Journal of Classical Analysis
Volume 5, Number 1 (2014), 1–14
doi:10.7153/jca-05-01

A CLASS OF ANALYTIC FUNCTIONS INVOLVING THE DZIOK–RAINA OPERATOR

NENG XU AND R. K. RAINA

Abstract. This paper first defines a class of analytic functions which is associated with the Dziok-Raina operator and related closely with the class of uniformly convex functions. Several characteristics for this class of functions are investigated which include certain inclusion relations, convolution properties and the order of starlikeness. Several cases and implications of the main results including the concept of subordinations are discussed and some consequent results are also pointed out.

1. Introduction

Let $\mathscr{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. For $\rho < 1$, a function $f \in \mathscr{A}$ is said to be starlike of order ρ in $\mathbb{U}$ if

$$\Re \frac{zf'(z)}{f(z)} > \rho \quad (z \in \mathbb{U}). \tag{1.2}$$

This class is denoted by $\mathscr{S}^*(\rho)$ $(\rho < 1)$. For $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ and $\rho < 1$, a function $f \in \mathscr{A}$ is said to be α-spirallike of order ρ in $\mathbb{U}$ if

$$\Re \left\{ e^{i\alpha} \frac{zf'(z)}{f(z)} \right\} > \rho \cos \alpha \quad (z \in \mathbb{U}). \tag{1.3}$$

When $0 \leqslant \rho < 1$, it is well known that all the starlike functions of order ρ and α-spirallike functions of order ρ are univalent in $\mathbb{U}$. A function $f \in \mathscr{A}$ is said to be convex univalent in $\mathbb{U}$ if

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in \mathbb{U}). \tag{1.4}$$

Mathematics subject classification (2010): 30C45.
Keywords and phrases: Uniformly convex functions, convex univalent functions, starlike functions, α-spirallike functions, Wright generalized hypergeometric function, Dziok-Raina operator, subordination.

Paper JCA-05-01

We denote this class by $\mathscr{K}$. Also, let $\mathscr{UC}(\subset \mathscr{K})$ be the class of uniformly convex functions in $\mathbb{U}$ introduced by Goodman [7]. It was shown in [16] that $f \in \mathscr{A}$ is in $\mathscr{UC}$ if and only if

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathbb{U}). \tag{1.5}$$

For $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, $0 < \beta \leqslant 1$, a function $f \in \mathscr{A}$ is said to be β uniformly convex α-spiral in $\mathbb{U}$ if

$$\Re\left\{e^{i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathbb{U}). \tag{1.6}$$

This class is denoted $\mathscr{UCSP}(\alpha,\beta)$. A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{SP}(\alpha,\beta)$ if and only if $f(z) = zg'(z)$ and $g \in \mathscr{UCSP}(\alpha,\beta)$. In [16], Rønning investigated the class $\mathscr{S}_p$ defined by

$$\mathscr{S}_p = \{f(z) \in \mathscr{S}^*(0) : f(z) = zg'(z), \quad g(z) \in \mathscr{UC}\}.$$

Note that $\mathscr{UCSP}(0,1) = \mathscr{UC}$ and $\mathscr{SP}(0,1) = \mathscr{S}_p$. The uniformly convex and related functions have been studied by many authors (see, e.g., [6–9, 11] and the references therein).

If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{A} \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathscr{A},$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let $\alpha_1, A_1, \cdots, \alpha_p, A_p$ and $\beta_1, B_1, \cdots, \beta_q, B_q (p,q \in \mathbb{N})$ be positive real parameters satisfying the inequality:

$$1 + \sum_{m=1}^{q} B_m \geqslant \sum_{m=1}^{p} A_m.$$

The Wright generalized hypergeometric function (see [23])

$${}_p\Psi_q[(\alpha_1,A_1),\cdots,(\alpha_p,A_p);(\beta_1,B_1),\cdots,(\beta_q,B_q);z] := {}_p\Psi_q[(\alpha_m,A_m)_{1,p}(\beta_m,B_m)_{1,q};z]$$

is defined by

$${}_p\Psi_q[(\alpha_m,A_m)_{1,p}(\beta_m,B_m)_{1,q};z] = \sum_{n=0}^{\infty}\left\{\frac{\prod_{m=1}^{p}\Gamma(\alpha_m+nA_m)}{\prod_{m=1}^{q}\Gamma(\beta_m+nB_m)}\right\}\frac{z^n}{n!} \quad (z \in \mathbb{U}).$$

If $A_m = 1 (m = 1,\cdots,p)$ and $B_m = 1 (m = 1,\cdots,q)$, then we have the following obvious relationship:

$$\begin{aligned}\Theta \cdot {}_p\Psi_q[(\alpha_m,1)_{1,p}(\beta_m,1)_{1,q};z] &\equiv {}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z)\\ &= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!} \quad (z \in \mathbb{U}),\end{aligned}$$

where ${}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z)$ is the generalized hypergeometric function , $(c)_n$ is the Pochhammer symbol defined by

$$(c)_n = \begin{cases} 1 & (n=0), \\ c(c+1)\cdots(c+n-1) & (n\in\mathbb{N}), \end{cases}$$

and Θ is given by

$$\Theta = \left(\prod_{m=0}^{p}\Gamma(\alpha_m)\right)^{-1}\left(\prod_{m=0}^{q}\Gamma(\beta_m)\right).$$

Corresponding to the function

$$z\cdot{}_p\Psi_q[(\alpha_m,A_m)_{1,p};(\beta_m,B_m)_{1,q};z],$$

the Dziok-Raina operator ([4], see also, for example [3], [12–15, 18])

$$W[(\alpha_m,A_m)_{1,p};(\beta_m,B_m)_{1,q}]:\mathscr{A}\to\mathscr{A}$$

is defined by the following Hadamard product:

$$W[(\alpha_m,A_m)_{1,p};(\beta_m,B_m)_{1,q}]f(z) := \Theta\{z\cdot{}_p\Psi_q[(\alpha_m,A_m)_{1,p}(\beta_m,B_m)_{1,q};z]\} * f(z).$$

We observe that for a function $f(z)$ defined by (1.1) we have

$$W[(\alpha_m,A_m)_{1,p};(\beta_m,B_m)_{1,q}]f(z) = z + \sum_{n=2}^{\infty}\Omega_n[\alpha_m;\beta_m]a_nz^n, \tag{1.7}$$

where

$$\Omega_n[\alpha_m;\beta_m] = \Theta\cdot\frac{\Gamma(\alpha_1+A_1(n-1))\cdots\Gamma(\alpha_p+A_p(n-1))}{\Gamma(\beta_1+B_1(n-1))\cdots\Gamma(\beta_q+B_q(n-1))(n-1)!}. \tag{1.8}$$

In order to make the notation simple, we write

$$W_p^q(\alpha_1)f(z) = W[(\alpha_m,A_m)_{1,p};(\beta_m,B_m)_{1,q}]f(z). \tag{1.9}$$

The linear operator $W_p^q(\alpha_1)$ contains the Dziok-Srivastava operator ([5]; see also [19–21] and [24]) and as its various special cases contain such linear operators as the Carlson-Shaffer operator [2], the Ruscheweyh derivative operator [17], the Bernardi-Libera-Livingston integral operator [1], etc. Also, it is worth noting that the linear operator (1.9) would also contain operators in terms of generalized Mittag-Leffler function and the Bessel-Maitland function (see, for example [13]).

In this paper, we introduce and investigate the following subclass of $\mathscr{A}$.

DEFINITION. A function $f\in\mathscr{A}$ is said to be in $\mathscr{W}_p^q(\alpha_1,\alpha,\beta)$ if it satisfies the condition

$$\Re\left\{e^{i\alpha}\frac{z(W_p^q(\alpha_1)f(z))'}{W_p^q(\alpha_1)f(z)}\right\} > \beta\left|\frac{z(W_p^q(\alpha_1)f(z))'}{W_p^q(\alpha_1)f(z)}-1\right| \quad (z\in\mathbb{U}), \tag{1.10}$$

where

$$-\frac{\pi}{2} < \alpha < \frac{\pi}{2} \quad \text{and} \quad 0 < \beta \leqslant 1. \tag{1.11}$$

REMARK 1. The function $f \in \mathscr{A}$ defined by

$$W_p^q(\alpha_1)f(z) = \frac{z}{(1-bz)^{2e^{i\alpha}\cos\alpha}} \tag{1.12}$$

is in $\mathscr{W}_p^q(\alpha_1, \alpha, \beta)$, *if and only if*

$$|b| \leqslant \frac{1}{1+2\beta}. \tag{1.13}$$

In fact, for $f(z)$ defined by (1.12), we have

$$\frac{z(W_p^q(\alpha_1)f(z))'}{W_p^q(\alpha_1)f(z)} = 1 + 2e^{i\alpha}\cos\alpha \frac{bz}{1-bz}.$$

From (1.10), we know that $f(z) \in \mathscr{W}_p^q(\alpha_1, \alpha, \beta)$, if and only if

$$\Re\frac{1+bz}{1-bz} > 2\beta\left|\frac{bz}{1-bz}\right| \quad (z \in \mathbb{U}). \tag{1.14}$$

Letting $z \to -|b|/b$ in (1.14), we get (1.13). Conversely, if the inequality (1.13) holds, then

$$\Re\frac{1+bz}{1-bz} \geqslant \frac{1-|bz|}{|1-bz|} > 2\beta\left|\frac{bz}{1-bz}\right| \quad (z \in \mathbb{U}).$$

This completes the proof.

REMARK 2. For specific values assigned to the parameters of the class defined by (1.10), the following relationships are easy to verify:

$$\mathscr{W}_0^1(1, \alpha, \beta) = \mathscr{S}\mathscr{P}(\alpha, \beta) \quad \text{and} \quad \mathscr{W}_0^1(2, \alpha, \beta) = \mathscr{U}\mathscr{C}\mathscr{S}\mathscr{P}(\alpha, \beta)$$

with $A_1 = 1$.

Throughout this paper we assume, unless otherwise stated, that α and β satisfy (1.11).

2. Subordination Theorem

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. We say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and we write $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that

$$|w(z)| \leqslant |z| \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

If $g(z)$ is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

THEOREM 1. *A function $f \in \mathscr{A}$ is in $\mathscr{W}_p^q(\alpha_1,\alpha,\beta)$, if and only if*

$$e^{i\alpha}\frac{z(W_p^q(\alpha_1)f(z))'}{W_p^q(\alpha_1)f(z)} \prec h_\beta(z)\cos\alpha + i\sin\alpha, \tag{2.1}$$

where

$$h_\beta(z) = 1+\frac{1}{2\sin^2\sigma}\left\{\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{2\sigma/\pi}+\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{2\sigma/\pi}-2\right\} \quad (z\in\mathbb{U}), \tag{2.2}$$

$\sigma = \arccos\beta\,(0<\beta<1)$, *when* $\beta = 1$,

$$h_1(z) = 1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \quad (z\in\mathbb{U}). \tag{2.3}$$

Proof. Let us define $w(z) = u+iv$ by

$$e^{i\alpha}\frac{z(W_p^q(\alpha_1)f(z))'}{W_p^q(\alpha_1)f(z)} = w(z)\cos\alpha + i\sin\alpha \quad (z\in\mathbb{U}), \tag{2.4}$$

then (1.9) in conjunction with (1.7) readily gives $w(0) = 1$.

When $0<\beta<1$, the inequality (1.10) can be rewritten as $u > \beta\sqrt{(u-1)^2+v^2}$, which is equivalent to

$$u^2 > \beta^2u^2 - 2\beta^2u + \beta^2 + \beta^2v^2$$

and

$$u > \frac{\beta}{1+\beta}.$$

That is

$$u^2 + \frac{2\beta^2u}{1-\beta^2} - \frac{\beta^2}{1-\beta^2}v^2 > \frac{\beta^2}{1-\beta^2}$$

and

$$u > \frac{\beta}{1+\beta}.$$

Hence, we have

$$\left(u+\frac{\beta^2}{1-\beta^2}\right)^2 - \frac{\beta^2}{1-\beta^2}v^2 > \left(\frac{\beta}{1-\beta^2}\right)^2 \tag{2.5}$$

and

$$u > \frac{\beta}{1+\beta}. \tag{2.6}$$

Thus, the domain of values of $w(z)$ for $z\in\mathbb{U}$ is contained in the hyperbolic region

$$D = \{w = u+iv : u \text{ and } v \text{ satisfy (2.5) and (2.6)}\},$$

and we also note that $h_\beta(0) = w(0) = 1$. In order to prove our theorem, it suffices to show that the function $h_\beta(z)$ given by (2.2) maps $\mathbb{U}$ conformally onto D.

Consider the transformations

$$w_1 = (1-\beta^2)w + \beta^2, \quad w_1 = \frac{1}{2}\left(w_2 + \frac{1}{w_2}\right),$$

$$w_3 = w_2^{\pi/\sigma}(\sigma = \arccos\beta), \quad t = \frac{1}{2}\left(w_3 + \frac{1}{w_3}\right).$$

It is not difficult to verify that the composite function $t = t(w)$ maps

$$D^+ = D \cap \{w = u + iv : v > 0\}$$

conformally onto the upper-half plane $\text{Im}\,(t) > 0$, so that $w = 1$ corresponds to $t = 1$ and $w = \beta/(1+\beta)$ to $t = -1$. Making use of the symmetry principle, this function $t = t(w)$ maps D onto $G = \{t : |\arg(t+1)| < \pi\}$. Since

$$t = 2\left(\frac{1+z}{1-z}\right)^2 - 1$$

maps $\mathbb{U}$ onto G, we see that

$$\begin{aligned} w &= 1 + \frac{1}{2(1-\beta^2)}\left\{\left(t + \sqrt{t^2-1}\right)^{\sigma/\pi} + \left(t + \sqrt{t^2-1}\right)^{-\sigma/\pi} - 2\right\} \\ &= 1 + \frac{1}{2\sin^2\sigma}\left\{\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{2\sigma/\pi} + \left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{2\sigma/\pi} - 2\right\} \\ &= h_\beta(z) \end{aligned}$$

maps $\mathbb{U}$ conformally onto D.

When $\beta = 1$, the inequality (1.10) can be written as $u > \frac{1+v^2}{2}$. It is known ([16]) that the function

$$h_1(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$$

maps $\mathbb{U}$ conformally onto the parabolic region

$$D_1 = \{w = u + iv; u > (v^2+1)/2\}.$$

Therefore, the proof of the theorem is complete. $\square$

COROLLARY 1. *Let $f \in \mathscr{W}_p^q(\alpha_1, \alpha, \beta)$, and $h_\beta(z)$ be given by (2.2) and (2.3). Then for $z \in \mathbb{U}$,*

$$\frac{W_p^q(\alpha_1)f(z)}{z} \prec \exp\left(e^{-i\alpha}\cos\alpha\int_0^z \frac{h_\beta(t)-1}{t}dt\right) \tag{2.7}$$

and

$$\exp\int_0^1 \frac{h_\beta(-\rho)-1}{\rho}d\rho < \left|\left(\frac{W_p^q(\alpha_1)f(z)}{z}\right)^{e^{i\alpha}\sec\alpha}\right| < \exp\int_0^1 \frac{h_\beta(\rho)-1}{\rho}d\rho. \tag{2.8}$$

The bounds in (2.8) are sharp.

Proof. From Theorem 1, we have

$$\frac{e^{i\alpha}}{\cos\alpha}\left(\frac{z(W_p^q(\alpha_1)f(z))'}{W_p^q(\alpha_1)f(z)}-1\right)\prec h_\beta(z)-1 \tag{2.9}$$

for $f(z)\in\mathscr{W}_p^q(\alpha_1,\alpha,\beta)$, where $h_\beta(z)$ is given by (2.2). Since the analytic function $h_\beta(z)-1$ is univalent and starlike (with respect to the origin) in $\mathbb{U}$, therefore, by (2.9) and the result due to Suffridge [22, Theorem 3], we get

$$\frac{e^{i\alpha}}{\cos\alpha}\log\frac{W_p^q(\alpha_1)f(z)}{z}=\frac{e^{i\alpha}}{\cos\alpha}\int_0^z\left(\frac{(W_p^q(\alpha_1)f(t))'}{W_p^q(\alpha_1)f(t)}-\frac{1}{t}\right)dt\prec\int_0^z\frac{h_\beta(t)-1}{t}dt. \tag{2.10}$$

This implies (2.7).

Noting that the univalent function $h_\beta(z)$ maps the disk $|z|<\rho\,(0<\rho\leqslant 1)$ onto a region which is convex and symmetric with respect to the real axis, we see that

$$\int_0^1\frac{h_\beta(-\rho)-1}{\rho}d\rho<\Re\int_0^1\frac{h_\beta(\rho z)-1}{\rho}d\rho<\int_0^1\frac{h_\beta(\rho)-1}{\rho}d\rho$$

for $z\in\mathbb{U}$. Consequently, the subordination (2.10) leads to

$$\int_0^1\frac{h_\beta(-\rho)-1}{\rho}d\rho<\log\left|\left(\frac{W_p^q(\alpha_1)f(z)}{z}\right)^{e^{i\alpha}\sec\alpha}\right|<\int_0^1\frac{h_\beta(\rho)-1}{\rho}d\rho$$

for $z\in\mathbb{U}$, which gives (2.8).

Obviously, the bounds in (2.8) are best possible for the function $f_0(z)\in\mathscr{W}_p^q(\alpha_1,\alpha,\beta)$, defined by

$$W_p^q(\alpha_1)f_0(z)=z\exp\left(e^{-i\alpha}\cos\alpha\int_0^z\frac{h_\beta(t)-1}{t}\right), \tag{2.11}$$

where $h_\beta(z)$ is given by (2.2). The proof is thus complete. □

From (1.7)–(1.9), (2.2), (2.3) and (2.10), we have

COROLLARY 2.
(i) If $f(z)\in\mathscr{W}_p^q(\alpha_1,\alpha,\beta)$ and $0<\beta<1$, then

$$f(z)=z\exp\left\{\frac{e^{-\alpha}\cos\alpha}{2\sin^2\sigma}\int_0^1\frac{1}{\rho}\left\{\left(\frac{1+\sqrt{\rho w(z)}}{1-\sqrt{\rho w(z)}}\right)^{2\sigma/\pi}+\left(\frac{1-\sqrt{\rho w(z)}}{1+\sqrt{\rho w(z)}}\right)^{2\sigma/\pi}-2\right\}d\rho\right\}\times$$

$$\times\left\{z+\sum_{n=2}^{\infty}\frac{(n-1)!\Gamma(\beta_1+B_1(n-1))\cdots\Gamma(\beta_q+B_q(n-1))}{\Gamma(\alpha_1+A_1(n-1))\cdots\Gamma(\alpha_p+A_p(n-1))}\cdot\frac{\prod_{m=1}^{p}(\alpha_m)}{\prod_{m=1}^{q}\Gamma(\beta_m)}\right\};$$

(ii) If $f(z) \in \mathscr{W}_p^q(\alpha_1, \alpha, \beta)$ *and* $\beta = 1$*, then*

$$f(z) = z\exp\left\{\frac{2\cos\alpha e^{-i\alpha}}{\pi^2}\int_0^1 \frac{1}{\rho}\left(\log\frac{1+\sqrt{\rho w(z)}}{1-\sqrt{\rho w(z)}}\right)^2 d\rho\right\}\times$$
$$\times\left\{z+\sum_{n=2}^{\infty}\frac{(n-1)!\Gamma(\beta_1+B_1(n-1))\cdots\Gamma(\beta_q+B_q(n-1))}{\Gamma(\alpha_1+A_1(n-1))\cdots\Gamma(\alpha_p+A_p(n-1))}\cdot\frac{\prod_{m=1}^{p}(\alpha_m)}{\prod_{m=1}^{q}\Gamma(\beta_m)}\right\},$$

where $w(z)$ *is analytic in* $\mathbb{U}$ *with* $w(0) = 0$ *and* $|w(z)| < 1 (z \in \mathbb{U})$.

3. Inclusion relations

We will need the following lemma on the Briot-Bouquet differential subordination.

LEMMA 1. ([10]) *Let* $h(z)$ *be convex univalent in* $\mathbb{U}$*, with* $\Re(\mu h(z)+\gamma) \geqslant 0$*. If* $p(z)$ *is analytic in* $\mathbb{U}$*, with* $p(0) = h(0)$*, then*

$$p(z)+\frac{zp'(z)}{\mu p(z)+\gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

THEOREM 2. *Let* $0 < |\sin\alpha| \leqslant \beta \leqslant 1$ *and*

$$\frac{\alpha_1}{A_1} \geqslant 1-\frac{\sqrt{\beta^2-\sin^2\alpha}}{\cos\alpha+\sqrt{\beta^2-\sin^2\alpha}}. \tag{3.1}$$

Then

$$\mathscr{W}_p^q(\alpha_1+1, \alpha, \beta) \subset \mathscr{W}_p^q(\alpha_1, \alpha, \beta).$$

Proof. For $f(z) \in \mathscr{W}_p^q(\alpha_1+1, \alpha, \beta)$, it follows from Theorem 1 that

$$e^{i\alpha}\frac{z(W_p^q(\alpha_1+1)f(z))'}{W_p^q(\alpha_1+1)f(z)} \prec h_\beta(z)\cos\alpha+i\sin\alpha \quad (z \in \mathbb{U}), \tag{3.2}$$

where $h_\beta(z)$ is given by (2.2).

From (1.7)–(1.9) we have

$$W_p^q(\alpha_1+1)f(z) = \left(1-\frac{A_1}{\alpha_1}\right)W_p^q(\alpha_1)f(z)+\frac{A_1}{\alpha_1}z(W_p^q(\alpha_1)f(z))', \tag{3.3}$$

that is

$$\frac{W_p^q(\alpha_1+1)f(z)}{W_p^q(\alpha_1)f(z)} = \left(1-\frac{A_1}{\alpha_1}\right)+\frac{A_1}{\alpha_1}\left(\frac{z(W_p^q(\alpha_1)f(z))'}{W_p^q(\alpha_1)}\right). \tag{3.4}$$

Differentiating (3.4) logarithmically, we obtain

$$e^{i\alpha}\frac{z(W_p^q(\alpha_1+1)f(z))'}{W_p^q(\alpha_1+1)f(z)} = p(z)+\frac{zp'(z)}{\frac{\alpha_1}{A_1}-1+e^{-i\alpha}p(z)}, \tag{3.5}$$

where

$$p(z) = e^{i\alpha}\frac{z(W_p^q(\alpha_1)f(z))'}{W_p^q(\alpha_1)f(z)}$$

is analytic in $\mathbb{U}$, with $p(0) = h_\beta(0)$.

From (3.2) and (3.5), we obtain

$$p(z) + \frac{zp'(z)}{\frac{\alpha_1}{A_1} - 1 + e^{-i\alpha}p(z)} \prec h_\beta(z)\cos\alpha + i\sin\alpha.$$

(i) Let $|\sin\alpha| < \beta < 1$, $h_\beta(z) = u + iv$, where $h_\beta(z)$ is given by (2.2). From the proof of Theorem 1, we find that $u > \beta\sqrt{(u-1)^2+v^2}$ and $u > \frac{\beta}{1+\beta}$. That is, $|v| < \frac{1}{\beta}\sqrt{u^2-\beta^2(u-1)^2}$ and $u > \frac{\beta}{1+\beta}$. Hence, we have

$$\min_{|z|=1}\Re\left\{e^{-i\alpha}h_\beta(z)\right\} = \min_{|z|=1}\{u\cos\alpha - v\sin\alpha\} = \min_{u\geqslant\beta/(1+\beta)} g(u),$$

where

$$g(u) = u\cos\alpha - \frac{|\sin\alpha|}{\beta}\sqrt{u^2-\beta^2(u-1)^2}.$$

Note that

$$\begin{aligned} g'(u) &= \frac{\beta\cos\alpha\sqrt{u^2-\beta^2(u-1)^2} - |\sin\alpha|(\beta^2+(1-\beta^2)u)}{\beta\sqrt{u^2-\beta^2(u-1)^2}} \\ &= \frac{(1-\beta^2)(\beta^2-\sin^2\alpha)u^2 + 2\beta^2(\beta^2-\sin^2\alpha)u - \beta^4}{\beta\sqrt{u^2-\beta^2(u-1)^2}(\beta\cos\alpha\sqrt{u^2-\beta^2(u-1)^2} + |\sin\alpha|(\beta^2+(1-\beta^2)u))} \end{aligned} \tag{3.6}$$

for $u > \beta/(1+\beta)$. Since $|\sin\alpha| < \beta < 1$, it follows from (3.6) that the function $g(u)$ $(u \geqslant \beta/(1+\beta))$ attains its minimum at $u = u_0$, where

$$u_0 = \frac{\beta^2}{1-\beta^2}\left(\frac{\cos\alpha}{\sqrt{\beta^2-\sin^2\alpha}} - 1\right) \geqslant \frac{\beta}{1+\beta}.$$

Now

$$\begin{aligned} g(u_0) &= u_0\cos\alpha - \frac{|\sin\alpha|}{\beta}\sqrt{u_0^2-\beta^2(u_0-1)^2} \\ &= \frac{\beta^2\cos\alpha}{1-\beta^2}\left(\frac{\cos\alpha}{\sqrt{\beta^2-\sin^2\alpha}} - 1\right) - \frac{\sin^2\alpha}{\sqrt{\beta^2-\sin^2\alpha}} \\ &= \frac{\sqrt{\beta^2-\sin^2\alpha} - \beta^2\cos\alpha}{1-\beta^2}, \end{aligned}$$

and therefore,

$$\min_{|z|=1} \Re\left\{e^{-i\alpha}(h_\beta(z)\cos\alpha + i\sin\alpha)\right\} = g(u_0)\cos\alpha + \sin^2\alpha$$
$$= \frac{\sqrt{\beta^2-\sin^2\alpha}}{\cos\alpha+\sqrt{\beta^2-\sin^2\alpha}}. \tag{3.7}$$

Since the function $Q(z) = e^{-i\alpha}(h_\beta(z)\cos\alpha + i\sin\alpha)$ is convex (and univalent) in $\mathbb{U}$ and

$$\Re\left\{\frac{\alpha_1}{A_1} - 1 + Q(z)\right\} > \frac{\alpha_1}{A_1} - 1 + \frac{\sqrt{\beta^2-\sin^2\alpha}}{\cos\alpha+\sqrt{\beta^2-\sin^2\alpha}},$$

it follows from (3.1) and Lemma 1 that

$$p(z) = e^{i\alpha}\frac{zf'(z)}{f(z)} \prec h_\beta(z)\cos\alpha + i\sin\alpha.$$

Therefore, $f(z) \in \mathscr{W}_p^q(\alpha_1,\alpha,\beta)$ $(|\sin\alpha| < \beta < 1)$.

(ii) When $0 < |\sin\alpha| = \beta < 1$, we have

$$g(u) = u\sqrt{1-\beta^2} - \sqrt{u^2-\beta^2(u-1)^2} \quad (u \geqslant \beta/(1+\beta)).$$

It is clear that $g'(u) < 0$ for $u > \beta/(1+\beta)$, and so

$$\inf_{u\geqslant\beta/(1+\beta)} g(u) = g(+\infty) = -\frac{\beta^2}{\sqrt{1-\beta^2}}.$$

Hence

$$\min_{|z|=1} \Re\left\{e^{-\alpha}(h_\beta(z)\cos\alpha + i\sin\alpha)\right\} = g(+\infty)\cos\alpha + \sin^2\alpha = 0. \tag{3.8}$$

It follows from (3.1) and Lemma 1 that

$$p(z) = e^{i\alpha}\frac{zf'(z)}{f(z)} \prec h_\beta(z)\cos\alpha + i\sin\alpha.$$

Therefore $f(z) \in \mathscr{W}_p^q(\alpha_1,\alpha,\beta)$ $(0 < |\sin\alpha| = \beta < 1)$.

(iii) When $\beta = 1$, $g_1(u) = u\cos\alpha - |\sin\alpha|\sqrt{2u-1}$ $(u \geqslant 1/2)$. Then

$$\min_{u\geqslant 1/2} g_1(u) = g_1\left(\frac{1}{2\cos^2\alpha}\right) = \frac{1-2\sin^2\alpha}{2\cos\alpha}$$

and hence

$$\min_{|z|=1} \Re\left\{e^{-i\alpha}(h_1(z)\cos\alpha + i\sin\alpha)\right\} = \cos\alpha \min_{u\geqslant 1/2} g_1(u) + \sin^2\alpha = 1/2. \tag{3.9}$$

It follows from (3.1) and Lemma 1 that

$$p(z) = e^{i\alpha}\frac{zf'(z)}{f(z)} \prec h_1(z)\cos\alpha + i\sin\alpha.$$

Hence, $f(z) \in \mathscr{W}_p^q(\alpha_1,\alpha,1)$. This completes the proof. □

THEOREM 3. *If* $0 < |\sin\alpha| \leqslant \beta \leqslant 1$, *then* $\mathscr{W}_p^q(\alpha_1,\alpha,\beta) \subset S^*(\rho(\alpha,\beta))$, *where*

$$\rho(\alpha,\beta) = \frac{\sqrt{\beta^2-\sin^2\alpha}}{\cos\alpha + \sqrt{\beta^2-\sin^2\alpha}}, \tag{3.10}$$

and the order $\rho(\alpha,\beta)$ *is sharp.*

Proof. Let $f(z) \in \mathscr{W}_p^q(\alpha_1,\alpha,\beta)$ $(|\sin\alpha| \leqslant \beta \leqslant 1)$. Then, by (3.7)–(3.9) and Theorem 1, we conclude that $f(z) \in S^*(\rho(\alpha,\beta))$, where $\rho(\alpha,\beta)$ is given by (3.10), and the order $\rho(\alpha,\beta)$ is sharp for the function $f_0(z)$ given by (2.11). □

Lastly, we examine the closure properties of the class $\mathscr{W}_p^q(\alpha_1,\alpha,\beta)$ under the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ which is defied by

$$L_cf(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad (\Re(c) > -1). \tag{3.11}$$

THEOREM 4. *Let* $f(z) \in \mathscr{W}_p^q(\alpha_1,\alpha,\beta)$ $(0 < |\sin\alpha| \leqslant \beta \leqslant 1)$ *and*

$$\Re(c) \geqslant -\frac{\sqrt{\beta^2-\sin^2\alpha}}{\cos\alpha + \sqrt{\beta^2-\sin^2\alpha}}. \tag{3.12}$$

Then $L_cf(z)$ *belongs to* $\mathscr{W}_p^q(\alpha_1,\alpha,\beta)$, *where* $L_cf(z)$ *is given by (3.11) and* $0 < |\sin\alpha| \leqslant \beta \leqslant 1$.

Proof. It follows from (3.11) that

$$(c+1)W_p^q(\alpha_1)f(z) = cW_p^q(\alpha_1)L_cf(z) + z(W_p^q(\alpha_1)L_cf(z))', \tag{3.13}$$

that is

$$\frac{W_p^q(\alpha_1)f(z)}{W_p^q(\alpha_1)L_cf(z)} = \frac{c}{c+1} + \frac{1}{c+1}\frac{z(W_p^q(\alpha_1)L_cf(z)}{W_p^q(\alpha_1)L_cf(z)}. \tag{3.14}$$

Differentiating (3.14) logarithmically, we obtain

$$e^{i\alpha}\frac{z(W_p^q(\alpha_1)f(z))'}{W_p^q(\alpha_1)f(z)} = q(z)+\frac{zq'(z)}{c+e^{-i\alpha}q(z)}, \tag{3.15}$$

where

$$q(z)=e^{i\alpha}\frac{z(W_p^q(\alpha_1)L_cf(z))'}{W_p^q(\alpha_1)L_cf(z)}$$

is analytic in $\mathbb{U}$, with $q(0)=h(0)$. Noting that $f(z)\in\mathscr{W}_p^q(\alpha_1,\alpha,\beta)$ $(0<|\sin\alpha|\leqslant\beta\leqslant 1)$, hence by Theorem 1 and (3.15), we obtain

$$q(z)+\frac{zq'(z)}{c+e^{-i\alpha}q(z)} \prec h_\beta(z)\cos\alpha+i\sin\alpha,$$

where $h_\beta(z)$ is given by (2.2) and (2.3).

Since the function $Q(z)=e^{-i\alpha}(h(z)\cos\alpha+i\sin\alpha)$ is convex (and univalent) in $\mathbb{U}$ and

$$\Re\{c+Q(z)\}>\Re c+\frac{\sqrt{\beta^2-\sin^2\alpha}}{\cos\alpha+\sqrt{\beta^2-\sin^2\alpha}}\geqslant 0 \quad (z\in\mathbb{U})$$

by (3.7)–(3.9) and (3.12), therefore, by Lemma 1, we have

$$q(z)=e^{i\alpha}\frac{z(W_p^q(\alpha_1)L_cf(z))'}{W_p^q(\alpha_1)L_cf(z)} \prec h(z)\cos\alpha+i\sin\alpha,$$

that is $L_cf(z)\in\mathscr{W}_p^q(\alpha_1,\alpha,\beta)$ on using Theorem 1. This completes the proof. □

THEOREM 5. *Let* $c=\frac{\alpha_1}{A_1}-1$ (>-1). *Then* $f(z)\in\mathscr{W}_p^q(\alpha_1,\alpha,\beta)$ *if and only if* $L_cf(z)\in\mathscr{W}_p^q(\alpha_1+1,\alpha,\beta)$.

Proof. If $f(z)\in\mathscr{W}_p^q(\alpha_1,\alpha,\beta)$, it follows from Theorem 1 that

$$e^{i\alpha}\frac{z(W_p^q(\alpha_1)f(z))'}{W_p^q(\alpha_1)f(z)} \prec h_\beta(z)\cos\alpha+i\sin\alpha,$$

where $h_\beta(z)$ is given by (2.2).

Let $c=\frac{\alpha_1}{A_1}-1$. It follows from (3.3) and (3.13) that

$$\begin{aligned} W_p^q(\alpha_1)f(z) &= \frac{cW_p^q(\alpha_1)L_cf(z)+z(W_p^q(\alpha_1)L_cf(z))'}{c+1} \\ &= \left(1-\frac{A_1}{\alpha_1}\right)W_p^q(\alpha_1)L_cf(z)+\frac{A_1}{\alpha_1}z(W_p^q(\alpha_1)L_cf(z))' \\ &= W_p^q(\alpha_1+1)L_cf(z) \quad (z\in\mathbb{U}). \end{aligned}$$

Hence

$$e^{i\alpha}\frac{z(W_p^q(\alpha_1+1)L_cf(z))'}{W_p^q(\alpha_1+1)L_cf(z)}=e^{i\alpha}\frac{z(W_p^q(\alpha_1)f(z))'}{W_p^q(\alpha_1)f(z)}\prec h_\beta(z)\cos\alpha+i\sin\alpha,$$

and it follows from Theorem 1 that $L_cf(z)\in\mathscr{W}_p^q(\alpha_1+1,\alpha,\beta)$.

Conversely, if $L_cf(z)\in\mathscr{W}_p^q(\alpha_1+1,\alpha,\beta)$, then it is easy to verify that $f(z)\in\mathscr{W}_p^q(\alpha_1,\alpha,\beta)$ also. This completes the proof. $\square$

Acknowledgements. This work was partially supported by *the National Natural Science Foundation of China* (Grant No. 11171045; 11471163)

REFERENCES

[1] S. D. BERNARDI, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc. **135** (1969), 429–446.

[2] B. C. CARLSON AND D. B. SHAFFER, *Starlike and prestarlike hypergeometric functions,* SIAM J. Math. Anal. **15** (1984), 737–745.

[3] J. DZIOK AND R. K. RAINA, *Some results based on first order differential subordination with the Wrights generalized hypergeometric function,* Comment. Math. Univ. St. Pauli **58**(2009), 87–94.

[4] J. DZIOK AND R. K. RAINA, *Families of analytic functions associated with theWright generalized hypergeometric function*, Demonstratio Math. **37** (2004), 533–542.

[5] J. DZIOK AND H. M. SRIVASTAVA, *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput. **103** (1999), 1–13.

[6] A. GANGADHARAN, T. N. SHANMUGAM AND H. M. SRIVASTAVA, *Generalized hypergeometric function associated with k-uniformly convex functions*, Comput. Math. Appl. **44** (2002), 1515–1526.

[7] A. W. GOODMAN, *On uniformly convex functions,* Ann. Polon. Math. **56** (1991), 87–92.

[8] A. W. GOODMAN, *On uniformly starlike functions,* J. Math. Anal. Appl. **155** (1991), 364–370.

[9] S. KANAS AND H. M. SRIVASTAVA, *Linear operators associated with k-uniformly convex functions,* Integral Transform. Spec. Funct. **9** (2000), 121–132.

[10] S. S. MILLER AND P. T. MOCANU, *Differential Subordinations: Theory and Application,* vol. **225** of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, (2000).

[11] S. OWA, *On uniformly convex functions*, Math. Japon. **48** (1998), 377–384.

[12] R. K. RAINA, *Certain subclasses of analytic functions with fixed argument of coefficients involving the Wright's function,* Tamsui Oxford J. Maths. **22** (2006), 51–59.

[13] R. K. RAINA AND P. SHARMA, *Harmonic univalent functions associated with Wright's generalized hypergeometric functions,* Integral Transform. Spec. Funct. **22**, 8 (2011), 561–572.

[14] R. K. RAINA AND G. MURUGUSUNDARAMOORTHY, *On a subclass of harmonic functions associated with Wright's generalized hypergeometric function,* Hacettepe J. Math. Stat. **38**, 2 (2009), 129–136.

[15] R. K. RAINA, G. MURUGUDUNDARMOORTHY AND N. MAGESH, *Subordination and superordination properties for analytic functions involving Wright's functions,* Le Matematiche **66**, 1 (2011), 65–67.

[16] F. RØNNING, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. **118** (1993), 189–196.

[17] S. RUSCHEWEYH, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109–115.

[18] N. SARKAR, P. GOSWAMI, J. DZIOK AND J. SOKÓ L, *Subordination for multivalent analytic functions associated with Wright generalized hypergeometric function*, Tamkang J. Math. **44**, 1 (2013), 61–71.

[19] J. SOKÓŁ, *On some applications of the Dziok-Srivastava operator*, Appl. Math. Comp. **201** (2008), 774–780.

[20] J. SOKÓŁ AND K. PIEJKO, *On the Dziok-Srivastava operator under multivalent analytic functions*, Appl. Math. Comp. **177** (2006), 839–843.

[21] H. M. SRIVASTAVA, DING-GONG YANG AND NENG XU, *Subordinations for Multivalent Analytic Functions Associated with the Dziok-Srivastava Operator,* Integral Transform Spec. Funct. **20** (2009), 581–606.

[22] T. J. SUFFRIDGE, *Some remarks on convex maps of the unit disk*, Duke Math. J. **37** (1970), 775–777.

[23] E. M. WRIGHT, *The asymptotic expansion of the generalized hypergeometric function,* Proc. London Math. Soc. **46** (1946), 389–408.

[24] N. XU, D.-G. YANG AND J. SOKÓL, *A Class of Analytic Functions Involving the Dziok-Srivastava Operator,* J. Inequal. Appl. **138** (2013), 1–15.

(Received January 20, 2014)

Neng Xu
Department of Mathematics, Changshu Institute of Technology
Changshu 215500, Jiangsu
People's Republic of China
e-mail: xun@cslg.edu.cn

R. K. Raina
M. P. University of Agriculture and Technology
Udaipur 313001, Rajasthan
India
and
10/11, Ganpati Vihar Opposite Sector 5
Udaipur-313002 Rajasthan
India
e-mail: rkraina_7@hotmail.com