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PROBABILITY DISTRIBUTIONS OF EXTREMES OF SELF–SIMILAR GAUSSIAN RANDOM FIELDS

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Abstract. We have obtained some upper bounds for probability distributions of extremes of a self-similar Gaussian random field with stationary rectangular increments, which is defined on a compact space. In the paper we also present the probability distributions of extremes for normalized self-similar Gaussian random fields with stationary rectangular increments defined on $\mathbb{R}_+^2$. In our work we have used the techniques developed for self-similar fields and based on the classical series analysis of the supremum distribution for Gaussian fields.

1. Introduction

A self-similar process is a stochastic process that is invariant in distribution under suitable scaling of time and space. A random process $\{X(t), t \in \mathbb{R}\}$ is a self-similar process with index $H > 0$ if for all $a > 0$: $\{X(t), t \in \mathbb{R}\} \stackrel{d}{=} \{a^{-H}X(at), t \in \mathbb{R}\}$, where "$\stackrel{d}{=}$" denotes equality of finite-dimensional distributions. We refer to Embrechts and Maejima [7] and Samorodnitsky and Taqqu [16] for extensive surveys of results and techniques for self-similar processes.

In this paper we consider self-similar random fields, which generalize the self-similar random processes. More precisely, we deal with anisotropic self-similar random fields, which means that their indexes of self-similarity are different for different coordinates. We denote $\mathbb{R}_+ = [0, +\infty)$.

DEFINITION 1. A real valued random field $\{X(\mathbf{t}), \mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}_+^n\}$ is a self-similar field with index $\mathbf{H} = (H_1, \ldots, H_n) \in (0, +\infty)^n$ if

$$\left\{X(a_1t_1, \ldots, a_nt_n),\ \ \mathbf{t} \in \mathbb{R}_+^n\right\} \stackrel{d}{=} \left\{a_1^{H_1} \cdots a_n^{H_n} X(\mathbf{t}),\ \ \mathbf{t} \in \mathbb{R}_+^n\right\}$$

for all $a_1 > 0, \ldots, a_n > 0$.

An interest to anisotropic self-similar random fields is motivated by applications coming from climatological and environmental sciences (see [13, 14]). Several authors have proposed to apply such random fields for modeling phenomena in spatial statistics, stochastic hydrology and image processing (see [3, 4, 5]).

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DEFINITION 2. A normalized fractional Brownian sheet with Hurst index $\mathbf{H} = (H_1, \ldots, H_n)$, $0 < H_i < 1$, $i = \overline{1,n}$, is a centered Gaussian random field $B_{\mathbf{H}} = \{B_{\mathbf{H}}(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^n\}$ with the following covariance function

$$\mathbf{E}(B_{\mathbf{H}}(\mathbf{t})B_{\mathbf{H}}(\mathbf{s})) = 2^{-n}\prod_{i=1}^{n}\left(|t_i|^{2H_i} + |s_i|^{2H_i} - |t_i - s_i|^{2H_i}\right), \quad \mathbf{t}, \mathbf{s} \in \mathbb{R}_+^n.$$

It is a self-similar field with index $\mathbf{H} = (H_1, \ldots, H_n)$ by Definition 1.

Further in the paper, we assume that the fields under consideration satisfy Definition 1. Moreover, we shall consider only the case $n = 2$ since switching to the parameter of a higher dimension is rather technical.

Denote $\mathbf{0} = (0,0)$.

DEFINITION 3. Let $X = \{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^2\}$ be a self-similar field with index $\mathbf{H} = (H_1, H_2) \in (0, +\infty)^2$. For any $\mathbf{u} = (u_1, u_2) \in \mathbb{R}_+^2$ and any $\mathbf{v} = (v_1, v_2) \in \mathbb{R}_+^2$ such that $v_1 > u_1,\ v_2 > u_2$ let us define

$$\Delta_{\mathbf{u}} X(\mathbf{v}) = X(v_1, v_2) - X(u_1, v_2) - X(v_1, u_2) + X(u_1, u_2).$$

The field X admits stationary rectangular increments if for any $\mathbf{u} = (u_1, u_1) \in \mathbb{R}_+^2$

$$\{\Delta_{\mathbf{u}} X(\mathbf{u} + \mathbf{h}), \mathbf{h} \in \mathbb{R}_+^2\} \stackrel{d}{=} \{\Delta_{\mathbf{0}} X(\mathbf{h}), \mathbf{h} \in \mathbb{R}_+^2\}.$$

The fractional Brownian sheet has stationary rectangular increments. The proof of this property for the $\mathbb{R}_+^2$ case can be found in the paper [2]. A similar property for the case $n > 2$ can be easily proved as well.

The aim of the paper is to obtain the upper bound for probability distributions of extremes of normalized self-similar Gaussian random fields with stationary rectangular increments. These probabilities can be used for estimation of asymptotic growth of sample paths of a fractional Brownian sheet. Furthermore, these probabilities can be applied in investigation of asymptotic behavior of a fractional derivative of a fractional Brownian motion, which is used in the analysis of a non-standard maximum likelihood estimate for unknown drift parameter in the stochastic differential equations driven by the fractional Brownian motion (see Kozachenko et al. [11]).

To achieve this goal we use the results from the theory of extremes for Gaussian processes (Kozachenko et al. [12]). This theory, in turn, is based on the theory of metric spaces. To apply these results we need to choose an appropriate compact metric space $(\mathbf{T}, \rho)$ and to estimate the variance of the increments. Since we work with anisotropic field we expect that chosen metric has different geometric characteristics along different directions. So, we use two metrics $\rho_1(\mathbf{t}, \mathbf{s}) = \max_{i=1,2}|t_i - s_i|$, $\mathbf{t},\ \mathbf{s} \in \mathbf{T} \subset \mathbb{R}_2^+$ and $\rho_2(\mathbf{t}, \mathbf{s}) = \sum_{i=1,2}|t_i - s_i|^{H_i}$, $\mathbf{t},\ \mathbf{s} \in \mathbf{T} \subset \mathbb{R}_2^+$, where $\mathbf{H} = (H_1, H_2) \in (0,1)^2$ is the index of self-similarity of the corresponding random field. The second metric has played an important role in studying anisotropic Gaussian fields and self-similar random fields (see [18]).

There exist many papers devoted to a study of the distribution of the supremum of Gaussian random processes (for example [1, 15]). Exponential estimates of the tails of supremum distributions have been established for certain classes of Gaussian random processes ([6, 9]). I this paper we establish some properties of the covariance functions for considered centered Gaussian self-similar random fields. Thus we obtain more accurate inequalities for supremum distributions for such fields.

The main point in the proofs of this paper is the self-similar property of considered fields. This yields the similar behavior of sample paths on compact subsets. From the theory of extremes for Gaussian processes we get the upper bounds for the probabilities defined on compact sets. Hence we expand $\mathbb{R}_+^2$ into the union of the compact subsets and apply the inequalities for probabilities in each subset. We use the techniques of the self-similar fields based on the classical series analysis for finding the upper bound of the supremum distribution for Gaussian fields. Several results in this paper are obtained by the optimization procedure.

The paper is organized as follows. In Section 2, we present the probability distributions of extremes of the Gaussian fields defined on compact spaces and a bound for the variance of increments in the case of a self-similar field. In Section 3 we establish the probability distributions of extremes of the fields defined on a compact metric space $(\mathbf{T},\rho_1)$ and derive the upper bounds for such probabilities of the normalized field defined on $\mathbb{R}_+^2$. In Section 4 we obtain the probability distribution of extremes of the normalized self-similar Gaussian field defined on a metric space $(\mathbf{T},\rho_2)$.

2. Probability distributions of extremes of a Gaussian field defined on a compact space

Let $(\Omega,\mathrm{F},\mathbf{P})$ be a complete probability space satisfying the standard assumptions. It is assumed that all processes under consideration are defined on this space.

The next theorem follows from Theorem 2.8 of [17] or it could be obtained from Lemma 3.2 of [12].

THEOREM 2.1. *Let $(\mathbf{T},\rho)$ be a compact metric space and $X=\{X(\mathbf{t}),\mathbf{t}\in\mathbf{T})\}$ be a separable centered Gaussian process. Suppose there exists such a continuous monotonically increasing function $\sigma:\mathbb{R}_+\to(0,+\infty)$, $\sigma(0)=0$ that the following inequality holds*

$$\sup_{\rho(\mathbf{t},\mathbf{s})\leqslant h}\left(\mathbf{E}(X(\mathbf{t})-X(\mathbf{s}))^2\right)^{1/2}\leqslant\sigma(h). \tag{1}$$

Let

$$\beta=\sigma\left(\inf_{\mathbf{s}\in\mathbf{T}}\sup_{\mathbf{t}\in\mathbf{T}}\rho(\mathbf{t},\mathbf{s})\right),\qquad \gamma=\sup_{\mathbf{u}\in\mathbf{T}}\left(\mathbf{E}[X^2(\mathbf{u})]\right)^{1/2}. \tag{2}$$

We denote as $N(\varepsilon)$ the minimal number of closed ρ-balls with radius ε needed to cover the space $(\mathbf{T},\rho)$. Let $r:[1,+\infty)\to(0,+\infty)$ be such a continuous function that a function $r(e^t)$, $t>0$ is convex. If

$$\int_0^{+\infty}r(N(\sigma^{(-1)}(u)))du<\infty,$$

then for all $\lambda > 0,\ 0 < p < 1,\ \varepsilon > 0$

$$I_{\mathbf{T}}(\varepsilon) := \mathbf{P}\left\{\sup_{\mathbf{t}\in\mathbf{T}} |X(\mathbf{t})| > \varepsilon\right\} \leqslant 2\exp\left\{\frac{1}{2}\frac{\lambda^2\gamma^2}{1-p} + p\frac{\lambda^2\beta^2}{2(1-p)^2} - \lambda\varepsilon\right\} \times \\ \times r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p} r\left(N(\sigma^{(-1)}(u))\right) du\right). \tag{3}$$

We shall minimize the right-hand side of (3) with respect to $\lambda > 0$.

COROLLARY 2.2. *Under the conditions of Theorem 2.1 we have*

$$I_{\mathbf{T}}(\varepsilon) \leqslant 2\exp\left\{-\frac{\varepsilon^2(1-p)}{2\left(\gamma^2 + \frac{\beta^2 p}{1-p}\right)}\right\} r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p} r\left(N(\sigma^{(-1)}(u))\right) du\right). \tag{4}$$

Proof. Consider the right-hand side of (3). To prove the corollary it is sufficient to minimize the following value

$$\frac{1}{2}\frac{\lambda^2\gamma^2}{1-p} + p\frac{\lambda^2\beta^2}{2(1-p)^2} - \lambda\varepsilon.$$

Differentiating this expression with respect to λ, we get

$$\frac{d}{d\lambda}\left(\frac{1}{2}\frac{\lambda^2\gamma^2}{1-p} + p\frac{\lambda^2\beta^2}{2(1-p)^2} - \lambda\varepsilon\right) = \frac{\lambda\gamma^2}{1-p} + p\frac{\lambda\beta^2}{(1-p)^2} - \varepsilon.$$

Then, the minimum is achieved if

$$\lambda = \lambda^* = -\frac{\varepsilon(1-p)}{2\left(\gamma^2 + \frac{\beta^2 p}{1-p}\right)}.$$

If we replace λ by λ^* in (3), we obtain (4). □

Throughout the paper the field $X = \{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^2_+\}$ is a Gaussian self-similar random field with index $\mathbf{H} = (H_1, H_2) \in (0,1)^2$ and with stationary rectangular increments. Denote $\mathbf{1} = (1,1)$. Evidently,

$$\mathbf{E}[X(\mathbf{t})]^2 = t_1^{2H_1} t_2^{2H_2}\mathbf{E}[X^2(\mathbf{1})], \quad \mathbf{t} = (t_1, t_2) \in \mathbb{R}^2_+.$$

In what follows we need some auxiliary results.

LEMMA 2.3. *For all* $\mathbf{s} = (s_1, s_2) \in \mathbb{R}^2_+,\ \mathbf{t} = (t_1, t_2) \in \mathbb{R}^2_+$ *we have*

$$\mathbf{E}\left[X(\mathbf{t}) - X(s_1, t_2)\right]^2 = |t_1 - s_1|^{2H_1} t_2^{2H_2}\mathbf{E}X^2(\mathbf{1}), \tag{5}$$

$$\mathbf{E}\left[X(s_1, t_2) - X(\mathbf{s})\right]^2 = |t_2 - s_2|^{2H_2} s_1^{2H_1}\mathbf{E}X^2(\mathbf{1}). \tag{6}$$

Proof. Without loss of generality suppose that $s_1 \leqslant t_1$. It follows from self-similarity that for any $s \in \mathbb{R}_+ :\ X(s,0) = X(0,s) = 0$ a.s. Then the left-hand side of (5) equals

$$\mathbf{E}\left(X(\mathbf{t}) - X(t_1,0) - X(s_1,t_2) + X(s_1,0)\right)^2 = \mathbf{E}\left(\Delta_{s_1,0} X(\mathbf{t})\right)^2.$$

Stationarity of the increments implies that

$$\mathbf{E}\left(\Delta_{s_1,0} X(\mathbf{t})\right)^2 = \mathbf{E}\left(\Delta_{\mathbf{0}} X(t_1 - s_1, t_2)\right)^2 = \mathbf{E}\left(X(t_1 - s_1, t_2)\right)^2.$$

Further, self-similarity implies that

$$\mathbf{E}\left(X(\mathbf{t}) - X(s_1,t_2)\right)^2 = \mathbf{E}\left(X(t_1 - s_1, t_2)\right)^2 = |t_1 - s_1|^{2H_1} t_2^{2H_2} \mathbf{E} X^2(\mathbf{1}).$$

The proof of the equality (6) can be done in a similar way. □

LEMMA 2.4. *Assume that* $\mathbf{E}X^2(\mathbf{1}) = 1$. *For all* $\mathbf{s} = (s_1,s_2) \in \mathbb{R}_+^2$, $\mathbf{t} = (t_1,t_2) \in \mathbb{R}_+^2$ *we have*

$$\left(\mathbf{E}\left[X(\mathbf{t}) - X(\mathbf{s})\right]^2\right)^{1/2} \leqslant |t_1 - s_1|^{H_1} t_2^{H_2} + |t_2 - s_2|^{H_2} s_1^{H_1}. \tag{7}$$

Proof. Using the Minkowski inequality, we get

$$\begin{aligned}\left(\mathbf{E}\left[X(\mathbf{t}) - X(\mathbf{s})\right]^2\right)^{1/2} &= \left(\mathbf{E}\left[X(\mathbf{t}) - X(s_1,t_2) + X(s_1,t_2) + X(\mathbf{s})\right]^2\right)^{1/2} \\ &\leqslant \left(\mathbf{E}\left[X(\mathbf{t}) - X(s_1,t_2)\right]^2\right)^{1/2} + \left(\mathbf{E}\left[X(s_1,t_2) - X(\mathbf{s})\right]^2\right)^{1/2}.\end{aligned}$$

It follows from Lemma 2.3 that

$$\mathbf{E}\left[X(\mathbf{t}) - X(s_1,t_2)\right]^2 = |t_1 - s_1|^{2H_1} t_2^{2H_2},$$

and

$$\mathbf{E}\left[X(s_1,t_2) - X(\mathbf{s})\right]^2 = |t_2 - s_2|^{2H_2} s_1^{2H_1}.$$

Hence, inequality (7) holds. □

3. Random fields on space $(\mathbf{T}, \rho_1)$

In this section we put $\rho(\mathbf{t},\mathbf{s}) = \rho_1(\mathbf{t},\mathbf{s}) = \max_{i=1,2} |t_i - s_i|$, $\mathbf{t}$, $\mathbf{s} \in \mathbf{T} \subset \mathbb{R}_2^+$.

COROLLARY 3.1. *Let* $\sigma(h) = Ch^\alpha$, $0 < \alpha \leqslant 1$, $C > 0$ *and* $\mathbf{T} = [0,T]^2$ *in Theorem 2.1. Then*

$$I_{[0,T]^2}(\varepsilon) \leqslant 8 \exp\left\{ - \frac{\varepsilon^2 (1-p)}{2\left(\gamma^2 + \frac{C^2 T^{2\alpha} p}{2^{2\alpha}(1-p)}\right)} \right\} \left(\frac{e}{p}\right)^{2/\alpha} \tag{8}$$

for all $0 < p < 1$ *and* $\varepsilon > 0$.

Proof. We have

$$\beta = C\left(\frac{T}{2}\right)^{\alpha}, \qquad N(u) \leqslant \left(\frac{TC^{1/\alpha}}{2u^{1/\alpha}} + 1\right)^{2}.$$

Put $r(v) = v^{\mu},\ v \in \mathbb{R}_{+},\ 0 < \mu < \alpha/2$. It follows from Corollary 2.2 that

$$I_{[0,T]^2}(\varepsilon) \leqslant 2\exp\left\{-\frac{\varepsilon^2}{2\left(\gamma^2 + \frac{C^2T^{2\alpha}p}{2^{2\alpha}(1-p)}\right)}\right\}Z(p),$$

where

$$Z(p) = \left(\frac{1}{\beta p}\int_0^{\beta p}\left(N(\sigma^{(-1)}(u))\right)^{\mu} du\right)^{1/\mu}. \tag{9}$$

Since $u \leqslant \beta p,$ we have

$$\frac{TC^{1/\alpha}}{2u^{1/\alpha}} \geqslant \frac{TC^{1/\alpha}}{2(\beta p)^{1/\alpha}} > \frac{2TC^{1/\alpha}}{2TC^{1/\alpha}} = 1.$$

Therefore, we obtain

$$Z(p) \leqslant \left(\frac{1}{\beta p}\int_0^{\beta p}\left(\frac{TC^{1/\alpha}}{2u^{1/\alpha}} + 1\right)^{2\mu} du\right)^{1/\mu} \leqslant \left(\frac{1}{\beta p}\int_0^{\beta p}\left(\frac{TC^{1/\alpha}}{u^{1/\alpha}}\right)^{2\mu} du\right)^{1/\mu}$$

$$= T^2C^{2/\alpha}\frac{1}{(\beta p)^{1/\mu}}\left(\int_0^{\beta p}\left(\frac{1}{u^{1/\alpha}}\right)^{2\mu} du\right)^{1/\mu} = T^2C^{2/\alpha}\frac{1}{(\beta p)^{2/\alpha}}\frac{1}{(1-2\mu/\alpha)^{1/\mu}}.$$

As $\mu \to 0$, we have

$$Z(p) \leqslant T^2C^{2/\alpha}\frac{1}{(\beta p)^{2/\alpha}}e^{2/\alpha} = 4\left(\frac{e}{p}\right)^{2/\alpha}.$$

The last inequality completes the proof. □

From now on we denote $H = \min\{H_1, H_2\}$, where $\mathbf{H} = (H_1, H_2) \in (0,1)^2$ is the index of self-similarity.

PROPOSITION 3.2. *Let* $\mathbf{T} = [0,1]^2$, $\rho = \rho_1$, *and* $X = \{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^2_+\}$ *be a centered Gaussian self-similar random field of order* $\mathbf{H} = (H_1, H_2) \in (0,1)^2$ *with stationary rectangular increments. Then for all* $0 < p < 1$ *we have*

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in[0,1]^2}|X(\mathbf{t})| > \varepsilon\right\} \leqslant 8\exp\left\{-\frac{\varepsilon^2(1-p)}{2\left(1 + \frac{4p}{2^{2H}(1-p)}\right)}\right\}\left(\frac{e}{p}\right)^{2/H}, \quad \varepsilon > 0. \tag{10}$$

Proof. We have from inequality (7) that for all $\mathbf{t},\mathbf{s}\in[0,1]^2$

$$\left(\mathbf{E}[X(\mathbf{t})-X(\mathbf{s})]^2\right)^{1/2}\leqslant|t_1-s_1|^{H_1}t_2^{H_2}+|t_2-s_2|^{H_2}s_1^{H_1}$$

$$\leqslant|t_1-s_1|^{H_1}+|t_2-s_2|^{H_2}\leqslant 2\max_{i=1,2}|t_i-s_i|^{H_i}\leqslant 2\max_{i=1,2}|t_i-s_i|^{H}=2[\rho(\mathbf{s},\mathbf{t})]^H.$$

Therefore, it follows from (1) that $\sigma(h)=2h^H$ and $\gamma=1$, where γ is defined in (2). Thus, inequality (10) follows from (8), where $C=2$, $T=1$, $\alpha=H$. □

Denote $S_{T_1T_2}=[0,T_1]\times[0,T_2]\subset\mathbb{R}_+^2$, $T_1>0$, $T_2>0$. The self-similarity of a random field gives a correspondence between the probability distributions of extremes defined on $[0,1]^2$ and in $S_{T_1T_2}$.

COROLLARY 3.3. *Under the conditions of Proposition 3.2, we have*

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in S_{T_1T_2}}\frac{|X(\mathbf{t})|}{T_1^{H_1}T_2^{H_2}}>\varepsilon\right\}=\mathbf{P}\left\{\sup_{\mathbf{t}\in[0,1]^2}|X(\mathbf{t})|>\varepsilon\right\}$$
$$\leqslant 8\exp\left\{-\frac{\varepsilon^2(1-p)}{2\left(1+\frac{4p}{2^{2H}(1-p)}\right)}\right\}\left(\frac{e}{p}\right)^{2/H}, \tag{11}$$

where $\varepsilon>0$, $p\in(0,1)$.

Proof. It follows from self-similarity that $\left\{T_1^{-H_1}T_2^{-H_2}X(T_1t_1,T_2t_2),\mathbf{t}\in\mathbb{R}_+\right\}$ and $\{X(\mathbf{t}),\mathbf{t}\in\mathbb{R}_+\}$ have the same finite dimensional distributions. Therefore,

$$\sup_{\mathbf{t}\in S_{T_1T_2}}\frac{|X(\mathbf{t})|}{T_1^{H_1}T_2^{H_2}}\stackrel{d}{=}\sup_{\mathbf{t}\in[0,1]^2}|X(\mathbf{t})|.$$

Hence, inequality (11) follows from Proposition 3.2. □

COROLLARY 3.4. *Let* $\varepsilon>2$. *Under the conditions of Proposition 3.2 we have*

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in[0,1]^2}|X(\mathbf{t})|>\varepsilon\right\}\leqslant 8e^{\frac{2}{H}+\frac{1}{2}}\varepsilon^{\frac{4}{H}}\exp\left\{-\frac{3\varepsilon^2}{2(4^{1-H}+3)}\right\}. \tag{12}$$

Proof.

Put $p=1/\varepsilon^2$ in (10). Then

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in[0,1]^2}|X(\mathbf{t})|>\varepsilon\right\}\leqslant 8\exp\left\{-\frac{\varepsilon^2-1}{2(1+4^{1-H}(\varepsilon^2-1)^{-1})}\right\}e^{2/H}\varepsilon^{4/H}$$

$$\leqslant 8\exp\left\{-\frac{3\varepsilon^2}{2(3+4^{1-H})}\right\}e^{2/H}\varepsilon^{4/H}e^{\frac{3}{2\left(3+4^{1-H}\right)}}\leqslant 8e^{\frac{2}{H}+\frac{1}{2}}\varepsilon^{\frac{4}{H}}\exp\left\{-\frac{3\varepsilon^2}{2(4^{1-H}+3)}\right\}.$$

The corollary is proved. $\square$

We obtained the upper bound for the probability of exceeding the level $\varepsilon > 2$ by a self-similar Gaussian random field defined on $[0,1]^2$.

Now we prove the upper bound for such probabilities for normalized fields defined on $\mathbb{R}_+^2$. Denote $x \vee y = \max\{x,y\}$.

THEOREM 3.5. *Let $X = \{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^2\}$ be a centered Gaussian self-similar random field with index $\mathbf{H} = (H_1,H_2) \in (0,1)^2$ and stationary rectangular increments. Let a function $c : (0,+\infty) \to (0,+\infty)$ and a sequence $\{b_n,\ n \in \mathbb{N}\bigcup\{0\}\}$ satisfy the following conditions*

(*i*) *c is increasing on $[1,+\infty)$, $c(t) \to \infty$, $t \to \infty$, and $c\left(\frac{1}{t}\right) = c(t)$, $t \geqslant 1$;*

(*ii*) *$b_0 = 1$, $b_n < b_{n+1}$, $n \in \mathbb{N}$, $b_n \to \infty$, $n \to \infty$, and*

$$M := \inf_{k \in 0 \cup \mathbb{N}} \left(\frac{b_k}{b_{k+1}}\right)^{H_1+H_2} c(b_k) > 0;$$

(*iii*) *for all $D > 0$ the following series converges*

$$\sum_{k=1}^{\infty} \exp\left\{-D\left(\frac{b_k^{H_1+H_2}}{b_{k+1}^{H_1+H_2}} c(b_k)\right)\right\} < +\infty.$$

Then for all $\varepsilon > 2/M$ we have

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in\mathbb{R}_+^2} \frac{|X(\mathbf{t})|}{(t_1 \vee t_2)^{H_1+H_2} c(t_1 \vee t_2)} > \varepsilon\right\}$$
$$\leqslant 16 e^{\frac{2}{H}+\frac{1}{2}} \varepsilon^{4/H} \sum_{k=0}^{\infty} \exp\left\{-\frac{3\varepsilon^2}{2(4^{1-H}+3)}\left(\frac{b_k^{H_1+H_2}}{b_{k+1}^{H_1+H_2}} c(b_k)\right)^2\right\}\left(\frac{b_k^{H_1+H_2}}{b_{k+1}^{H_1+H_2}} c(b_k)\right)^{\frac{4}{H}}$$
$$=: \widetilde{Z}(\varepsilon). \tag{13}$$

Proof. Denote

$$B_k = [0,b_{k+1}]^2 \setminus [0,b_k)^2, \ \ k \geqslant 0, \quad B_{-k} = \left[0,\frac{1}{b_k}\right]^2 \setminus \left[0,\frac{1}{b_{k+1}}\right)^2, \quad k \geqslant 1.$$

Let us remark that $\mathbf{T} = [0,+\infty)^2 = \bigcup_{k=-\infty}^{+\infty} B_k$. Denote

$$\widetilde{P}(\mathbf{T},\varepsilon) = \mathbf{P}\left\{\sup_{\mathbf{t}\in\mathbf{T}} \frac{|X(\mathbf{t})|}{(t_1 \vee t_2)^{H_1+H_2} c(t_1 \vee t_2)} > \varepsilon\right\}.$$

Evidently, we get

$$\widetilde{P}(\mathbb{R}_+^2,\varepsilon) \leqslant \widetilde{P}(\mathbb{R}_+^2 \setminus [0,1)^2,\varepsilon) + \widetilde{P}([0,1]^2,\varepsilon).$$

Firstly, consider $\widetilde{P}(\mathbb{R}_+^2 \setminus [0,1)^2, \varepsilon)$. Note that, if $\mathbf{t} \in B_k$, $k \geqslant 0$ then $b_k \leqslant t_1 \vee t_2 \leqslant b_{k+1}$ and $c(b_k) \leqslant c(t_1 \vee t_2) \leqslant c(b_{k+1})$, $k \geqslant 1$. Therefore, we get

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in\mathbb{R}_+^2\setminus[0,1)^2} \frac{|X(\mathbf{t})|}{(t_1\vee t_2)^{H_1+H_2} c(t_1\vee t_2)} > \varepsilon\right\} \leqslant \sum_{k=0}^{\infty} \mathbf{P}\left\{\sup_{\mathbf{t}\in B_k} \frac{|X(\mathbf{t})|}{(t_1\vee t_2)^{H_1+H_2} c(t_1\vee t_2)} > \varepsilon\right\}$$

$$\leqslant \sum_{k=0}^{\infty} \mathbf{P}\left\{\sup_{\mathbf{t}\in B_k} \frac{|X(\mathbf{t})|}{b_{k+1}^{H_1+H_2}} \frac{b_{k+1}^{H_1+H_2}}{b_k^{H_1+H_2} c(b_k)} > \varepsilon\right\} \leqslant \sum_{k=0}^{\infty} \mathbf{P}\left\{\sup_{\mathbf{t}\in[0,b_{k+1}]^2} \frac{|X(\mathbf{t})|}{b_{k+1}^{H_1+H_2}} \frac{b_{k+1}^{H_1+H_2}}{b_k^{H_1+H_2} c(b_k)} > \varepsilon\right\}$$

$$\leqslant \sum_{k=0}^{\infty} \mathbf{P}\left\{\sup_{\mathbf{t}\in[0,b_{k+1}]^2} \frac{|X(\mathbf{t})|}{b_{k+1}^{H_1+H_2}} > \varepsilon \frac{b_k^{H_1+H_2} c(b_k)}{b_{k+1}^{H_1+H_2}}\right\}.$$

From corollaries 3.3 and 3.4 we obtain that for $\varepsilon > 2/M$

$$\widetilde{P}(\mathbb{R}_+^2 \setminus [0,1)^2, \varepsilon) \leqslant \sum_{k=0}^{\infty} \mathbf{P}\left\{\sup_{\mathbf{t}\in[0,1]^2} |X(\mathbf{t})| > \varepsilon \frac{b_k^{H_1+H_2} c(b_k)}{b_{k+1}^{H_1+H_2}}\right\}$$

$$\leqslant 8e^{\frac{2}{H}+\frac{1}{2}} \varepsilon^{4/H} \sum_{k=0}^{\infty} \exp\left\{-\frac{3\varepsilon^2}{2(4^{1-H}+3)} \left(\frac{b_k^{H_1+H_2}}{b_{k+1}^{H_1+H_2}} c(b_k)\right)^2\right\} \left(\frac{b_k}{b_{k+1}}\right)^{4\frac{H_1+H_2}{H}} (c(b_k))^{4/H}.$$

Consider $\widetilde{P}([0,1]^2, \varepsilon)$. Note that, if $\mathbf{t} \in B_{-k}, k \geqslant 1$ then $\frac{1}{b_{k+1}} \leqslant t_1 \vee t_2 \leqslant \frac{1}{b_k}$ and $c(b_k) \leqslant c(t_1 \vee t_2) = c(\frac{1}{t_1 \vee t_2}) \leqslant c(b_{k+1})$, $k \geqslant 1$. Therefore, we have

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in[0,1]^2} \frac{|X(\mathbf{t})|}{(t_1\vee t_2)^{H_1+H_2} c(t_1\vee t_2)} > \varepsilon\right\} \leqslant \sum_{k=1}^{\infty} \mathbf{P}\left\{\sup_{\mathbf{t}\in B_{-k}} \frac{|X(\mathbf{t})|}{(t_1\vee t_2)^{H_1+H_2} c(t_1\vee t_2)} > \varepsilon\right\}$$

$$\leqslant \sum_{k=1}^{\infty} \mathbf{P}\left\{\sup_{\mathbf{t}\in B_{-k}} \frac{|X(\mathbf{t})|}{b_{k+1}^{-H_1-H_2}} \frac{b_k^{-H_1-H_2}}{b_k^{-H_1-H_2} c(b_k)} > \varepsilon\right\}$$

$$\leqslant \sum_{k=1}^{\infty} \mathbf{P}\left\{\sup_{\mathbf{t}\in[0,b_k^{-1}]^2} \frac{|X(\mathbf{t})|}{b_k^{-H_1-H_2}} \frac{b_{k+1}^{H_1+H_2}}{b_k^{H_1+H_2} c(b_k)} > \varepsilon\right\}$$

$$\leqslant \sum_{k=1}^{\infty} \mathbf{P}\left\{\sup_{\mathbf{t}\in[0,b_k^{-1}]^2} \frac{|X(\mathbf{t})|}{b_k^{-H_1-H_2}} > \varepsilon \frac{b_k^{H_1+H_2} c(b_k)}{b_{k+1}^{H_1+H_2}}\right\}.$$

From corollaries 3.3 and 3.4 we obtain that for $\varepsilon > 2/M$

$$\widetilde{P}([0,1]^2, \varepsilon) \leqslant \sum_{k=1}^{\infty} \mathbf{P}\left\{\sup_{\mathbf{t}\in[0,1]^2} |X(t_1,t_2)| > \varepsilon \frac{b_k^{H_1+H_2} c(b_k)}{b_{k+1}^{H_1+H_2}}\right\}$$

$$\leqslant 8e^{\frac{2}{H}+\frac{1}{2}} \varepsilon^{4/H} \sum_{k=1}^{\infty} \exp\left\{-\frac{3\varepsilon^2}{2(4^{1-H}+3)} \left(\frac{b_k^{H_1+H_2}}{b_{k+1}^{H_1+H_2}} c(b_k)\right)^2\right\} \left(\frac{b_k}{b_{k+1}}\right)^{4\frac{H_1+H_2}{H}} (c(b_k))^{4/H}.$$

The theorem is proved. $\square$

The following corollary is an immediate consequence of Theorem 3.5.

COROLLARY 3.6. *Let* $M = \inf_{k\in\{0\}\cup\mathbb{N}} \left(\frac{b_k}{b_{k+1}}\right)^{H_1+H_2} c(b_k) > 0$. *Denote*

$$u = \frac{3}{(4^{1-H}+3)}\frac{M^2}{4} \quad \text{and} \quad v_k = \frac{2}{M^2}\left(\frac{b_k^{H_1+H_2}}{b_{k+1}^{H_1+H_2}}c(b_k)\right)^2, \quad k \geqslant 0.$$

If for any $\mathbf{H} \in (0,1)^2$ *the series* $\sum_{k=0}^{\infty} \frac{v_k^{2/H}}{e^{v_k}}$ *converges, then for any* $\varepsilon > \frac{2}{M}\sqrt{\frac{2}{3}(4^{1-H}+3)}$

$$\mathbf{P}\left\{\sup_{t\in\mathbb{R}_+^2} \frac{|X(\mathbf{t})|}{(t_1\vee t_2)^{H_1+H_2}c(t_1\vee t_2)} > \varepsilon\right\} \leqslant 16\sqrt{2}\left(\frac{e}{2}\right)^{2/H} \varepsilon^{4/H}\left(\sum_{k=0}^{\infty}\frac{v_k^{2/H}}{e^{v_k}}\right) M^{4/H} e^{-u\varepsilon^2}. \tag{14}$$

Proof. It is clear that $u\varepsilon^2 > 2$ and $v_k > 2$, $k \geqslant 0$. Recall that for $u\varepsilon^2, v_k > 2$ we have $u\varepsilon^2 + v_k \leqslant u\varepsilon^2 v_k$. It follows from (13) that for $\varepsilon > \frac{2}{M}\sqrt{\frac{2}{3}(4^{1-H}+3)} > \frac{2}{M}$ we have

$$\widetilde{Z}(\varepsilon) = 16 e^{\frac{2}{H}+\frac{1}{2}} \varepsilon^{4/H} \sum_{k=0}^{\infty} \frac{M^{4/H}}{2^{2/H}} \frac{v_k^{2/H}}{\exp\{u\varepsilon^2 v_k\}} \leqslant 16\sqrt{2}\left(\frac{e}{2}\right)^{2/H} \varepsilon^{4/H}\left(\sum_{k=0}^{\infty}\frac{v_k^{2/H}}{e^{v_k}}\right) M^{4/H} e^{-u\varepsilon^2}.$$

The corollary is proved. $\square$

Consider an example of applying Corollary 3.6.

EXAMPLE 1. Put $b_k = e^k$, $k = 0,1,\ldots$, and $c(t) = \sqrt{\ln(|\ln t| + e)}$, $t \geqslant 1$ in Theorem 3.5. Then $M = \inf_{k\in 0\cup\mathbb{N}}\left(\frac{b_k}{b_{k+1}}\right)^{H_1+H_2} c(b_k) = e^{-(H_1+H_2)}$, and

$$u = \frac{3}{(4^{1-H}+3)}\frac{s^2}{4} = \frac{3}{4(4^{1-H}+3)} e^{-2(H_1+H_2)},$$

$$v_k = 2e^{2(H_1+H_2)}\frac{\ln(k+e)}{e^{2(H_1+H_2)}} = 2\ln(k+e), \quad k \geqslant 0.$$

Then inequality (14) has the form

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in\mathbb{R}_+^2} \frac{|X(\mathbf{t})|}{(t_1\vee t_2)^{H_1+H_2}\sqrt{\ln(|\ln(t_1\vee t_2)|+e)}} > \varepsilon\right\}$$
$$\leqslant 16\sqrt{2}e^{2/H}\varepsilon^{4/H}\left(\sum_{k=0}^{\infty}\frac{(\ln(k+e))^{2/H}}{(k+e)^2}\right) e^{-4\frac{H_1+H_2}{H}} \exp\{-u\varepsilon^2\}$$
$$\leqslant 16\sqrt{2}e^{2/H-8}\varepsilon^{4/H}\left(\sum_{k=0}^{\infty}\frac{(\ln(k+e))^{2/H}}{(k+e)^2}\right) \exp\left\{-\frac{3\varepsilon^2}{4(4^{1-H}+3)}e^{-2(H_1+H_2)}\right\}.$$

Thus, we obtain the upper bound for probability distribution of extremes of a normalized self-similar Gaussian random field with stationary rectangular increments, which is defined on $\mathbb{R}_+^2$.

4. Random fields on $(\mathbf{T},\rho_2)$

Recall the notation of the metric $\rho_2(\mathbf{t},\mathbf{s}) = \sum_{i=1,2}|t_i - s_i|_i^H$, $\mathbf{t} = (t_1,t_2) \in \mathbb{R}_+^2$, $\mathbf{s} = (s_1,s_2) \in \mathbb{R}_+^2$, where $\mathbf{H} = (H_1,H_2) \in (0,1)^2$ is the index of self-similarity of a field X. Now we want to obtain result witch is similar to Proposition 3.2, but with metric ρ_2.

Let us remember that $N(u)$ is the minimal number of closed ρ-balls with radius u needed to cover a space $(\mathbf{T},\rho)$. First let us prove the estimate for $N(u)$ in the case $\rho = \rho_2$ and $\mathbf{T} = S_{T_1T_2}$.

LEMMA 4.1. *Let $\rho = \rho_2$ and $\mathbf{T} = S_{T_1T_2}$. Then*

$$N(u) \leqslant 2\left(\frac{T_1}{4K_1u^{\frac{1}{H_1}}} + \frac{3}{2}\right)\left(\frac{T_2}{4K_2u^{\frac{1}{H_2}}} + \frac{3}{2}\right), u > 0,$$

where

$$K_1 = \left(\frac{H_2}{H_1+H_2}\right)^{\frac{1}{H_1}}, \quad K_2 = \left(\frac{H_1}{H_1+H_2}\right)^{\frac{1}{H_2}}.$$

Proof. Consider an auxiliary metric $\rho_3 = \{\rho_3(\mathbf{x},\mathbf{y}) = \frac{|y_1-x_1|}{a_1} + \frac{|y_2-x_2|}{a_2}$, $\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2$, $\mathbf{y} = (y_1,y_2) \in \mathbb{R}^2\}$, with $a_1 > 0$, $a_2 > 0$. A closed ρ_3-ball with radius 1 in space $(\mathbf{T},\rho_3)$ is a set $V_{\rho_3}(1) = \left\{\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2,\ \frac{|x_1|}{a_1} + \frac{|x_2|}{a_2} \leqslant 1\right\}$. The minimum number of $V_{\rho_3}(1)$ needed to cover space $(\mathbf{T},\rho_3)$ is less than $2\left(\frac{T_1+a_1}{2a_1}+1\right)\left(\frac{T_2+a_2}{2a_2}+1\right)$.

Put

$$a_1 = 2\left(\frac{H_2}{H_1+H_2}\right)^{\frac{1}{H_1}} \varepsilon^{\frac{1}{H_1}} = 2K_1\varepsilon^{\frac{1}{H_1}},$$

$$a_2 = 2\left(\frac{H_1}{H_1+H_2}\right)^{\frac{1}{H_2}} \varepsilon^{\frac{1}{H_2}} = 2K_2\varepsilon^{\frac{1}{H_2}}.$$

It is not hard to prove that $V_{\rho_3}(1) \subset V_{\rho_2}(\varepsilon)$. Hence,

$$N_{\rho_2}(\varepsilon) \leqslant N_{\rho_3}(1) \leqslant 2\left(\frac{T_1}{4K_1\varepsilon^{\frac{1}{H_1}}} + \frac{3}{2}\right)\left(\frac{T_2}{4K_2\varepsilon^{\frac{1}{H_2}}} + \frac{3}{2}\right). \quad \square$$

To prove the next statement, we need some notation. Denote

$$T_\eta = \max\{T_1^{H_1}, T_2^{H_2}\}, \quad H = \min\{H_1,H_2\}, \quad Q = \frac{1}{H_1} + \frac{1}{H_2},$$

$$N_1 = \left(\frac{H_1+H_2}{H_2}\right)^{\frac{1}{H_1}} + 3, \quad N_2 = \left(\frac{H_1+H_2}{H_1}\right)^{\frac{1}{H_2}} + 3.$$

PROPOSITION 4.2. *Let* $(\mathbf{T},\rho) = (S_{T_1T_2},\rho_2)$, $T_1 \geqslant 1$, $T_2 \geqslant 1$ *and* $X = \{X(\mathbf{t}),\mathbf{t} \in \mathbb{R}^2_+\}$ *be a centered self-similar Gaussian random field with stationary rectangular increments. Under the conditions of Theorem 2.1, for all* $0 < p < 1$ *we have*

$$I_{\mathbf{T}}(\varepsilon) = \mathbf{P}\left\{\sup_{\mathbf{t}\in\mathbf{T}} |X(t)| > \varepsilon\right\} \leqslant N_1N_2\left(\frac{e}{p}\right)^Q \exp\left\{-\frac{\varepsilon^2(1-p)}{2\left(T_1^{2H_1}T_2^{2H_2} + \frac{p}{1-p}4^{1-H}T_\eta^4\right)}\right\}, \quad \varepsilon > 0. \tag{15}$$

Proof. Recall that $\rho_2(\mathbf{s},\mathbf{t}) = |s_1-t_1|^{H_1} + |s_2-t_2|^{H_2}$, $\mathbf{s} = (s_1,s_2)$, $\mathbf{t} = (t_1,t_2)$, $\mathbf{t}$, $\mathbf{s} \in \mathbf{T}$. From Lemma 2.4 we get

$$\sup_{\rho(\mathbf{s},\mathbf{t})\leqslant h}\left(\mathbf{E}(X(\mathbf{t})-X(\mathbf{s}))^2\right)^{1/2} \leqslant \sup_{\rho(\mathbf{s},\mathbf{t})\leqslant h}\left(t_2^{H_2}|s_1-t_1|^{H_1} + t_1^{H_1}|s_2-t_2|^{H_2}\right) \leqslant T_\eta h.$$

Thus, we can put $\sigma(h) = T_\eta h$ in Theorem 2.1. From (2) we have

$$\beta = \sigma\left(\left(\frac{T_1}{2}\right)^{H_1} + \left(\frac{T_2}{2}\right)^{H_2}\right) = T_\eta\left(\left(\frac{T_1}{2}\right)^{H_1} + \left(\frac{T_2}{2}\right)^{H_2}\right).$$

It is clear that

$$\gamma^2 = \sup_{\mathbf{t}\in\mathbf{T}} \mathbf{E}X^2(\mathbf{t}) = T_1^{2H_1}T_2^{2H_2}\mathbf{E}X^2(\mathbf{1}) = T_1^{2H_1}T_2^{2H_2}.$$

From Lemma 4.1 we have

$$N(u) \leqslant 2\left(\frac{T_1}{4K_1u^{\frac{1}{H_1}}} + \frac{3}{2}\right)\left(\frac{T_2}{4K_2u^{\frac{1}{H_2}}} + \frac{3}{2}\right),$$

and therefore

$$N(\sigma^{-1}(u)) \leqslant 2\left(\frac{T_1T_\eta^{\frac{1}{H_1}}}{4K_1u^{\frac{1}{H_1}}} + \frac{3}{2}\right)\left(\frac{T_2T_\eta^{\frac{1}{H_2}}}{4K_2u^{\frac{1}{H_2}}} + \frac{3}{2}\right).$$

It follows from $\beta > \beta p \geqslant u$ that

$$1 < \left(\left(\frac{T_1}{2}\right)^{H_1} + \left(\frac{T_2}{2}\right)^{H_2}\right)^{\frac{1}{H_i}} \frac{T_\eta^{\frac{1}{H_i}}}{u^{\frac{1}{H_i}}}, \quad i = 1,2.$$

Recall that $0 < H_i < 1$ and

$$\frac{T_i}{2} = \left(\left(\frac{T_i}{2}\right)^{H_i}\right)^{\frac{1}{H_i}} \leqslant \left(\left(\frac{T_1}{2}\right)^{H_1} + \left(\frac{T_2}{2}\right)^{H_2}\right)^{\frac{1}{H_i}}, \quad i = 1,2.$$

Then

$$\left(\frac{T_iT_\eta^{\frac{1}{H_i}}}{4K_iu^{\frac{1}{H_i}}}+\frac{3}{2}\right)\leqslant\frac{T_iT_\eta^{\frac{1}{H_i}}}{4K_iu^{\frac{1}{H_i}}}+\left(\left(\frac{T_1}{2}\right)^{H_1}+\left(\frac{T_2}{2}\right)^{H_2}\right)^{\frac{1}{H_1}}\frac{3(T_\eta)^{\frac{1}{H_i}}}{2u^{\frac{1}{H_i}}}$$
$$\leqslant\left(\left(\frac{T_1}{2}\right)^{H_1}+\left(\frac{T_2}{2}\right)^{H_2}\right)^{\frac{1}{H_i}}\frac{T_\eta^{\frac{1}{H_i}}}{u^{\frac{1}{H_i}}}\left(\frac{1}{2K_i}+\frac{3}{2}\right).$$

Therefore, we have the following inequality for $Z(p)$, where $Z(p)$ is defined in (9). For each $0<\mu<1/Q$ we obtain

$$Z(p)\leqslant\left(\frac{1}{\beta p}\int_0^{\beta p}\left(\left(\left(\frac{T_1}{2}\right)^{H_1}+\left(\frac{T_2}{2}\right)^{H_2}\right)^{Q}\frac{T_\eta^Q}{u^Q}\frac{N_1N_2}{2}\right)^{\mu}du\right)^{1/\mu}$$
$$=2N_1N_2\left(\left(\frac{T_1}{2}\right)^{H_1}+\left(\frac{T_2}{2}\right)^{H_2}\right)^{Q}\frac{T_\eta^Q}{(\beta p)^{1/\mu}}\left(\int_0^{\beta p}\frac{1}{u^{Q\mu}}\right)^{1/\mu}$$
$$=\frac{N_1N_2}{2}\left(\left(\frac{T_1}{2}\right)^{H_1}+\left(\frac{T_2}{2}\right)^{H_2}\right)^{Q}\frac{T_\eta^Q}{(\beta p)^{Q}}\left(\frac{1}{1-Q\mu}\right)^{1/\mu}.$$

As $\mu\to 0$, we have

$$Z(p)\leqslant\frac{N_1N_2}{2}\left(\left(\frac{T_1}{2}\right)^{H_1}+\left(\frac{T_2}{2}\right)^{H_2}\right)^{Q}\frac{T_\eta^Q}{(\beta p)^{Q}}e^Q=\frac{N_1N_2}{2}\left(\frac{e}{p}\right)^{Q}.$$

Finally, from (4) we obtain

$$I_{\mathbf{T}}(\varepsilon)\leqslant N_1N_2\left(\frac{e}{p}\right)^{Q}\exp\left\{-\frac{\varepsilon^2(1-p)}{2\left(T_1^{2H_1}T_2^{2H_2}+\frac{p}{1-p}T_\eta^2\left(\left(\frac{T_1}{2}\right)^{H_1}+\left(\frac{T_2}{2}\right)^{H_2}\right)^2\right)}\right\}$$
$$\leqslant N_1N_2\left(\frac{e}{p}\right)^{Q}\exp\left\{-\frac{\varepsilon^2(1-p)}{2\left(T_1^{2H_1}T_2^{2H_2}+\frac{p}{1-p}4^{1-H}T_\eta^4\right)}\right\}.\quad\square$$

COROLLARY 4.3. *Under the conditions of Proposition 4.2 we have*

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in\mathbf{T}}|X(\mathbf{t})|>\varepsilon\right\}\leqslant N_1N_2\varepsilon^{2Q}\exp\left\{Q+\frac{3}{2T_1^{2H_1}T_2^{2H_2}(3+4^{1-H})}\right\}$$
$$\times\exp\left\{-\frac{3\varepsilon^2}{2\left(3T_1^{2H_1}T_2^{2H_2}+4^{1-H}T_\eta^4\right)}\right\},\quad\varepsilon>2. \tag{16}$$

Proof. The corollary follows from (15) if we put $p = 1/\varepsilon^2$. □

Consider probability distribution of extremes defined on $[0,1]^2$.

COROLLARY 4.4. *Let* $(\mathbf{T},\rho) = ([0,1]^2,\rho_2)$. *Under the conditions of Proposition 4.2 for all* $\varepsilon > 2$ *we have*

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in[0,1]^2} |X(\mathbf{t})| > \varepsilon\right\} \leqslant N_1N_2\varepsilon^{2Q}\exp\left\{Q+\frac{3}{2\left(3+4^{1-H}\right)}\right\}\exp\left\{-\frac{3\varepsilon^2}{2\left(3+4^{1-H}\right)}\right\},$$

$$\varepsilon > 2.$$

Proof. In this case $T_1 = T_2 = 1$, so the corollary follows from (16). □

We want to find an upper bound for probability distribution of extremes defined on $[1,+\infty)^2$. For this goal we obtain probabilities defined on $[1,2]^2$.

PROPOSITION 4.5. *Let* $\mathbf{T} = [1,2]^2$, $\rho = \rho_2$ *and* $X = \{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^2\}$ *be a centered self-similar Gaussian random field with stationary rectangular increments. Under the conditions of Theorem 2.1 for all* $0 < p < 1$ *we have*

$$\begin{aligned} I_{[1,2]^2}(\varepsilon) &= \mathbf{P}\left\{\sup_{\mathbf{t}\in[1,2]^2} |X(\mathbf{t})| > \varepsilon\right\} \\ &\leqslant N_1N_2\left(\frac{e}{p}\right)^Q \exp\left\{-\frac{\varepsilon^2(1-p)}{2\left(4^{H_1+H_2}+\left(1+2^{|H_1-H_2|}\right)^2\frac{p}{1-p}\right)}\right\}. \end{aligned} \tag{17}$$

Proof. We prove the proposition in the same way as Proposition 4.2. Denote $\eta = \max\{H_1,H_2\}$ and $H = \min\{H_1,H_2\}$. It is clear that $\sigma(h) = 2^\eta h$ and

$$\beta = \sigma\left(\left(\frac{1}{2}\right)^{H_1}+\left(\frac{1}{2}\right)^{H_2}\right) = 2^\eta\left(2^{-H_1}+2^{-H_2}\right) = 1+2^{|H_1-H_2|},$$

$$\gamma^2 = 4^{H_1+H_2}.$$

From Lemma 4.1 we have

$$N(\sigma^{-1}(u)) \leqslant 2\left(\frac{2^{\eta/H_1}}{4K_1u^{1/H_1}}+\frac{3}{2}\right)\left(\frac{2^{\eta/H_2}}{4K_2u^{1/H_2}}+\frac{3}{2}\right).$$

It follows from $\beta > \beta p \geqslant u > 0$ that

$$1 \leqslant \frac{\beta^{1/H_1}}{u^{1/H_1}} = \frac{\left(1+2^{|H_1-H_2|}\right)^{1/H_1}}{2u^{1/H_1}}.$$

Then for $i=1,2$

$$\left(\frac{2^{\eta/H_i}}{4K_i u^{1/H_1}}+\frac{3}{2}\right)\leqslant\frac{2^{\eta/H_i}}{4K_i u^{1/H_i}}+\frac{3\left(1+2^{|H_1-H_2|}\right)^{1/H_i}}{2u^{1/H_i}}\leqslant\frac{\left(1+2^{|H_1-H_2|}\right)^{1/H_i}}{u^{1/H_i}}\left(\frac{1}{2K_i}+\frac{3}{2}\right).$$

Further, from definition (9) of $Z(p)$ we get the following inequality:

$$Z(p)\leqslant\left(\frac{1}{\beta p}\int_0^{\beta p}\left(\left(1+2^{|H_1-H_2|}\right)^Q\frac{N_1N_2}{2u^Q}\right)^\mu du\right)^{1/\mu}$$

$$=\frac{N_1N_2}{2}\left(1+2^{|H_1-H_2|}\right)^Q\frac{1}{(\beta p)^{1/\mu}}\left(\int_0^{\beta p}\frac{1}{u^{Q\mu}}\right)^{1/\mu}=\frac{N_1N_2}{2p^Q}\left(\frac{1}{1-Q\mu}\right)^{1/\mu}.$$

As $\mu\to 0$, we have

$$Z(p)\leqslant\frac{N_1N_2}{2}\left(\frac{e}{p}\right)^Q.$$

Thus, we obtain

$$I_{[1,2]^2}(\varepsilon)\leqslant N_1N_2\left(\frac{e}{p}\right)^Q\exp\left\{-\frac{\varepsilon^2(1-p)}{2\left(4^{H_1+H_2}+\left(1+2^{|H_1-H_2|}\right)^2\frac{p}{1-p}\right)}\right\}.\quad\square$$

As before, denote $\eta=\max\{H_1,H_2\}$.

COROLLARY 4.6. *Under the conditions of Proposition 4.5 for $\varepsilon>2$ we have*

$$I_{[1,2]^2}(\varepsilon)\leqslant N_1N_2\exp\{Q+\frac{1}{2\left(4^{H_1+H_2}+1\right)}\}\varepsilon^{2Q}\exp\left\{-\frac{3\varepsilon^2}{2\cdot4^\eta\left(4^H3+4^{1-H}\right)}\right\}.\tag{18}$$

Proof. The corollary follows from (17), if we put $p=1/\varepsilon^2$. $\square$

THEOREM 4.7. *Let $\mathbf{T}=[1,\infty)^2$, $\rho=\rho_2$ and $X=\{X(\mathbf{t}),\mathbf{t}=(t_1,t_2)\in\mathbb{R}_+^2\}$ be a centered self-similar Gaussian random field with stationary rectangular increments. Let $\varphi:(0,+\infty)^2\to(0,+\infty)$ be an increasing function in each coordinate. Suppose that for any $D>0$*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\exp\{-D\varphi(2^n,2^m)\}<+\infty.\tag{19}$$

Denote

$$C_1=N_1N_2\exp\{Q+\frac{1}{2(4^{H_1+H_2}+1)}\}\quad\textit{and}\quad C_2=\frac{3}{2\cdot4^\eta\left(4^H3+4^{1-H}\right)}.$$

If $\varepsilon>\frac{2}{\varphi(\mathbf{1})}$, then

$$Y(\varepsilon):=\mathbf{P}\left\{\sup_{\mathbf{t}\in[1,+\infty)^2}\frac{|X(\mathbf{t})|}{t_1^{H_1}t_2^{H_2}\varphi(\mathbf{t})}>\varepsilon\right\}\leqslant C_1\varepsilon^{2Q}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\varphi^{2Q}(2^n,2^m)}{\exp\{C_2\varepsilon^2\varphi^2(2^n,2^m)\}}.\tag{20}$$

Proof. At first, we have the following obvious inequality

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in[1,+\infty)^2}\frac{|X(\mathbf{t})|}{t_1^{H_1}t_2^{H_2}\varphi(\mathbf{t})}>\varepsilon\right\}\leqslant\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathbf{P}\left\{\sup_{\substack{t_1\in[2^{n-1},2^n]\\ t_2\in[2^{m-1},2^m]}}\frac{|X(\mathbf{t})|}{t_1^{H_1}t_2^{H_2}\varphi(\mathbf{t})}>\varepsilon\right\}.$$

Then from monotonicity of φ we get for all $n,m>1$:

$$\mathbf{P}\left\{\sup_{\substack{t_1\in[2^{n-1},2^n]\\ t_2\in[2^{m-1},2^m]}}\frac{|X(\mathbf{t})|}{t_1^{H_1}t_2^{H_2}\varphi(\mathbf{t})}>\varepsilon\right\}\leqslant\mathbf{P}\left\{\sup_{\substack{t_1\in[2^{n-1},2^n]\\ t_2\in[2^{m-1},2^m]}}\frac{2^{(1-n)H_1}2^{(1-m)H_2}|X(\mathbf{t})|}{\varphi(2^{n-1},2^{m-1})}>\varepsilon\right\}.$$

By self-similarity, we obtain the following equality for all $n,m\geqslant 1$:

$$\mathbf{P}\left\{\sup_{\substack{t_1\in[2^{n-1},2^n]\\ t_2\in[2^{m-1},2^m]}}\frac{2^{(1-n)H_1}2^{(1-m)H_2}|X(\mathbf{t})|}{\varphi(2^{n-1},2^{m-1})}>\varepsilon\right\}=\mathbf{P}\left\{\sup_{\mathbf{t}\in[1,2]^2}\frac{|X(\mathbf{t})|}{\varphi(2^{n-1},2^{m-1})}>\varepsilon\right\}.$$

Thus,

$$\begin{aligned}Y(\varepsilon)&\leqslant\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathbf{P}\left\{\sup_{\mathbf{t}\in[1,2]^2}\frac{|X(\mathbf{t})|}{\varphi(2^{n-1},2^{m-1})}>\varepsilon\right\}\\&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathbf{P}\left\{\sup_{\mathbf{t}\in[1,2]^2}|X(\mathbf{t})|>\varepsilon\varphi\left(2^{n-1},2^{m-1}\right)\right\}.\end{aligned}$$

It follows from Corollary 4.6 that for $\varepsilon>\frac{2}{\varphi(\mathbf{1})}$ we have

$$Y(\varepsilon)\leqslant C_1\varepsilon^{2Q}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varphi^{2Q}\left(2^n,2^m\right)\exp\left\{-C_2\varepsilon^2\varphi^2\left(2^n,2^m\right)\right\}.$$

This completes the proof. □

COROLLARY 4.8. *If for any* $\mathbf{H}\in(0,1)^2$ *the series*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\varphi^{2Q}\left(2^n,2^m\right)}{\exp\left\{2\frac{\varphi^2(2^n,2^m)}{\varphi^2(\mathbf{1})}\right\}}<+\infty,$$

then for $\varepsilon>\frac{2}{\varphi(\mathbf{1})}\sqrt{\frac{2}{4^{\eta}3}\left(4^H3+4^{1-H}\right)}$,

$$Y(\varepsilon)\leqslant C_1\varepsilon^{2Q}\exp\left\{-\frac{\varepsilon^2}{2}\frac{\varphi^2(\mathbf{1})}{C_2}\right\}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\varphi^{2Q}\left(2^n,2^m\right)}{\exp\left\{2\frac{\varphi^2(2^n,2^m)}{\varphi^2(\mathbf{1})}\right\}}. \tag{21}$$

Proof. Denote

$$u = \frac{3}{4 \cdot 4^{\eta} \left(4^{H}3 + 4^{1-H}\right)} \varphi^{2}(\mathbf{1}) \quad \text{and} \quad v_{n,m} = 2\frac{\varphi^{2}(2^{n}, 2^{m})}{\varphi^{2}(\mathbf{1})}, \ n, m \geqslant 0.$$

It can easily be checked that $u\varepsilon^2 > 2$ and $v_{n,m} > 2$, n, $m \geqslant 0$. Recall that for $u\varepsilon^2$, $v_{n,m} > 2$ we have $u\varepsilon^2 + v_{n,m} \leqslant u\varepsilon^2 v_{n,m}$.

It follows from (20) that for $\varepsilon > \frac{2}{\varphi(\mathbf{1})}\sqrt{\frac{2}{4^{\eta}3}\left(4^{H}3 + 4^{1-H}\right)} > \frac{2}{\varphi(\mathbf{1})}$ we have

$$Y(\varepsilon) \leqslant C_1 \varepsilon^{2Q} \exp\left\{\frac{3\varepsilon^2}{4 \cdot 4^{\eta} \left(4^{H}3 + 4^{1-H}\right)} \varphi^{2}(\mathbf{1})\right\} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\varphi^{2Q}(2^{n}, 2^{m})}{\exp\left\{2\frac{\varphi^{2}(2^{n},2^{m})}{\varphi^{2}(\mathbf{1})}\right\}}.$$

The corollary is proved. □

We present the example of applying Corollary 4.8.

EXAMPLE 2. Let φ_1, φ_2 be the positive functions from $\mathbb{R}_{+}^{2}$ to $\mathbb{R}$ such that

$$\varphi_1(\mathbf{x}) = \sqrt{(2+\delta)}\sqrt{\ln(\log_2(x_1x_2) + e)}, \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}_{+}^{2}$$

and

$$\varphi_2(\mathbf{x}) = \sqrt{(2+\delta)}\sqrt{\ln(e + \log_2 x_1) + \ln(e + \log_2 x_1)}, \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}_{+}^{2}.$$

Then

$$\varphi_1(2^n, 2^m) = \sqrt{\ln(n+m+e)}, n, m \in \{0\} \cup \mathbb{N},$$
$$\varphi_2(2^n, 2^m) = \sqrt{\ln(n+e) + \ln(m+e)}, n, m \in \{0\} \cup \mathbb{N},$$

and $\varphi_1(\mathbf{1}) = \varphi_2(\mathbf{1}) = 1$.

Therefore, from (21) we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\varphi_1^{2Q}(2^{n}, 2^{m})}{\exp\left\{2\frac{\varphi_1^{2}(2^{n},2^{m})}{\varphi_1^{2}(\mathbf{1})}\right\}} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\ln(n+m+e))^{Q}}{(n+m+e)^2},$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\varphi_2^{2Q}(2^{n}, 2^{m})}{\exp\left\{2\frac{\varphi_2^{2}(2^{n},2^{m})}{\varphi_2^{2}(\mathbf{1})}\right\}} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\ln(n+e)(m+e))^{Q}}{(n+e)^2(m+e)^2}.$$

Hence, from Corollary 4.8 we have

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in[1,+\infty)^2} \frac{|X(\mathbf{t})|}{t_1^{H_1}t_2^{H_2}\varphi_1(\mathbf{t})} > \varepsilon\right\} \leqslant C_1\varepsilon^{2Q}\exp\left\{-\frac{C_2}{2}\varepsilon^2\right\}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\ln(n+m+e))^{Q}}{(n+m+e)^2},$$

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in[1,+\infty)^2} \frac{|X(\mathbf{t})|}{t_1^{H_1}t_2^{H_2}\varphi_2(\mathbf{t})} > \varepsilon\right\} \leqslant C_1\varepsilon^{2Q}\exp\left\{-\frac{C_2}{2}\varepsilon^2\right\}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\ln(n+e)(m+e))^{Q}}{(n+e)^2(m+e)^2}.$$

Thus, we obtain probability distributions for extremes of a normalized self-similar Gaussian random field with stationary rectangular increments defined on $[1, +\infty)^2$.

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