A CLASS OF ANALYTIC FUNCTIONS INVOLVING THE DZIOK–RAINA OPERATOR

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Abstract. This paper first defines a class of analytic functions which is associated with the Dziok-Raina operator and related closely with the class of uniformly convex functions. Several characteristics for this class of functions are investigated which include certain inclusion relations, convolution properties and the order of starlikeness. Several cases and implications of the main results including the concept of subordinations are discussed and some consequent results are also pointed out.

1. Introduction

Let \( \mathcal{A} \) be the class of functions \( f(z) \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk \( \mathbb{U} = \{ z : |z| < 1 \} \). For \( \rho < 1 \), a function \( f \in \mathcal{A} \) is said to be starlike of order \( \rho \) in \( \mathbb{U} \) if

\[
\Re \frac{zf'(z)}{f(z)} > \rho \quad (z \in \mathbb{U}).
\]

This class is denoted by \( \mathcal{S}^*(\rho) \) (\( \rho < 1 \)). For \( -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \) and \( \rho < 1 \), a function \( f \in \mathcal{A} \) is said to be \( \alpha \)-spirallike of order \( \rho \) in \( \mathbb{U} \) if

\[
\Re \left\{ e^{i\alpha} \frac{zf''(z)}{f'(z)} \right\} > \rho \cos \alpha \quad (z \in \mathbb{U}).
\]

When \( 0 \leq \rho < 1 \), it is well known that all the starlike functions of order \( \rho \) and \( \alpha \)-spirallike functions of order \( \rho \) are univalent in \( \mathbb{U} \). A function \( f \in \mathcal{A} \) is said to be convex univalent in \( \mathbb{U} \) if

\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in \mathbb{U}).
\]


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We denote this class by $\mathcal{K}$. Also, let $\mathcal{UC}(\subset \mathcal{K})$ be the class of uniformly convex functions in $\mathbb{U}$ introduced by Goodman [7]. It was shown in [16] that $f \in \mathcal{A}$ is in $\mathcal{UC}$ if and only if
\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{U}).
\] (1.5)

For $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, $0 < \beta \leq 1$, a function $f \in \mathcal{A}$ is said to be $\beta$ uniformly convex $\alpha$-spiral in $\mathbb{U}$ if
\[
\Re \left\{ e^{i\alpha} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \beta \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{U}).
\] (1.6)

This class is denoted $\mathcal{UC}\mathcal{SP}(\alpha, \beta)$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{SP}(\alpha, \beta)$ if and only if $f(z) = zg'(z)$ and $g \in \mathcal{UC}\mathcal{SP}(\alpha, \beta)$. In [16], Ronning investigated the class $\mathcal{I}_p$ defined by
\[
\mathcal{I}_p = \{ f(z) \in \mathcal{P}(0, 1) : f(z) = zg'(z), \quad g(z) \in \mathcal{UC} \}.
\]

Note that $\mathcal{UC}\mathcal{SP}(0, 1) = \mathcal{UC}$ and $\mathcal{SP}(0, 1) = \mathcal{I}_p$. The uniformly convex and related functions have been studied by many authors (see, e.g., [6–9, 11] and the references therein).

If
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A} \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A},
\]
then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by
\[
(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.
\]

Let $\alpha_1, A_1, \ldots, \alpha_p, A_p$ and $\beta_1, B_1, \ldots, \beta_q, B_q (p, q \in \mathbb{N})$ be positive real parameters satisfying the inequality:
\[
1 + \sum_{m=1}^{q} B_m \geq \sum_{m=1}^{p} A_m.
\]

The Wright generalized hypergeometric function (see [23])
\[
p\Psi_q[(\alpha_1, A_1), \ldots, (\alpha_p, A_p); (\beta_1, B_1), \ldots, (\beta_q, B_q); z] := p\Psi_q[(\alpha_m, A_m)_{1,p}(\beta_m, B_m)_{1,q}; z]
\]
is defined by
\[
p\Psi_q[(\alpha_m, A_m)_{1,p}(\beta_m, B_m)_{1,q}; z] = \sum_{n=0}^{\infty} \left\{ \frac{\prod_{m=1}^{p} \Gamma(\alpha_m + nA_m)}{\prod_{m=1}^{q} \Gamma(\beta_m + nB_m)} \right\} \frac{z^n}{n!} \quad (z \in \mathbb{U}).
\]

If $A_m = 1 (m = 1, \ldots, p)$ and $B_m = 1 (m = 1, \ldots, q)$, then we have the following obvious relationship:
\[
\Theta \cdot p\Psi_q[(\alpha_1, 1)_{1,p}(\beta_1, 1)_{1,q}; z] = p\Psi_q[(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q); z] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n} \ldots (\alpha_p)_{n}}{(\beta_1)_{n} \ldots (\beta_q)_{n}} \frac{z^n}{n!} \quad (z \in \mathbb{U}),
\]
where $pF_q(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z)$ is the generalized hypergeometric function, $(c)_n$ is the Pochhammer symbol defined by

$$(c)_n = \begin{cases} 1 & (n = 0), \\ c(c + 1) \cdots (c + n - 1) & (n \in \mathbb{N}), \end{cases}$$

and $\Theta$ is given by

$$\Theta = \left( \prod_{m=0}^{p} \Gamma(\alpha_m) \right)^{-1} \left( \prod_{m=0}^{q} \Gamma(\beta_m) \right).$$

Corresponding to the function

$$z \cdot p \Psi_q[(\alpha_m, A_m)_{1,p}; (\beta_m, B_m)_{1,q}; z],$$
the Dziok-Raina operator ([4], see also, for example [3], [12–15, 18])

$$W[(\alpha_m, A_m)_{1,p}; (\beta_m, B_m)_{1,q}] : \mathcal{A} \to \mathcal{A}$$

is defined by the following Hadamard product:

$$W[(\alpha_m, A_m)_{1,p}; (\beta_m, B_m)_{1,q}]f(z) := \Theta\{z \cdot p \Psi_q[(\alpha_m, A_m)_{1,p}(\beta_m, B_m)_{1,q}; z]\} \ast f(z).$$

We observe that for a function $f(z)$ defined by (1.1) we have

$$W[(\alpha_m, A_m)_{1,p}; (\beta_m, B_m)_{1,q}]f(z) = z + \sum_{n=2}^{\infty} \Omega_n[\alpha_m; \beta_m]a_nz^n,$$ (1.7)

where

$$\Omega_n[\alpha_m; \beta_m] = \Theta \cdot \frac{\Gamma(\alpha_1 + A_1(n - 1)) \cdots \Gamma(\alpha_p + A_p(n - 1))}{\Gamma(\beta_1 + B_1(n - 1)) \cdots \Gamma(\beta_q + B_q(n - 1))(n - 1)!}. \quad (1.8)$$

In order to make the notation simple, we write

$$W_p^q(\alpha_1) f(z) = W[(\alpha_m, A_m)_{1,p}; (\beta_m, B_m)_{1,q}]f(z).$$ (1.9)

The linear operator $W_p^q(\alpha_1)$ contains the Dziok-Srivastava operator ([5]; see also [19–21] and [24]) and as its various special cases contain such linear operators as the Carlson-Shaffer operator [2], the Ruscheweyh derivative operator [17], the Bernardi-Libera-Livingston integral operator [1], etc. Also, it is worth noting that the linear operator (1.9) would also contain operators in terms of generalized Mittag-Leffler function and the Bessel-Maitland function (see, for example [13]).

In this paper, we introduce and investigate the following subclass of $\mathcal{A}$.

**Definition.** A function $f \in \mathcal{A}$ is said to be in $W_p^q(\alpha_1, \alpha, \beta)$ if it satisfies the condition

$$\Re \left\{ e^{i\alpha} \frac{z(W_p^q(\alpha_1)f(z))'}{W_p^q(\alpha_1)f(z)} \right\} > \beta \left| \frac{z(W_p^q(\alpha_1)f(z))'}{W_p^q(\alpha_1)f(z)} - 1 \right| (z \in \mathbb{U}), \quad (1.10)$$
where
\[-\frac{\pi}{2} < \alpha < \frac{\pi}{2}\] and \(0 < \beta \leq 1\). \hfill (1.11)

**Remark 1.** The function \(f \in \mathcal{A}\) defined by
\[W^q_p(\alpha_1)f(z) = \frac{z}{(1-bz)^2e^{i\alpha} \cos \alpha}\] \hfill (1.12)
is in \(W^q_p(\alpha_1, \alpha, \beta)\), if and only if
\[|b| \leq \frac{1}{1+2\beta}.\] \hfill (1.13)

In fact, for \(f(z)\) defined by (1.12), we have
\[
\frac{z(W^q_p(\alpha_1)f(z))'}{W^q_p(\alpha_1)f(z)} = 1 + 2e^{i\alpha} \cos \alpha \frac{b\bar{z}}{1-b\bar{z}}.
\]
From (1.10), we know that \(f(z) \in W^q_p(\alpha_1, \alpha, \beta)\), if and only if
\[
\Re \left[ \frac{1+b\bar{z}}{1-b\bar{z}} > 2\beta \left| \frac{b\bar{z}}{1-b\bar{z}} \right| \right] (z \in \mathbb{U}).
\] \hfill (1.14)

Letting \(z \to -|b|/b\) in (1.14), we get (1.13). Conversely, if the inequality (1.13) holds, then
\[
\Re \left[ \frac{1+b\bar{z}}{1-b\bar{z}} \geq 1 - \frac{|b\bar{z}|}{|1-b\bar{z}|} > 2\beta \left| \frac{b\bar{z}}{1-b\bar{z}} \right| \right] (z \in \mathbb{U}).
\]
This completes the proof.

**Remark 2.** For specific values assigned to the parameters of the class defined by (1.10), the following relationships are easy to verify:
\[
W^1_0(1, \alpha, \beta) = \mathcal{S}\mathcal{P}(\alpha, \beta) \quad \text{and} \quad W^1_0(2, \alpha, \beta) = \mathcal{W}\mathcal{C}\mathcal{S}\mathcal{P}(\alpha, \beta)
\]
with \(A_1 = 1\).
Throughout this paper we assume, unless otherwise stated, that \(\alpha\) and \(\beta\) satisfy (1.11).

## 2. Subordination Theorem

Let \(f(z)\) and \(g(z)\) be analytic in \(\mathbb{U}\). We say that the function \(f(z)\) is subordinate to \(g(z)\) in \(\mathbb{U}\), and we write \(f(z) \prec g(z)\), if there exists an analytic function \(w(z)\) in \(\mathbb{U}\) such that
\[|w(z)| \leq |z| \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in \mathbb{U}).\]
If \(g(z)\) is univalent in \(\mathbb{U}\), then
\[f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).\]
THEOREM 1. A function \( f \in \mathcal{A} \) is in \( \mathcal{W}_p^q(\alpha_1, \alpha, \beta) \), if and only if

\[
e^{i\alpha z} \frac{z(W_p^q(\alpha_1) f(z))'}{W_p^q(\alpha_1) f(z)} \prec h_\beta(z) \cos \alpha + i\sin \alpha,
\]

where

\[
h_\beta(z) = 1 + \frac{1}{2 \sin^2 \sigma} \left\{ \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^{2\sigma/\pi} + \left( \frac{1 - \sqrt{z}}{1 + \sqrt{z}} \right)^{2\sigma/\pi} - 2 \right\} (z \in \mathbb{U}),
\]

\( \sigma = \arccos \beta (0 < \beta < 1) \), when \( \beta = 1 \),

\[
h_1(z) = 1 + \frac{2}{\pi^2} \left( \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2 (z \in \mathbb{U}).
\]

**Proof.** Let us define \( w(z) = u + iv \) by

\[
e^{i\alpha z} \frac{z(W_p^q(\alpha_1) f(z))'}{W_p^q(\alpha_1) f(z)} = w(z) \cos \alpha + i\sin \alpha \quad (z \in \mathbb{U}),
\]

then (1.9) in conjunction with (1.7) readily gives \( w(0) = 1 \).

When \( 0 < \beta < 1 \), the inequality (1.10) can be rewritten as \( u > \beta \sqrt{(u-1)^2 + v^2} \), which is equivalent to

\[ u^2 > \beta^2 u^2 - 2\beta^2 u + \beta^2 + \beta^2 v^2 \]

and

\[ u > \frac{\beta}{1 + \beta}. \]

That is

\[ u^2 + \frac{2\beta^2 u}{1 - \beta^2} - \frac{\beta^2}{1 - \beta^2} v^2 > \frac{\beta^2}{1 - \beta^2} \]

and

\[ u > \frac{\beta}{1 + \beta}. \]

Hence, we have

\[ \left( u + \frac{\beta^2}{1 - \beta^2} \right)^2 - \frac{\beta^2}{1 - \beta^2} v^2 > \left( \frac{\beta}{1 - \beta^2} \right)^2 \]

and

\[ u > \frac{\beta}{1 + \beta}. \]

Thus, the domain of values of \( w(z) \) for \( z \in \mathbb{U} \) is contained in the hyperbolic region

\[ D = \{ w = u + iv : u \text{ and } v \text{ satisfy (2.5) and (2.6)} \}, \]

and we also note that \( h_\beta(0) = w(0) = 1 \). In order to prove our theorem, it suffices to show that the function \( h_\beta(z) \) given by (2.2) maps \( \mathbb{U} \) conformally onto \( D \).
Consider the transformations
\[ w_1 = (1 - \beta^2)w + \beta^2, \quad w_1 = \frac{1}{2} \left( w_2 + \frac{1}{w_2} \right), \]
\[ w_3 = w_2^{\pi/\sigma} (\sigma = \arccos \beta), \quad t = \frac{1}{2} \left( w_3 + \frac{1}{w_3} \right). \]

It is not difficult to verify that the composite function \( t = t(w) \) maps
\[ D^+ = D \cap \{ w = u + iv : v > 0 \} \]
conformally onto the upper-half plane \( \text{Im}(t) > 0 \), so that \( w = 1 \) corresponds to \( t = 1 \) and \( w = \beta / (1 + \beta) \) to \( t = -1 \). Making use of the symmetry principle, this function \( t = t(w) \) maps \( D \) onto \( G = \{ t : |\text{arg}(t + 1)| < \pi \} \). Since
\[ t = 2 \left( \frac{1 + z}{1 - z} \right)^2 - 1 \]
maps \( \mathbb{U} \) onto \( G \), we see that
\[ w = 1 + \frac{1}{2(1 - \beta^2)} \left\{ \left( t + \sqrt{t^2 - 1} \right)^{\sigma/\pi} + \left( t + \sqrt{t^2 - 1} \right)^{-\sigma/\pi} - 2 \right\} \]
\[ = 1 + \frac{1}{2 \sin^2 \sigma} \left\{ \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^{2\sigma/\pi} + \left( \frac{1 - \sqrt{z}}{1 + \sqrt{z}} \right)^{2\sigma/\pi} - 2 \right\} \]
\[ = h_\beta(z) \]
maps \( \mathbb{U} \) conformally onto \( D \).

When \( \beta = 1 \), the inequality (1.10) can be written as \( u > \frac{1 + v^2}{2} \). It is known ([16]) that the function
\[ h_1(z) = 1 + \frac{2}{\pi^2} \left( \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2 \]
maps \( \mathbb{U} \) conformally onto the parabolic region
\[ D_1 = \{ w = u + iv ; u > (v^2 + 1)/2 \} \].

Therefore, the proof of the theorem is complete. \( \square \)

**COROLLARY 1.** Let \( f \in \mathcal{W}_p^q(\alpha_1, \alpha, \beta) \), and \( h_\beta(z) \) be given by (2.2) and (2.3). Then for \( z \in \mathbb{U} \),
\[ \frac{W_p^q(\alpha_1)f(z)}{z} < \exp \left( e^{-i\alpha} \cos \alpha \int_0^1 \frac{h_\beta(t) - 1}{t} dt \right) \quad (2.7) \]
and
\[ \exp \int_0^1 \frac{h_\beta(-\rho) - 1}{\rho} d\rho < \left| \left( \frac{W_p^q(\alpha_1)f(z)}{z} \right)^{e^{i\alpha} \sec \alpha} \right| < \exp \int_0^1 \frac{h_\beta(\rho) - 1}{\rho} d\rho. \quad (2.8) \]
The bounds in (2.8) are sharp.

Proof. From Theorem 1, we have

\[
\frac{e^{i\alpha}}{\cos \alpha} \left( \frac{z (W_p^q(\alpha_1) f(z))'}{W_p^q(\alpha_1) f(z)} - 1 \right) < h_\beta(z) - 1
\]

for \( f(z) \in \mathcal{W}_p^q(\alpha, \alpha, \beta) \), where \( h_\beta(z) \) is given by (2.2). Since the analytic function \( h_\beta(z) - 1 \) is univalent and starlike (with respect to the origin) in \( \mathbb{U} \), therefore, by (2.9) and the result due to Suffridge [22, Theorem 3], we get

\[
\frac{e^{i\alpha}}{\cos \alpha} \log \frac{W_p^q(\alpha_1) f(z)}{z} = \frac{e^{i\alpha}}{\cos \alpha} \int_0^z \frac{(W_p^q(\alpha_1) f(t))'}{W_p^q(\alpha_1) f(t)} - \frac{1}{t} \ dt < \int_0^z \frac{h_\beta(t) - 1}{t} \ dt.
\]

This implies (2.7).

Noting that the univalent function \( h_\beta(z) \) maps the disk \( |z| < \rho (0 < \rho \leq 1) \) onto a region which is convex and symmetric with respect to the real axis, we see that

\[
\int_0^1 \frac{h_\beta(-\rho) - 1}{\rho} \ d\rho < \Re \int_0^1 \frac{h_\beta(\rho z) - 1}{\rho} \ d\rho < \int_0^1 \frac{h_\beta(\rho) - 1}{\rho} \ d\rho
\]

for \( z \in \mathbb{U} \). Consequently, the subordination (2.10) leads to

\[
\int_0^1 \frac{h_\beta(-\rho) - 1}{\rho} \ d\rho < \log \left| \frac{W_p^q(\alpha_1) f(z)}{z} \right|_{\frac{e^{i\alpha}}{\cos \alpha}} < \int_0^1 \frac{h_\beta(\rho) - 1}{\rho} \ d\rho
\]

for \( z \in \mathbb{U} \), which gives (2.8).

Obviously, the bounds in (2.8) are best possible for the function \( f_0(z) \in \mathcal{W}_p^q(\alpha_1, \alpha, \beta) \), defined by

\[
W_p^q(\alpha_1) f_0(z) = z \exp \left( e^{-i\alpha} \cos \alpha \int_0^z \frac{h_\beta(t) - 1}{t} \ dt \right),
\]

where \( h_\beta(z) \) is given by (2.2). The proof is thus complete. \( \Box \)

From (1.7)–(1.9), (2.2), (2.3) and (2.10), we have

**Corollary 2.**

(i) If \( f(z) \in \mathcal{W}_p^q(\alpha, \alpha, \beta) \) and \( 0 < \beta < 1 \), then

\[
f(z) = z \exp \left\{ \frac{e^{-i\alpha} \cos \alpha}{2 \sin^2 \sigma} \left( 1 + \frac{\sqrt{\rho w(z)}}{1 - \sqrt{\rho w(z)}} \right)^{2\sigma / \pi} \left( 1 - \frac{\sqrt{\rho w(z)}}{1 + \sqrt{\rho w(z)}} \right)^{-2} \right\} \times \left\{ \sum_{n=2}^{\infty} \frac{(n-1)! \Gamma(\beta_1 + B_1(n-1)) \cdots \Gamma(\beta_q + B_q(n-1))}{\Gamma(\alpha_1 + A_1(n-1)) \cdots \Gamma(\alpha_p + A_p(n-1))} \prod_{m=1}^p \left( \frac{\alpha_m}{\Gamma(\beta_m)} \right)^{\rho \alpha_m} \right\}
\]
Differentiating (3.4) logarithmically, we obtain
\[ e^{i\alpha} \frac{z(W_p^q(\alpha_1+1)f(z))'}{W_p^q(\alpha_1+1)f(z)} = p(z) + \frac{zp'(z)}{\alpha_1} - 1 + e^{-i\alpha} p(z), \]

where \( w(z) \) is analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1(z \in U) \).

3. Inclusion relations

We will need the following lemma on the Briot-Bouquet differential subordination.

**Lemma 1.** ([10]) Let \( h(z) \) be convex univalent in \( U \), with \( \Re(\mu h(z) + \gamma) \geq 0 \). If \( p(z) \) is analytic in \( U \), with \( p(0) = h(0) \), then
\[ p(z) + \frac{zp'(z)}{\mu p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z). \]

**Theorem 2.** Let \( 0 < |\sin \alpha| \leq \beta \leq 1 \) and
\[ \frac{\alpha_1}{A_1} \geq 1 - \frac{\sqrt{\beta^2 - \sin^2 \alpha}}{\cos \alpha + \sqrt{\beta^2 - \sin^2 \alpha}}. \quad (3.1) \]
Then
\[ \mathcal{W}_p^q(\alpha_1 + 1, \alpha, \beta) \subset \mathcal{W}_p^q(\alpha_1, \alpha, \beta). \]

**Proof.** For \( f(z) \in \mathcal{W}_p^q(\alpha_1 + 1, \alpha, \beta) \), it follows from Theorem 1 that
\[ e^{i\alpha} \frac{z(W_p^q(\alpha_1+1)f(z))'}{W_p^q(\alpha_1+1)f(z)} < h_\beta(z) \cos \alpha + i\sin \alpha \quad (z \in U), \quad (3.2) \]
where \( h_\beta(z) \) is given by (2.2).

From (1.7)–(1.9) we have
\[ W_p^q(\alpha_1+1)f(z) = \left( 1 - \frac{A_1}{\alpha_1} \right) W_p^q(\alpha_1)f(z) + \frac{A_1}{\alpha_1} z(W_p^q(\alpha_1)f(z))', \quad (3.3) \]
that is
\[ \frac{W_p^q(\alpha_1+1)f(z)}{W_p^q(\alpha_1)f(z)} = \left( 1 - \frac{A_1}{\alpha_1} \right) + \frac{A_1}{\alpha_1} \left( \frac{z(W_p^q(\alpha_1)f(z))'}{W_p^q(\alpha_1)} \right). \quad (3.4) \]

Differentiating (3.4) logarithmically, we obtain
\[ e^{i\alpha} \frac{z(W_p^q(\alpha_1+1)f(z))'}{W_p^q(\alpha_1+1)f(z)} = p(z) + \frac{zp'(z)}{\alpha_1} - 1 + e^{-i\alpha} p(z), \quad (3.5) \]
where
\[ p(z) = e^{i\alpha z(W_p^q(\alpha_1)f(z))'} \]
is analytic in \( U \), with \( p(0) = h_\beta(0) \).

From (3.2) and (3.5), we obtain
\[ p(z) + \frac{zp'(z)}{\alpha_1 - 1 + e^{-i\alpha}p(z)} < h_\beta(z) \cos \alpha + i \sin \alpha. \]

(i) Let \(|\sin \alpha| < \beta < 1\), \( h_\beta(z) = u + iv \), where \( h_\beta(z) \) is given by (2.2). From the proof of Theorem 1, we find that \( u > \beta \sqrt{(u-1)^2 + v^2} \) and \( u > \frac{\beta}{1+\beta} \). That is, \(|v| < \frac{1}{\beta} \sqrt{u^2 - \beta^2(u-1)^2} \) and \( u > \frac{\beta}{1+\beta} \). Hence, we have
\[ \min_{|z|=1} \Re \{e^{-i\alpha}h_\beta(z)\} = \min_{|z|=1} \{\cos \alpha - v \sin \alpha\} = \min_{u \geq \beta/(1+\beta)} g(u), \]
where
\[ g(u) = u \cos \alpha - \frac{|\sin \alpha|}{\beta} \sqrt{u^2 - \beta^2(u-1)^2}. \]

Note that
\[ g'(u) = \frac{\beta \cos \alpha \sqrt{u^2 - \beta^2(u-1)^2} - |\sin \alpha|(\beta^2 + (1 - \beta^2)u)}{\beta \sqrt{u^2 - \beta^2(u-1)^2}} \]
\[ = \frac{(1-\beta^2)(\beta^2 - \sin^2 \alpha)u^2 + 2\beta^2(\beta^2 - \sin^2 \alpha)u - \beta^4}{\beta \sqrt{u^2 - \beta^2(u-1)^2}(\beta \cos \alpha \sqrt{u^2 - \beta^2(u-1)^2} + |\sin \alpha|(\beta^2 + (1 - \beta^2)u))} \]
\[ \tag{3.6} \]
for \( u > \beta/(1+\beta) \). Since \(|\sin \alpha| < \beta < 1\), it follows from (3.6) that the function \( g(u) \) \((u \geq \beta/(1+\beta))\) attains its minimum at \( u = u_0 \), where
\[ u_0 = \frac{\beta^2}{1 - \beta^2} \left( \frac{\cos \alpha}{\sqrt{\beta^2 - \sin^2 \alpha}} - 1 \right) > \frac{\beta}{1+\beta}. \]

Now
\[ g(u_0) = u_0 \cos \alpha - \frac{|\sin \alpha|}{\beta} \sqrt{u_0^2 - \beta^2(u_0-1)^2} \]
\[ = \frac{\beta^2 \cos \alpha}{1 - \beta^2} \left( \frac{\cos \alpha}{\sqrt{\beta^2 - \sin^2 \alpha}} - 1 \right) - \frac{\sin^2 \alpha}{\sqrt{\beta^2 - \sin^2 \alpha}} \]
\[ = \frac{\sqrt{\beta^2 - \sin^2 \alpha} - \beta^2 \cos \alpha}{1 - \beta^2}, \]
and therefore,

$$\min_{|z|=1} \Re \{ e^{-i\alpha} (h_\beta(z) \cos \alpha + i \sin \alpha) \} = g(u_0) \cos \alpha + \sin^2 \alpha$$

$$= \frac{\sqrt{\beta^2 - \sin^2 \alpha}}{\cos \alpha + \sqrt{\beta^2 - \sin^2 \alpha}}. \quad (3.7)$$

Since the function $Q(z) = e^{-i\alpha} (h_\beta(z) \cos \alpha + i \sin \alpha)$ is convex (and univalent) in $U$ and

$$\Re \left\{ \frac{\alpha_1}{A_1} - 1 + Q(z) \right\} > \frac{\alpha_1}{A_1} - 1 + \frac{\sqrt{\beta^2 - \sin^2 \alpha}}{\cos \alpha + \sqrt{\beta^2 - \sin^2 \alpha}},$$

it follows from (3.1) and Lemma 1 that

$$p(z) = e^{i\alpha} \frac{zf'(z)}{f(z)} < h_\beta(z) \cos \alpha + i \sin \alpha.$$ 

Therefore, $f(z) \in \mathcal{W}_p^\alpha(\alpha_1, \alpha, \beta) \ (|\sin \alpha| < \beta < 1)$.

(ii) When $0 < |\sin \alpha| = \beta < 1$, we have

$$g(u) = u \sqrt{1 - \beta^2} - \sqrt{u^2 - \beta^2(u - 1)^2} \ (u \geq \beta/(1 + \beta)).$$

It is clear that $g'(u) < 0$ for $u > \beta/(1 + \beta)$, and so

$$\inf_{u \geq \beta/(1 + \beta)} g(u) = g(+\infty) = -\frac{\beta^2}{\sqrt{1 - \beta^2}}.$$ 

Hence

$$\min_{|z|=1} \Re \{ e^{-\alpha} (h_\beta(z) \cos \alpha + i \sin \alpha) \} = g(+\infty) \cos \alpha + \sin^2 \alpha = 0. \quad (3.8)$$

It follows from (3.1) and Lemma 1 that

$$p(z) = e^{i\alpha} \frac{zf'(z)}{f(z)} < h_\beta(z) \cos \alpha + i \sin \alpha.$$ 

Therefore $f(z) \in \mathcal{W}_p^\alpha(\alpha_1, \alpha, \beta) \ (0 < |\sin \alpha| = \beta < 1)$.

(iii) When $\beta = 1$, $g_1(u) = u \cos \alpha - |\sin \alpha| \sqrt{2u - 1} \ (u \geq 1/2)$. Then

$$\min_{u \geq 1/2} g_1(u) = g_1 \left( \frac{1}{2 \cos^2 \alpha} \right) = \frac{1 - 2 \sin^2 \alpha}{2 \cos \alpha}$$
and hence
\[
\min_{|z|=1} \Re \left\{ e^{-i\alpha}(h_1(z) \cos \alpha + i \sin \alpha) \right\} = \cos \alpha \min_{u \geq 1/2} g_1(u) + \sin^2 \alpha = 1/2. \tag{3.9}
\]

It follows from (3.1) and Lemma 1 that
\[
p(z) = e^{i\alpha} \frac{zf''(z)}{f(z)} < h_1(z) \cos \alpha + i \sin \alpha.
\]

Hence, \( f(z) \in \mathcal{W}_p^q(\alpha_1, \alpha, 1) \). This completes the proof. \( \Box \)

**THEOREM 3.** If \( 0 < |\sin \alpha| \leq \beta \leq 1 \), then \( \mathcal{W}_p^q(\alpha_1, \alpha, \beta) \subset S^*(\rho(\alpha, \beta)) \), where
\[
\rho(\alpha, \beta) = \frac{\sqrt{\beta^2 - \sin^2 \alpha}}{\cos \alpha + \sqrt{\beta^2 - \sin^2 \alpha}}, \tag{3.10}
\]
and the order \( \rho(\alpha, \beta) \) is sharp.

**Proof.** Let \( f(z) \in \mathcal{W}_p^q(\alpha_1, \alpha, \beta) \) \((|\sin \alpha| \leq \beta \leq 1)\). Then, by (3.7)–(3.9) and Theorem 1, we conclude that \( f(z) \in S^*(\rho(\alpha, \beta)) \), where \( \rho(\alpha, \beta) \) is given by (3.10), and the order \( \rho(\alpha, \beta) \) is sharp for the function \( f_0(z) \) given by (2.11). \( \Box \)

Lastly, we examine the closure properties of the class \( \mathcal{W}_p^q(\alpha_1, \alpha, \beta) \) under the generalized Bernardi-Libera-Livingston integral operator \( L_c(f) \) which is defined by
\[
L_c f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (\Re(c) > -1). \tag{3.11}
\]

**THEOREM 4.** Let \( f(z) \in \mathcal{W}_p^q(\alpha_1, \alpha, \beta) \) \((0 < |\sin \alpha| \leq \beta \leq 1)\) and
\[
\Re(c) \geq -\frac{\sqrt{\beta^2 - \sin^2 \alpha}}{\cos \alpha + \sqrt{\beta^2 - \sin^2 \alpha}}. \tag{3.12}
\]

Then \( L_c f(z) \) belongs to \( \mathcal{W}_p^q(\alpha_1, \alpha, \beta) \), where \( L_c f(z) \) is given by (3.11) and \( 0 < |\sin \alpha| \leq \beta \leq 1 \).

**Proof.** It follows from (3.11) that
\[
(c+1)W_p^q(\alpha_1)f(z) = cW_p^q(\alpha_1)L_c f(z) + z(W_p^q(\alpha_1)L_c f(z))', \tag{3.13}
\]
that is
\[
\frac{W_p^q(\alpha_1)f(z)}{W_p^q(\alpha_1)L_c f(z)} = \frac{c}{c+1} + \frac{1}{c+1} \frac{z(W_p^q(\alpha_1)L_c f(z))'}{W_p^q(\alpha_1)L_c f(z)}. \tag{3.14}
\]
Differentiating (3.14) logarithmically, we obtain

\[ e^{i\alpha} z \frac{W_p^q(\alpha_1) f(z)}{W_p^q(\alpha_1) f(z)} = q(z) + \frac{z q'(z)}{c + e^{-i\alpha} q(z)}, \]  

(3.15)

where

\[ q(z) = e^{i\alpha} z \frac{W_p^q(\alpha_1) L_c f(z)}{W_p^q(\alpha_1) L_c f(z)} \]

is analytic in \( \mathbb{U} \), with \( q(0) = h(0) \). Noting that \( f(z) \in \mathcal{W}_p^q(\alpha_1, \alpha, \beta) \) \((0 < |\sin \alpha| \leq \beta \leq 1)\), hence by Theorem 1 and (3.15), we obtain

\[ q(z) + \frac{z q'(z)}{c + e^{-i\alpha} q(z)} < h_\beta(z) \cos \alpha + i \sin \alpha, \]

where \( h_\beta(z) \) is given by (2.2) and (2.3).

Since the function \( Q(z) = e^{-i\alpha} (h(z) \cos \alpha + i \sin \alpha) \) is convex (and univalent) in \( \mathbb{U} \) and

\[ \Re \{c + Q(z)\} > \Re c + \frac{\sqrt{\beta^2 - \sin^2 \alpha}}{\cos \alpha + \sqrt{\beta^2 - \sin^2 \alpha}} \geq 0 \quad (z \in \mathbb{U}) \]

by (3.7)–(3.9) and (3.12), therefore, by Lemma 1, we have

\[ q(z) = e^{i\alpha} z \frac{W_p^q(\alpha_1) L_c f(z)}{W_p^q(\alpha_1) L_c f(z)} < h(z) \cos \alpha + i \sin \alpha, \]

that is \( L_c f(z) \in \mathcal{W}_p^q(\alpha_1, \alpha, \beta) \) on using Theorem 1. This completes the proof. \( \square \)

**THEOREM 5.** Let \( c = \frac{\alpha_1}{A_1} - 1 \ (> -1) \). Then \( f(z) \in \mathcal{W}_p^q(\alpha_1, \alpha, \beta) \) if and only if \( L_c f(z) \in \mathcal{W}_p^q(\alpha_1 + 1, \alpha, \beta) \).

*Proof.* If \( f(z) \in \mathcal{W}_p^q(\alpha_1, \alpha, \beta) \), it follows from Theorem 1 that

\[ e^{i\alpha} z \frac{W_p^q(\alpha_1) f(z)}{W_p^q(\alpha_1) f(z)} < h_\beta(z) \cos \alpha + i \sin \alpha, \]

where \( h_\beta(z) \) is given by (2.2).

Let \( c = \frac{\alpha_1}{A_1} - 1 \). It follows from (3.3) and (3.13) that

\[ W_p^q(\alpha_1) f(z) = \frac{c W_p^q(\alpha_1) L_c f(z) + z (W_p^q(\alpha_1) L_c f(z))'}{c + 1} \]

\[ = \left( 1 - \frac{A_1}{\alpha_1} \right) W_p^q(\alpha_1) L_c f(z) + \frac{A_1}{\alpha_1} z (W_p^q(\alpha_1) L_c f(z))' \]

\[ = W_p^q(\alpha_1 + 1) L_c f(z) \quad (z \in \mathbb{U}). \]
Hence
\[
e^{i\alpha} \frac{z(W_p^q(\alpha_1 + 1)L_c f(z))'}{W_p^q(\alpha_1 + 1)L_c f(z)} = e^{i\alpha} \frac{z(W_p^q(\alpha_1)f(z))'}{W_p^q(\alpha_1)f(z)} < h_\beta(z) \cos \alpha + i \sin \alpha,
\]
and it follows from Theorem 1 that \( L_c f(z) \in \mathcal{W}_p^q(\alpha_1 + 1, \alpha, \beta) \).

Conversely, if \( L_c f(z) \in \mathcal{W}_p^q(\alpha_1 + 1, \alpha, \beta) \), then it is easy to verify that \( f(z) \in \mathcal{W}_p^q(\alpha_1, \alpha, \beta) \) also. This completes the proof. □

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REFERENCES


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