1+\alpha_1)t_1+...+(2x_r+\alpha_r)t_r\right)}$$

and they verify the following orthogonality relation

$$\begin{aligned} &\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} e^{-\left(x_1+\frac{\alpha_1}{2}\right)^2-...-\left(x_r+\frac{\alpha_r}{2}\right)^2} \phi_{n_1,...,n_r}(x_1,...,x_r)\, \phi_{k_1,...,k_r}(x_1,...,x_r)\, dx_1...dx_r \\ &= 2^{n_1+...+n_r}(n_1)!...(n_r)!\pi^{r/2}\delta_{n_1,k_1}...\delta_{n_r,k_r}. \end{aligned}$$

From Corollary 4.4, they satisfy the following recurrence relations for $1 \leqslant i \leqslant r$

$$\phi_{\mathbf{n}+e_i}(x_1,...,x_r) + (2x_i+\alpha_i)\,\phi_{\mathbf{n}}(x_1,...,x_r) + 2n_i\phi_{\mathbf{n}-e_i}(x_1,...,x_r) = 0,$$

$$\frac{\partial}{\partial x_i}\phi_{\mathbf{n}}(x_1,...,x_r) + 2n_i\phi_{\mathbf{n}-e_i}(x_1,...,x_r) = 0$$

and

$$\phi_{\mathbf{n}+e_i}(x_1,...,x_r) = \frac{\partial}{\partial x_i}\phi_{\mathbf{n}}(x_1,...,x_r) - (2x_i+\alpha_i)\;\phi_{\mathbf{n}}(x_1,...,x_r),$$

which give the results presented by Altın *et.al* in [3].

REMARK 5.3. The case of $\psi_k(x_1,...,x_r) = x_1^{n_1}...x_r^{n_r}$ and $\varphi_m(x_1,...,x_r) = x_1 + ... + x_r$ in (1.6) gives

$$\begin{aligned} \phi_{n_1,...,n_r}(x_1,...,x_r) &= e^{x_1+...+x_r} \frac{\partial^{|\mathbf{n}|}}{\partial x_1^{n_1}...\partial x_r^{n_r}} \left\{ x_1^{n_1}...x_r^{n_r} e^{-x_1-...-x_r} \right\} \\ &= \left(e^{x_1} \frac{\partial^{n_1}}{\partial x_1^{n_1}} x_1^{n_1} e^{-x_1} \right) ... \left(e^{x_r} \frac{\partial^{n_r}}{\partial x_r^{n_r}} x_r^{n_r} e^{-x_r} \right) \\ &= n_1!...n_r! L_{n_1}(x_1)...L_{n_r}(x_r); \quad n_1,...,n_r = 0,1,..., \end{aligned}$$

which verifies the following orthogonality relation from (1.4) (see also [7])

$$\int_0^\infty \ldots \int_0^\infty e^{-x_1-\ldots-x_r} \phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\, \phi_{k_1,\ldots,k_r}(x_1,\ldots,x_r)\, dx_1\ldots dx_r$$
$$= (n_1!)^2 \ldots (n_r!)^2\, \delta_{n_1,k_1}\ldots\delta_{n_r,k_r}$$

where $L_{n_1}(x_1),\ldots,L_{n_r}(x_r)$ are Laguerre polynomials of degree $n_1,\ldots,n_r,$ respectively.

REMARK 5.4. If we take $\psi_k(x_1,\ldots,x_r) = x_1^{n_1}\ldots x_r^{n_r}$ and $\varphi_m(x_1,\ldots,x_r) = p_1 x_1^{k_1} + \ldots + p_r x_r^{k_r}$ in (1.6), we have

$$\phi_{n_1,\ldots,n_r}(x_1,\ldots,x_r) = e^{p_1x_1^{k_1}+\ldots+p_rx_r^{k_r}} \frac{\partial^{|\mathbf{n}|}}{\partial x_1^{n_1}\ldots\partial x_r^{n_r}} \left\{ x_1^{n_1}\ldots x_r^{n_r} e^{-\left(p_1x_1^{k_1}+\ldots+p_rx_r^{k_r}\right)} \right\}$$
$$= L_{\mathbf{n}}^{(0,0,\ldots,0)}(\mathbf{x};\mathbf{p};\mathbf{k})$$

where the family of polynomials $L_{\mathbf{n}}^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x};\mathbf{p};\mathbf{k})$ is defined by (see [10])

$$L_{\mathbf{n}}^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x};\mathbf{p};\mathbf{k}) = \prod_{i=1}^{r} e^{p_ix_i^{k_i}} \frac{\partial^{n_i}}{\partial x_i^{n_i}} x_i^{n_i} e^{-p_ix_i^{k_i}}$$

where $\mathbf{p} = (p_1,\ldots,p_r)$; $\mathbf{k} = (k_1,\ldots,k_r)$ and $p_i, k_i \in \mathbb{N}$ for $i = 1,2,\ldots,r$.

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