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THE ORDER OF MAGNITUDE FOR HAUSDORFF AND NÖRLUND SUMMABILITY OF AN ORTHOGONAL SERIES

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Abstract. In this paper, we estimate the order of magnitude for Hausdorff and Nörlund summability of an orthogonal series.

1. Introduction

NOTATION. $(X,\mathfrak{M},\mu)$ = Measure space, $\mathbb{N}$ = Natural numbers, $\mathbb{Z}^+ = \mathbb{N}\cup\{0\}$, $\mathbb{R}$ = Real numbers, $\mathbb{C}$ = Complex numbers, $BV[0,1]$ = Set of all functions of bounded variation on $[0,1]$.

DEFINITION 1. Let $\{\phi_n\}_{n=0}^{\infty}$ be an orthonormal system in $L_2(X)$. We shall consider an orthogonal series

$$\sum_{i=0}^{\infty} c_i\phi_i, \tag{1}$$

where $\{c_i\}_{i=0}^{\infty} \subset \mathbb{C}$. We define the m^{th} partial sum of the series (1) by

$$S_m = \sum_{i=0}^{m} c_i\phi_i. \tag{2}$$

For a complex matrix $A = (a_{m\,n})_{m,n\in\mathbb{Z}^+}$, we define

$$\sigma_m = \sum_{i=0}^{m} a_{m\,i}S_i = \sum_{u=0}^{m} b_{m\,u}c_u\phi_u,$$

where $b_{m\,u} = \sum_{i=u}^{m} a_{m\,i}$.

Hausdorff matrix. For $\Phi \in BV[0,1]$, we define the Hausdorff matrix $(h^{\Phi}_{n\,k})_{n,k\in\mathbb{Z}^+}$, where

$$h^{\Phi}_{n\,k} := \begin{cases} \int_0^1 \binom{n}{k} r^k(1-r)^{n-k}d\Phi(r) & \text{if } 0 \leqslant k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

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In particular,

(i) if $\dfrac{d\Phi}{dr} = \alpha(1-r)^{\alpha-1}$, then the Hausdorff matrix is a Cesàro $(\mathscr{C},\alpha)$ matrix;

(ii) if $\dfrac{d\Phi}{dr} = \dfrac{1}{\Gamma(\alpha)} \ln\left(\dfrac{1}{r}\right)^{\alpha-1}$, then the Hausdorff matrix is a Hölder (H,α) matrix.

(iii) For $t \in [0,1]$, $\Phi_t : [0,1] \to \mathbb{R}$ is defined by

$$\Phi_0(r) = \begin{cases} 0 & \text{if } r=0, \\ 1 & \text{if } 0<r\leqslant 1 \end{cases} \quad \text{and} \quad \Phi_t(r) = \begin{cases} 0 \text{ if } 0\leqslant r<t, \\ 1 \text{ if } t\leqslant r\leqslant 1. \end{cases}$$

Then the corresponding Hausdorff matrix is a Euler matrix.

Note that Hausdorff matrix is regular iff $\int_0^1 d\Phi(r) = 1$. Since $\Phi = \Phi_1 - \Phi_2$ with Φ_1, Φ_2 are monotonically increasing on $[0,1]$ and any Hausdorff matrix is constant multiple of a regular Hausdorff matrix, we assume that Φ is monotonically increasing on $[0,1]$ and our Hausdorff matrix is regular.

Nörlund matrix. Let $\{q_n\}_{n=0}^{\infty} \subset \mathbb{R}^+$ and $Q_n = \sum_{k=0}^{n} q_k$. We define the Nörlund matrix $(a_{n\,k})_{n,k\in\mathbb{Z}^+}$, where

$$a_{n\,k} := \begin{cases} \dfrac{q_k}{Q_n} & \text{if } 0\leqslant k\leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

We assume that $(q_n)_{n=0}^{\infty}$ is an increasing sequence of positive real numbers.

In 1922, Rademacher [2] proved that for any $(c_n)_{n=0}^{\infty} \in \ell_2(\mathbb{Z}^+)$, $S_n = o_x(\log(n+2))$ a.e. In 1957, Tandori [4] proved for any $(c_n)_{n=0}^{\infty} \in \ell_2(\mathbb{Z}^+)$, $\sigma_n = o_x(\log\log(n+2))$ a.e., where σ_n is n^{th} Cesàro mean of an orthogonal series.

In this paper, we shall estimate the order of convergence of an orthogonal series with respect to Hausdorff and Nörlund means. In particular, our result also holds for Euler, Hölder means.

2. Preliminaries

First we shall state some known results which are needed in the sequel.

THEOREM A. *For any* $(c_n)_{n=0}^{\infty} \in \ell_2(\mathbb{Z}^+)$, $S_{2^p} = o_x(\log(p+2))$ *a.e. as* $p\to\infty$.

THEOREM B. *For* $q>0$ $\exists$ $C_q>0$ *such that*

$$\sum_{n=m}^{\infty} \binom{n}{m} q^{n-m}(1+q)^{-n} \leqslant C_q, \ \forall\, m\in\mathbb{Z}^+.$$

Proofs of theorems A and B may be found in [1] and [6].

3. Helpful Lemmas

LEMMA 1. *Let* $(a_{m\,n})_{m,n\in\mathbb{Z}^+}$ *be a Hausdorff matrix. Then for* $u\in\mathbb{Z}^+$, $\exists\, C>0$ *such that*

(i) $1-b_{2^p\,u}=\int\limits_0^1\sum\limits_{i=0}^{u-1}\binom{2^p}{i}r^i(1-r)^{2^p-i}d\Phi(r)$;

(ii) $\sum\limits_{m=u}^{\infty}|b_{m\,u}-b_{m-1\,u}|^2\leqslant C,$

where $b_{m\,u}=\sum\limits_{i=u}^{m}a_{m\,i}.$

Proof. (i) For $0\leqslant u\leqslant 2^p$, we consider

$$\begin{aligned}1-b_{2^p\,u}&=1-\sum_{i=u}^{2^p}a_{2^p\,i}\\&=\int\limits_0^1 d\Phi(r)-\int\limits_0^1\sum_{i=u}^{2^p}\binom{2^p}{i}r^i(1-r)^{2^p-i}d\Phi(r)\\&=\int\limits_0^1\sum_{i=0}^{2^p}\binom{2^p}{i}r^i(1-r)^{2^p-i}d\Phi(r)-\int\limits_0^1\sum_{i=u}^{2^p}\binom{2^p}{i}r^i(1-r)^{2^p-i}d\Phi(r)\\&=\int\limits_0^1\sum_{i=0}^{u-1}\binom{2^p}{i}r^i(1-r)^{2^p-i}d\Phi(r).\end{aligned}$$

(ii) For $0\leqslant i\leqslant m$, we consider

$$\begin{aligned}a_{m\,i}-a_{m-1\,i}&=\int_0^1 r^i(1-r)^{m-i}\left[\binom{m}{i}-\binom{m-1}{i}\frac{1}{1-r}\right]d\Phi(r)\\&=\int_0^1 r^i(1-r)^{m-i-1}\left[\binom{m-1}{i-1}-r\binom{m}{i}\right]d\Phi(r).\end{aligned}$$

Noting $\sum\limits_{i=u}^{m}r^i(1-r)^{-i}\left[\binom{m-1}{i-1}-r\binom{m}{i}\right]=\binom{m-1}{u-1}\frac{r^u}{(1-r)^{u-1}}$, we obtain

$$\begin{aligned}\sum_{i=u}^{m}(a_{m\,i}-a_{m-1\,i})&=\int_0^1\sum_{i=u}^{m}r^i(1-r)^{m-i-1}\left[\binom{m-1}{i-1}-r\binom{m}{i}\right]d\Phi(r)\\&=\int_0^1 r^u(1-r)^{m-u}\binom{m-1}{u-1}d\Phi(r).\end{aligned}$$

Since Φ is monotonically increasing on $[0,1]$, we have

$$b_{m\,u}-b_{m-1\,u}=\binom{m-1}{u-1}\mu^u(1-\mu)^{m-u}[\Phi(1)-\Phi(0)]\quad\text{for some }\ \mu\in(0,1).$$

Hence

$$\sum_{m=u}^{\infty} |b_{m\ u} - b_{m-1\ u}|^2 \leqslant C \quad \text{by theorem B.} \quad \square$$

LEMMA 2. *Let* $(a_{m\ n})_{m,n\in\mathbb{Z}^+}$ *be a Nörlund matrix. Then for* $u \in \mathbb{Z}^+$,

$$\sum_{p|2^p\geqslant u} |1 - b_{2^p\ u}|^2 \leqslant 2 \text{ where } b_{m\ u} = \sum_{i=u}^{m} a_{m\ i}.$$

Proof. For $0 \leqslant u \leqslant 2^p$, we consider

$$\begin{aligned} 1 - b_{2^p\ u} &= 1 - \sum_{i=u}^{2^p} a_{2^p\ i} \\ &= \sum_{i=0}^{2^p} \frac{q_i}{Q_{2^p}} - \sum_{i=u}^{2^p} \frac{q_i}{Q_{2^p}} = \frac{Q_{u-1}}{Q_{2^p}}. \end{aligned}$$

Thus

$$\sum_{p|2^p\geqslant u} |1 - b_{2^p\ u}|^2 = (Q_{u-1})^2 \sum_{p|2^p\geqslant u} \left(\frac{1}{Q_{2^p}}\right)^2 \leqslant \sum_{p=0}^{\infty} \frac{1}{2^p} = 2. \quad \square$$

LEMMA 3. *Let* $\{\phi_n\}_{n=0}^{\infty} \subset L_2(X)$ *be an orthonormal system and* $(c_n)_{n=0}^{\infty} \in \ell_2(\mathbb{Z}^+)$.

(i) *For* $2^p < m \leqslant 2^{p+1}$, *with* $p \in \mathbb{Z}^+$, *let* $(a^*_{2^p,m,u})_{p,u\in\mathbb{Z}^+}$ *be a complex matrix with* $\sum_{p|2^p\geqslant u} |a^*_{2^p,m,u}|^2 \leqslant C_1$, $C_1 > 0$, $\forall\ u$. *Then* $F := \left\{ \sum_{p=0}^{\infty} \left| \sum_{u=0}^{2^p} a^*_{2^p,m,u} c_u \phi_u \right|^2 \right\}^{\frac{1}{2}}$ *is in* $L_2(X)$ *and* $\lim_{p\to\infty} \left| \sum_{u=0}^{2^p} a^*_{2^p,m,u} c_u \phi_u(x) \right| = 0$ *a.e.*

(ii) *If* $(b^*_{m\ n})_{m,n\in\mathbb{Z}^+}$ *is a complex matrix with* $\sum_{m=u}^{\infty} |b^*_{m\ u}|^2 \leqslant C_2$, $C_2 > 0$, $\forall\ u$. *Then*

$$G := \left\{ \sum_{m=0}^{\infty} \left| \sum_{u=0}^{m} b^*_{m\ u} c_u \phi_u \right|^2 \right\}^{\frac{1}{2}} \in L_2(X)$$

and

$$\lim_{p\to\infty} \sum_{m=2^p+1}^{2^{p+1}} \left| \sum_{u=0}^{m} b^*_{m\ u} c_u \phi_u(x) \right|^2 = 0 \ \text{a.e.}$$

Proof. (i) Using Parseval's identity, we have

$$\begin{aligned} \sum_{p=0}^{\infty} \int_X \left| \sum_{u=0}^{2^p} a^*_{2^p,m,u} c_u \phi_u(x) \right|^2 d\mu(x) &= \sum_{p=0}^{\infty} \sum_{u=0}^{2^p} |a^*_{2^p,m,u}|^2 |c_u|^2 \\ &\leqslant \sum_{u=0}^{\infty} \sum_{p|2^p\geqslant u} |a^*_{2^p,m,u}|^2 |c_u|^2 \leqslant C_1 \sum_{u=0}^{\infty} |c_u|^2 < \infty. \end{aligned}$$

Thus $F \in L_2(X)$ and

$$\lim_{p\to\infty}\left|\sum_{u=0}^{2^p} a^*_{2^p,m,u} c_u \phi_{u\ v}(x)\right| = 0 \text{ a.e.}$$

(ii) Again by Parseval's identity, we have

$$\begin{aligned}\sum_{m=0}^{\infty}\int_X\left|\sum_{u=0}^{m} b^*_{m\ u} c_u \phi_u(x)\right|^2 d\mu(x) &= \sum_{m=0}^{\infty}\sum_{u=0}^{m}|b^*_{m\ u}|^2|c_u|^2\\ &= \sum_{u=0}^{\infty}\sum_{m=u}^{\infty}|b^*_{m\ u}|^2|c_u|^2\\ &\leqslant C_2\sum_{u=0}^{\infty}|c_u|^2<\infty.\end{aligned}$$

Hence $G \in L_2(X)$ and

$$\lim_{p\to\infty}\sum_{m=2^p+1}^{2^{p+1}}\left|\sum_{u=0}^{m} b^*_{m\ u} c_u \phi_u(x)\right|^2 = 0 \text{ a.e.} \quad \square$$

4. Main Theorems

THEOREM 1. *If $A = (a_{n\ k})_{n,k\in\mathbb{Z}^+}$ is a Hausdorff matrix and $(c_n)_{n=0}^{\infty} \in \ell_2(\mathbb{Z}^+)$, then*

$$|S_{2^p} - \sigma_{2^p}| = o_x(\sqrt{2^p}) \textit{ a.e. as } p \to \infty.$$

Proof. First we observe that

$$\begin{aligned} S_{2^p}(x) - \sigma_{2^p}(x) &= \sum_{u=0}^{2^p}(1-b_{2^p\ u})c_u\phi_u(x)\\ \Rightarrow \frac{S_{2^p}(x)-\sigma_{2^p}(x)}{\sqrt{2^p}} &= \sum_{u=0}^{2^p}\frac{(1-b_{2^p\ u})}{\sqrt{2^p}}c_u\phi_u(x). \end{aligned} \tag{3}$$

By lemma 1 (i) and 3 (i), with $a^*_{2^p,m,u} = \dfrac{(1-b_{2^p\ u})}{\sqrt{2^p}}$, we obtain

$$\lim_{p\to\infty}\left|\sum_{u=0}^{2^p}\frac{(1-b_{2^p\ u})}{\sqrt{2^p}}c_u\phi_u(x)\right| = 0 \text{ a.e.} \tag{4}$$

Using (4) in (3), we obtain

$$|S_{2^p} - \sigma_{2^p}| = o_x(\sqrt{2^p}) \text{ a.e. as } p \to \infty. \quad \square$$

THEOREM 2. *If* $(a_{n\,k})_{n,k\in\mathbb{Z}^+}$ *is a Hausdorff matrix and* $(c_n)_{n=0}^{\infty}\in\ell_2(\mathbb{Z}^+)$, *then*

$$\max_{2^p<m\leqslant 2^{p+1}}|\sigma_m-\sigma_{2^p}|=o_x(\sqrt{2^p}) \text{ a.e. as } p\to\infty.$$

Proof. For $2^p<m\leqslant 2^{p+1}$, with $p\in\mathbb{Z}^+$, we have

$$\begin{aligned}|\sigma_m-\sigma_{2^p}| &\leqslant \sum_{m=2^p+1}^{2^{p+1}}|\sigma_m-\sigma_{m-1}|\\ &= \sum_{m=2^p+1}^{2^{p+1}}\left|\sum_{u=0}^{m}(b_{m\,u}-b_{m-1\,u})c_u\phi_u\right|\\ &\leqslant \left\{\sum_{m=2^p+1}^{2^{p+1}}\left|\sum_{u=0}^{m}(b_{m\,u}-b_{m-1\,u})c_u\phi_u\right|^2\right\}^{\frac{1}{2}}\sqrt{2^p}\end{aligned}$$

$$\Rightarrow \max_{2^p<m\leqslant 2^{p+1}}|\sigma_m-\sigma_{2^p}|\leqslant \left\{\sum_{m=2^p+1}^{2^{p+1}}\left|\sum_{u=0}^{m}(b_{m\,u}-b_{m-1\,u})c_u\phi_u\right|^2\right\}^{\frac{1}{2}}\sqrt{2^p}. \tag{5}$$

By lemmas 1 (ii) and 3 (ii), with $b^*_{m\,u}=b_{m\,u}-b_{m-1\,u}$, we obtain

$$\lim_{p\to\infty}\sum_{m=2^p+1}^{2^{p+1}}\left|\sum_{u=0}^{m}(b_{m\,u}-b_{m-1\,u})c_u\phi_u(x)\right|^2=0 \text{ a.e.} \tag{6}$$

Using (6) in (5), we see that

$$\max_{2^p<m\leqslant 2^{p+1}}|\sigma_m-\sigma_{2^p}|=o_x(\sqrt{2^p}) \text{ a.e. as } p\to\infty. \quad \square$$

THEOREM 3. *If* $A=(a_{n\,k})_{n,k\in\mathbb{Z}^+}$ *is a Hausdorff matrix and* $(c_n)_{n=0}^{\infty}\in\ell_2(\mathbb{Z}^+)$, *then*

$$\sigma_n=o_x(\sqrt{n}\log\log(n+2)) \text{ a.e. as } n\to\infty.$$

Proof. For $2^p<n\leqslant 2^{p+1}$, with $p\in\mathbb{Z}^+$, we have

$$\begin{aligned}\sigma_n(x) &= S_{2^p}(x)+(\sigma_n(x)-\sigma_{2^p}(x))+(\sigma_{2^p}(x)-S_{2^p}(x))\\ \frac{|\sigma_n(x)|}{|\sqrt{n}\log\log(n+2)|} &\leqslant \left|\frac{S_{2^p}(x)}{\sqrt{n}\log\log(n+2)}\right|+\left|\frac{\sigma_n(x)-\sigma_{2^p}(x)}{\sqrt{n}\log\log(n+2)}\right|+\left|\frac{\sigma_{2^p}(x)-S_{2^p}(x)}{\sqrt{n}\log\log(n+2)}\right|\\ &\leqslant \left|\frac{S_{2^p}(x)}{\sqrt{2^p}\log(p+2)}\right|+\left|\frac{\sigma_n(x)-\sigma_{2^p}(x)}{\sqrt{2^p}\log(p+2)}\right|+\left|\frac{\sigma_{2^p}(x)-S_{2^p}(x)}{\sqrt{2^p}\log(p+2)}\right|\\ &\to 0 \quad \text{a.e. as } p\to\infty\end{aligned}$$

by theorems A, 1 and 2. □

THEOREM 4. *If* $(a_{n\,k})_{n,k\in\mathbb{Z}^+}$ *is a Nörlund matrix and* $(c_n)_{n=0}^{\infty}\in\ell_2(\mathbb{Z}^+)$, *then*

$$|S_{2^p}-\sigma_{2^p}|=o_x(1) \text{ a.e. as } p\to\infty.$$

Proof. Note that

$$S_{2^p}(x)-\sigma_{2^p}(x)=\sum_{u=0}^{2^p}(1-b_{2^p\,u})c_u\phi_u(x).$$

By lemmas 2 and 3 (i), with $a^*_{2^p,m,u}=1-b_{2^p\,u}$, we obtain

$$\lim_{p\to\infty}\left|\sum_{u=0}^{2^p}(1-b_{2^p\,u})c_u\phi_u(x)\right|=0 \quad \text{a.e.}$$

Hence

$$|S_{2^p}-\sigma_{2^p}|=o_x(1) \text{ a.e. as } p\to\infty. \qquad \square$$

THEOREM 5. *If* $(a_{n\,k})_{n,k\in\mathbb{Z}^+}$ *is a Nörlund matrix and* $(c_n)_{n=0}^{\infty}\in\ell_2(\mathbb{Z}^+)$, *then for* $2^p<m\leqslant 2^{p+1}$, *with* $p\in\mathbb{Z}^+$, *we have*

$$|\sigma_m-\sigma_{2^p}|=o_x(1) \text{ a.e. as } p\to\infty.$$

Proof. For $2^p<m\leqslant 2^{p+1}$, with $p\in\mathbb{Z}^+$, we have

$$\begin{aligned}\sigma_m(x)-\sigma_{2^p}(x)&=\sum_{u=0}^{m}b_{m\,u}c_u\phi_u(x)-\sum_{u=0}^{2^p}b_{2^p\,u}c_u\phi_u(x)\\&=\sum_{u=0}^{2^p}(b_{m\,u}-b_{2^p\,u})c_u\phi_u(x)+\sum_{u=2^p+1}^{m}b_{m\,u}c_u\phi_u(x).\end{aligned}$$

Thus

$$|\sigma_m(x)-\sigma_{2^p}(x)|\leqslant\left|\sum_{u=0}^{2^p}(b_{m\,u}-b_{2^p\,u})c_u\phi_u(x)\right|+\left|\sum_{u=2^p+1}^{m}b_{m\,u}c_u\phi_u(x)\right|. \tag{7}$$

Using Parseval's identity, we have

$$\begin{aligned}\sum_{p=0}^{\infty}\int_X\left|\sum_{u=2^p+1}^{m}b_{m\,u}c_u\phi_u(x)\right|^2d\mu(x)&=\sum_{p=0}^{\infty}\sum_{u=2^p+1}^{m}\left(1-\frac{Q_{u-1}}{Q_m}\right)^2|c_u|^2\\&\leqslant\sum_{p=0}^{\infty}\sum_{u=2^p+1}^{2^{p+1}}|c_u|^2=\sum_{u=2}^{\infty}|c_u|^2.\end{aligned}$$

Hence

$$\lim_{p\to\infty}\left|\sum_{u=2^p+1}^{m}b_{m\,u}c_u\phi_u(x)\right|=0 \quad \text{a.e.} \tag{8}$$

Using lemmas 2, 3 (i), with $a^*_{2^p,m,u} = b_{m\ u} - b_{2^p\ u}$, and (8) in (7), we obtain for $2^p < m \leqslant 2^{p+1}$,

$$|\sigma_m - \sigma_{2^p}| = o_x(1) \text{ a.e. as } p \to \infty. \quad \square$$

THEOREM 6. *If* $(a_{n\ k})_{n,k\in\mathbb{Z}^+}$ *is a Nörlund matrix and* $(c_n)_{n=0}^{\infty} \in \ell_2(\mathbb{Z}^+)$, *then*

$$\sigma_m = o_x(\log\log(m+2)) \text{ a.e. as } m \to \infty.$$

Proof. For $2^p < m \leqslant 2^{p+1}$, with $p \in \mathbb{Z}^+$, we have

$$\sigma_m = S_{2^p} + (\sigma_m - \sigma_{2^p}) + (\sigma_{2^p} - S_{2^p}).$$

Thus

$$\left|\frac{\sigma_m}{\log\log(m+2)}\right| \leqslant \left|\frac{S_{2^p}}{\log(p+2)}\right| + \left|\frac{\sigma_m - \sigma_{2^p}}{\log(p+2)}\right| + \left|\frac{\sigma_{2^p} - S_{2^p}}{\log(p+2)}\right| \to 0 \text{ a.e. as } p \to \infty$$

by theorems A, 4 and 5. $\square$

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