Journal of
Classical
Analysis
Volume 5, Number 1 (2014), 71–80
doi:10.7153/jca-05-06

ON SOME FURTHER HYPERGEOMETRIC SERIES IDENTITIES OBTAINED VIA FRACTIONAL CALCULUS

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Abstract. In this paper we present a generalization of a result obtained recently by Rathie and Kiliçman (A. K. Rathie and A. Kiliçman, On certain new hypergeometric identities, Preprint 2014) involving hypergeometric identities. The result is obtained by suitably applying fractional calculus technique to a generalization of a quadratic transformation for the Gauss hypergeometric function due to Gauss.

1. Introduction

The generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters is defined by (see, e.g. [4, Chapter 4]; see also [20, pp. 71-72])

$$
\begin{aligned}
{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{matrix}\, z\right] &= {}_pF_q[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z] \\
&= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!},
\end{aligned} \tag{1.1}
$$

$$
\Big(p \leqslant q \text{ and } |z|<\infty;\ p=q+1 \text{ and } |z|<1;
$$

$$
p=q+1,\ |z|=1 \text{ and } \Re(\omega)>0\Big)
$$

where

$$
\omega := \sum_{j=1}^{q} b_i - \sum_{j=1}^{p} a_i
$$

and $(\alpha)_n$ denotes the Pochhammer symbol defined in terms of the Gamma function by

$$
(\alpha)_n := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases}\alpha(\alpha+1)\cdots(\alpha+n-1) & (n\in\mathbb{N};\alpha\in\mathbb{C})\\ 1 & (n=0;\alpha\in\mathbb{C}\setminus\{0\}),\end{cases}
$$

$\mathbb{N}$ and $\mathbb{C}$ being the sets of positive integers and complex numbers, respectively.

Mathematics subject classification (2010): Primary 26A33, 33C90, 33C05; Secondary 33C70.
Keywords and phrases: Fractional derivatives, generalized hypergeometric function, Gauss quadratic transformation formula, beta integral method.

Many years ago, Gauss [2, 4] obtained the following quadratic transformation:

$$(1-z)^{-2a}\,{}_2F_1\left[\begin{matrix}a, & b; \\ 2b; & \end{matrix}\; -\frac{4z}{(1-z)^2}\right] = {}_2F_1\left[\begin{matrix}a, & a-b+\frac{1}{2}; \\ b+\frac{1}{2}; & \end{matrix}\; z^2\right], \tag{1.2}$$

with $z \in \mathbb{U}$, where $\mathbb{U}$ denotes the *open* unit disk, that is, $\mathbb{U} = \{z: \ z \in \mathbb{C} \text{ and } |z| < 1\}$.

The following generalization of the Gauss quadratic transformation formula (1.2) was given recently by Rakha [17]:

$$\begin{aligned}(1-z)^{-2a}\,{}_2F_1&\left[\begin{matrix}a, & b; \\ 2b+\ell; & \end{matrix}\; -\frac{4z}{(1-z)^2}\right] \\ &= \sum_{n=0}^{\infty} D_\ell(n)\,\frac{(a)_n\left(a-b+\frac{1}{2}-\left[\frac{\ell}{2}\right]\right)_n}{\left(b+\frac{1}{2}+\left[\frac{\ell}{2}\right]\right)_n}\,\frac{z^{2n}}{n!} \\ &\quad + 2a\sum_{n=0}^{\infty} E_\ell(n)\,\frac{(a+1)_n\left(a-b+\frac{3}{2}-\left[\frac{\ell+1}{2}\right]\right)_n}{\left(b+\frac{1}{2}+\left[\frac{\ell+1}{2}\right]\right)_n}\,\frac{z^{2n+1}}{n!}\end{aligned} \tag{1.3}$$

for $\ell = 0, \pm1, \pm2$. Here, $[x]$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$. The coefficients $D_\ell(n)$ and $E_\ell(n)$ are given in Table 1.

Table 1: *Coefficients* $D_\ell(n)$ *and* $E_\ell(n)$

ℓ	-2	-1	0	1	2
$D_\ell(n)$	$1-\frac{4n(a+n)}{(b-1)(2b-2a-3)}$	1	1	1	$1-\frac{4n(a+n)}{(b+1)(2b-2a+1)}$
$E_\ell(n)$	$-\frac{1}{b-1}$	$-\frac{1}{2b-1}$	0	$\frac{1}{2b+1}$	$\frac{1}{b+1}$

REMARK 1. The special cases of (1.3) when $\ell = \pm1$ was also obtained by Rathie and Kim [19].

A large number of transformation formulas involving hypergeometric functions were obtained, recently, by using the so-called Beta integral method [3, 8, 11, 25]. The beta function $B(\alpha,\beta)$ is defined by the following integral representation:

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt \quad (\Re(\alpha) > 0,\ \Re(\beta) > 0). \tag{1.4}$$

The so-called Beta integral method consists essentially of an integral from 0 to 1 of expressions which contain terms in the form $z^a(1-z)^b$ to obtain new transformations formulas.

With the help of this technique in conjunction with formula (1.3), Rathie and Kiliçman [18] obtained a generalization of a result due to Krattenthaler and Rao [11]. In particular, they obtained the following theorem:

THEOREM 1. *Let a or d be a non-positive integer. Then, the following generalization of Krattenthaler-Rao formula holds true:*

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, & b, & d, & e; \\ 2b+\ell, & \frac{1}{2}d+\frac{1}{2}e, & \frac{1}{2}d+\frac{1}{2}e+\frac{1}{2}; \end{matrix} 1\right] \\
&\quad = \frac{\Gamma(1-e)\Gamma(1-e-d+2a)}{\Gamma(1-e-d)\Gamma(1-e+2a)} \\
&\qquad \cdot \sum_{n=0}^{\infty} \frac{D_\ell(n)}{n!} \frac{(a)_n \left(a-b+\frac{1}{2}-\left[\frac{\ell}{2}\right]\right)_n \left(\frac{1}{2}d\right)_n \left(\frac{1}{2}d+\frac{1}{2}\right)_n}{\left(b+\frac{1}{2}+\left[\frac{\ell}{2}\right]\right)_n \left(\frac{1}{2}-\frac{1}{2}e+a\right)_n \left(1-\frac{1}{2}e+a\right)_n} \\
&\quad + \frac{2ad}{1-e+2a} \\
&\qquad \cdot \sum_{n=0}^{\infty} \frac{E_\ell(n)}{n!} \frac{(a+1)_n \left(a-b+\frac{3}{2}-\left[\frac{\ell+1}{2}\right]\right)_n \left(\frac{1}{2}d+1\right)_n \left(\frac{1}{2}d+\frac{1}{2}\right)_n}{\left(b+\frac{1}{2}+\left[\frac{\ell+1}{2}\right]\right)_n \left(1-\frac{1}{2}e+a\right)_n \left(\frac{3}{2}-\frac{1}{2}e+a\right)_n}
\end{aligned} \tag{1.5}
$$

for $\ell = 0, \pm 1, \pm 2$. *The coefficients* $D_\ell(n)$ *and* $E_\ell(n)$ *are those given in* Table 1.

The purpose of this work is to obtain a more general hypergeometric identity which contains, as a special case, the relationship (1.5) proved by Rathie and Kiliçman [18]. This is done by using the fractional calculus method which was applied by the authors [6, 5, 7] to succesfully obtain new hypergeometric identities more general than those obtained by the use of the Beta integral method. Several special cases are also obtained.

2. Pochhammer contour integral representation for fractional derivative

The most familiar representation for the fractional derivative of order α of $z^p f(z)$ is the Riemann-Liouville integral [10] (see also [1, 9, 14]), that is,

$$
\mathscr{D}_z^{\alpha}\{z^p f(z)\} = \frac{1}{\Gamma(-\alpha)} \int_0^z f(\xi)\xi^p(\xi - z)^{-\alpha-1} \mathrm{d}\xi \tag{2.1}
$$

$$
\left(\Re(\alpha) < 0;\ \Re(p) > 1\right),
$$

where the integration is carried out along a straight line from 0 to z in the complex ξ-plane. By integrating by part m times, we obtain

$$
\mathscr{D}_z^{\alpha}\{z^p f(z)\} = \frac{\mathrm{d}^m}{\mathrm{d}z^m}\left\{\mathscr{D}_z^{\alpha-m}\{z^p f(z)\}\right\}. \tag{2.2}
$$

This allows us to modify the restriction $\Re(\alpha) < 0$ to $\Re(\alpha) < m$ (see [14]).

Another representation for the fractional derivative is based on the Cauchy integral formula. This representation, too, has been widely used in many interesting papers (see, for example, the works of Osler [21, 24, 23, 22]).

The relatively less restrictive representation of the fractional derivative according to parameters appears to be the one based on the Pochhammer's contour integral introduced by Tremblay [13, 16].

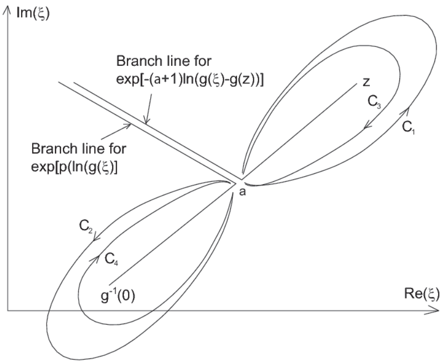

Figure 1: *Pochhammer's contour*

DEFINITION 1. Let $f(z)$ be analytic in a simply-connected region $\mathscr{R}$ of the complex z-plane. Let $g(z)$ be regular and univalent on $\mathscr{R}$ and let $g^{-1}(0)$ be an interior point of $\mathscr{R}$. Then, if α is not a negative integer, p is not an integer, and z is in $\mathscr{R}\setminus\{g^{-1}(0)\}$, we define the fractional derivative of order α of $g(z)^p f(z)$ with respect to $g(z)$ by

$$D^{\alpha}_{g(z)}\{[g(z)]^p f(z)\} = \frac{\mathrm{e}^{-\mathrm{i}\pi p}\Gamma(1+\alpha)}{4\pi\sin(\pi p)}\int_{C(z+,g^{-1}(0)+,z-,g^{-1}(0)-;F(a),F(a))} \cdot\frac{f(\xi)[g(\xi)]^p g'(\xi)}{[g(\xi)-g(z)]^{\alpha+1}}\,\mathrm{d}\xi. \tag{2.3}$$

For non-integers α and p, the functions $g(\xi)^p$ and $[g(\xi)-g(z)]^{-\alpha-1}$ in the integrand have two branch lines which begin, respectively, at $\xi=z$ and $\xi=g^{-1}(0)$, and both branches pass through the point $\xi=a$ without crossing the Pochhammer contour $P(a)=\{C_1\cup C_2\cup C_3\cup C_4\}$ at any other point as shown in Figure 1. Here $F(a)$ denotes the principal value of the integrand in (2.3) at the beginning and the ending point of the Pochhammer contour $P(a)$ which is closed on the Riemann surface of the multiple-valued function $F(\xi)$.

REMARK 2. In Definition 1, the function $f(z)$ must be analytic at $\xi=g^{-1}(0)$. However, it is interesting to note here that, if we could also allow $f(z)$ to have an essential singularity at $\xi=g^{-1}(0)$, then Equation (2.3) would still be valid.

REMARK 3. Since Pochhammer contour never crosses the singularities at $\xi=g^{-1}(0)$ and $\xi=z$ in (2.3), then we know that the integral is analytic for all p and for all α and for z in $\mathscr{R}\setminus\{g^{-1}(0)\}$. Indeed, in this case, the only possible singularities of $D^{\alpha}_{g(z)}\{[g(z)]^p f(z)\}$ are $\alpha=-1,-2,-3,\cdots$ and $p=0,\pm1,\pm2,\cdots$, which can directly be identified from the coefficient of the integral (2.3). However, by integrating by parts

N times the integral in (2.3) by two different ways, we can show that $\alpha = -1, -2, \cdots$ and $p = 0, 1, 2, \cdots$ are removable singularities (see, for details, [13]) .

It is well known that [15, p. 83, Equation (2.4)]

$$D_z^{\alpha}\{z^p\} = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} z^{p-\alpha} \qquad \left(\Re(p) > -1\right). \tag{2.4}$$

Adopting the Pochhammer based representation for the fractional derivative modifies the restriction to the case when p is not a negative integer.

3. The well poised fractional calculus operator ${}_zO_{\beta}^{\alpha}$

In this section, we recall some of the important properties of the fractional calculus operator ${}_zO_{\beta}^{\alpha}$ that was introduced by Tremblay [16]:

$${}_zO_{\beta}^{\alpha} := \frac{\Gamma(\beta)}{\Gamma(\alpha)} z^{1-\beta} D_z^{\alpha-\beta}\left\{z^{\alpha-1}\right\} \quad (\beta \text{ not a negative integer}). \tag{3.1}$$

We chose to simply list them since the proofs are readily obtainable.

1) Linearity

$${}_zO_{\beta}^{\alpha}\{\lambda_1 f(z) + \lambda_2 g(z)\} = \lambda_1\, {}_zO_{\beta}^{\alpha}\{f(z)\} + \lambda_2\, {}_zO_{\beta}^{\alpha}\{g(z)\}. \tag{3.2}$$

2) Identity

$${}_zO_{\alpha}^{\alpha} = I. \tag{3.3}$$

3) Reductions

$${}_zO_{\beta}^{\alpha}\, {}_zO_{\gamma}^{\beta} = {}_zO_{\gamma}^{\alpha}, \tag{3.4}$$

$${}_zO_{\beta}^{\alpha}\, {}_zO_{\alpha}^{\gamma} = {}_zO_{\beta}^{\gamma}. \tag{3.5}$$

4) Elementary cases

$${}_zO_{\beta}^{\alpha}\{1\} = 1, \tag{3.6}$$

$${}_zO_{\beta}^{\alpha}\{z^n\} = \frac{(\alpha)_n}{(\beta)_n} z^n, \tag{3.7}$$

$${}_zO_{\beta}^{\alpha}\left\{(1-z)^{-\gamma}\right\} = {}_2F_1\left[\begin{matrix}\gamma, & \alpha; \\ \beta; & \end{matrix}\; z\right]. \tag{3.8}$$

5) Useful case

$$ {}_zO_\beta^\alpha\left\{z^\lambda f(z)\right\}=\frac{\Gamma(\beta)\Gamma(\alpha+\lambda)}{\Gamma(\alpha)\Gamma(\beta+\lambda)}z^\lambda\ {}_zO_{\beta+\lambda}^{\alpha+\lambda}\left\{f(z)\right\}. \tag{3.9}$$

It is worthy to mention that operator ${}_zO_\beta^\alpha$ has a lot more interesting properties and applications. Tremblay introduced this operator in order to deal with special functions more efficiently and to facilitate the obtention of new relations such as hypergeometric transformations.

In this work, one of the most important properties of the operator ${}_zO_\beta^\alpha$ is given by the following relation:

$$ B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)}\ {}_zO_\beta^{\alpha+\beta}\left\{z^\gamma\right\}\Big|_{z=1}. \tag{3.10}$$

This relation shows that the so-called beta integral method consists in a fractional derivative evaluated at the point $z=1$.

4. Main result

The main result of the present paper is contained in the following theorem:

THEOREM 2. *The following generalization of the Rathie-Kiliçman formula* (1.5) *holds true:*

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(a)_k(b)_k(\alpha)_k(-4z)^k\ {}_2F_1\left[\begin{matrix}\gamma+2k,\ \alpha+k;\\ \beta+k;\end{matrix}\ z\right]}{(2b+\ell)_k(\beta)_k\ k!}\\
&=\sum_{n=0}^{\infty}D_\ell(n)\frac{(\alpha)_{2n}(a)_n\left(a-b+\frac{1}{2}-\left[\frac{\ell}{2}\right]\right)_n z^{2n}\ {}_2F_1\left[\begin{matrix}\gamma-2a,\ \alpha+2n;\\ \beta+2n;\end{matrix}\ z\right]}{(\beta)_{2n}\left(b+\frac{1}{2}+\left[\frac{\ell}{2}\right]\right)_n n!}\\
&\quad+\frac{2a\alpha}{\beta}\sum_{n=0}^{\infty}E_\ell(n)\frac{(\alpha+1)_{2n}(a+1)_n\left(a-b+\frac{3}{2}-\left[\frac{\ell+1}{2}\right]\right)_n z^{2n+1}}{(\beta+1)_{2n}\left(b+\frac{1}{2}+\left[\frac{\ell+1}{2}\right]\right)_n n!}\\
&\qquad\cdot\ {}_2F_1\left[\begin{matrix}\gamma-2a,\ \alpha+1+2n;\\ \beta+1+2n;\end{matrix}\ z\right],
\end{aligned}\tag{4.1}$$

where $|z|<1$ *and* $\ell=0,\pm1,\pm2$. *The coefficients* $D_\ell(n)$ *and* $E_\ell(n)$ *are those given in* Table 1.

Proof. Multiplying relation (1.3) by $(1-z)^{-\gamma+2a}$ where γ is a complex number, expressing ${}_2F_1$ involved as a series, applying the operator ${}_zO_\beta^\alpha$ and changing the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the involved series, gives

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(a)_k(b)_k\,(-4)^k}{(2b+\ell)_k\,k!}\,{}_zO_{\beta}^{\alpha}\left\{z^k(1-z)^{-2k-\gamma}\right\} &= \sum_{n=0}^{\infty} D_\ell(n)\,\frac{(a)_n\left(a-b+\frac{1}{2}-\left[\frac{\ell}{2}\right]\right)_n}{\left(b+\frac{1}{2}+\left[\frac{\ell}{2}\right]\right)_n}\,\frac{{}_zO_{\beta}^{\alpha}\left\{z^{2n}(1-z)^{-\gamma+2a}\right\}}{n!}\\ &\quad+2a\sum_{n=0}^{\infty} E_\ell(n)\,\frac{(a+1)_n\left(a-b+\frac{3}{2}-\left[\frac{\ell+1}{2}\right]\right)_n}{\left(b+\frac{1}{2}+\left[\frac{\ell+1}{2}\right]\right)_n}\,\frac{{}_zO_{\beta}^{\alpha}\left\{z^{2n+1}(1-z)^{-\gamma+2a}\right\}}{n!} \qquad (4.2)\end{aligned}$$

for $\ell=0,\pm1,\pm2$.
With the help of (3.8) and (3.9), we find

$$ {}_zO_{\beta}^{\alpha}\left\{z^k(1-z)^{-2k-\gamma}\right\} = \frac{(\alpha)_k}{(\beta)_k}\,z^k\,{}_2F_1\left[\begin{matrix}2k+\gamma,\ \alpha+k;\\ \beta+k;\end{matrix}\,z\right], \qquad (4.3)$$

$$ {}_zO_{\beta}^{\alpha}\left\{z^{2n}(1-z)^{-\gamma+2a}\right\} = \frac{(\alpha)_{2n}}{(\beta)_{2n}}\,z^{2n}\,{}_2F_1\left[\begin{matrix}\gamma-2a,\ \alpha+2n;\\ \beta+2n;\end{matrix}\,z\right] \qquad (4.4)$$

and

$$\begin{aligned}&{}_zO_{\beta}^{\alpha}\left\{z^{2n+1}(1-z)^{-\gamma+2a}\right\}\\ &\quad= \frac{\alpha\,(\alpha+1)_{2n}}{\beta\,(\beta+1)_{2n}}\,z^{2n+1}\,{}_2F_1\left[\begin{matrix}\gamma-2a,\ \ \alpha+1+2n;\\ \beta+1+2n;\end{matrix}\,z\right], \qquad (4.5)\end{aligned}$$

which leads us to the asserted result (4.1) after elementary simplifications. □

5. Corollaries and consequences

This section is devoted to present some corollaries of our main result (4.1). First, let us show that formula (4.1) reduces to the one of Rathie and Kiliçman (1.5).

Letting $z=1$, $\gamma=0$, $\alpha=d$ and $\beta=1-e$ in formula (4.1), we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(a)_k(b)_k(d)_k(-4)^k\,{}_2F_1\left[\begin{matrix}2k,\ \ d+k;\\ 1-e+k;\end{matrix}\,1\right]}{(2b+\ell)_k(1-e)_k\,k!} &= \sum_{n=0}^{\infty} D_\ell(n)\frac{(d)_{2n}(a)_n\left(a-b+\frac{1}{2}-\left[\frac{\ell}{2}\right]\right)_n\,{}_2F_1\left[\begin{matrix}-2a,\ \ d+2n;\\ 1-e+2n;\end{matrix}\,1\right]}{(1-e)_{2n}\left(b+\frac{1}{2}+\left[\frac{\ell}{2}\right]\right)_n\,n!}\\ &\quad+\frac{2ad}{(1-e)}\sum_{n=0}^{\infty} E_\ell(n)\frac{(d+1)_{2n}(a+1)_n\left(a-b+\frac{3}{2}-\left[\frac{\ell+1}{2}\right]\right)_n}{(2)_{2n}\left(b+\frac{1}{2}+\left[\frac{\ell+1}{2}\right]\right)_n\,n!}\end{aligned}$$

$$\cdot\ {}_2F_1\left[\begin{matrix}-2a, & d+1+2n; \\ 2-e+2n; & \end{matrix}\ 1\right]. \tag{5.1}$$

where $\ell = 0, \pm 1, \pm 2$.

Let d by a non positive integer. Then, the hypergeometric series appearing in the left-hand side of (5.1) terminates and, thus, appealing to Gauss summation formula [4]:

$${}_2F_1\left[\begin{matrix}a,\ b; \\ c;\end{matrix}\ 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (\mathrm{Re}(c-a-b) > 0), \tag{5.2}$$

we find respectively that

$${}_2F_1\left[\begin{matrix}2k, & d+k; \\ 1-e+k; & \end{matrix}\ 1\right] = \frac{\Gamma(1-e+k)\Gamma(1-e-d-2k)}{\Gamma(1-e-k)\Gamma(1-e-d)}, \tag{5.3}$$

$${}_2F_1\left[\begin{matrix}-2a, & d+2n; \\ 1-e+2n; & \end{matrix}\ 1\right] = \frac{\Gamma(1-e+2n)\Gamma(1-e-d+2a)}{\Gamma(1-e+2a+2n)\Gamma(1-e-d)}, \tag{5.4}$$

and

$${}_2F_1\left[\begin{matrix}-2a, & d+1+2n; \\ 2-e+2n; & \end{matrix}\ 1\right] = \frac{\Gamma(2-e+2n)\Gamma(1-e-d+2a)}{\Gamma(2-e+2a+2n)\Gamma(1-e-d)}. \tag{5.5}$$

Replacing the three hypergeometric series involved in (5.1) respectively by (5.3), (5.4) and (5.5) leads to the asserted result (1.5).

Furthermore, by specializing ℓ, we recover all the corollaries given by Rathie and Kiliçman [18].

Let us examine a very interesting special case of formula (4.1). Setting $z = -1$ and $\gamma = \alpha + \beta - 1$, in formula (4.1), we obtain

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k (\alpha)_k\, 4^k\ {}_2F_1\left[\begin{matrix}\alpha+\beta-1+2k,\ \alpha+k; \\ \beta+k;\end{matrix}\ -1\right]}{(2b+\ell)_k (\beta)_k\, k!}$$
$$= \sum_{n=0}^{\infty} D_\ell(n) \frac{(\alpha)_{2n} (a)_n \left(a-b+\frac{1}{2}-\left[\frac{\ell}{2}\right]\right)_n\ {}_2F_1\left[\begin{matrix}\alpha+\beta-1-2a,\ \alpha+2n; \\ \beta+2n;\end{matrix}\ -1\right]}{(\beta)_{2n} \left(b+\frac{1}{2}+\left[\frac{\ell}{2}\right]\right)_n n!}$$
$$- \frac{2a\alpha}{\beta} \sum_{n=0}^{\infty} E_\ell(n) \frac{(\alpha+1)_{2n} (a+1)_n \left(a-b+\frac{3}{2}-\left[\frac{\ell+1}{2}\right]\right)_n}{(\beta+1)_{2n} \left(b+\frac{1}{2}+\left[\frac{\ell+1}{2}\right]\right)_n n!}$$
$$\cdot\ {}_2F_1\left[\begin{matrix}\alpha+\beta-1-2a,\ \alpha+1+2n; \\ \beta+1+2n;\end{matrix}\ -1\right], \tag{5.6}$$

where $\ell = 0, \pm 1, \pm 2$.

Letting α be a non positive integer and making use of the following summation formula due to Kummer [12, p. 134, Entry 1]:

$${}_2F_1\left[\begin{matrix}a, & b; \\ a-b+1; & \end{matrix}\ -1\right] = \frac{2^{-a}\,\Gamma\left(\frac{1}{2}\right)\Gamma(a-b+1)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b+1\right)} \quad (\mathrm{Re}(b) < 1), \tag{5.7}$$

we have

$$\begin{aligned}
&\frac{2^{1-\alpha-\beta}\,\Gamma(\beta)}{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}\beta-\frac{1}{2}\alpha+\frac{1}{2}\right)}\,{}_3F_2\left[\begin{matrix}a, & b, & \alpha; \\ 2b+\ell, & \frac{1}{2}\alpha+\frac{1}{2}\beta; & \end{matrix}\;1\right]\\
&=\sum_{n=0}^{\infty} D_\ell(n)\frac{(\alpha)_{2n}(a)_n\left(a-b+\frac{1}{2}-\left[\frac{\ell}{2}\right]\right)_n\,{}_2F_1\left[\begin{matrix}\alpha+\beta-1-2a, \; \alpha+2n; \\ \beta+2n;\end{matrix}\;-1\right]}{(\beta)_{2n}\left(b+\frac{1}{2}+\left[\frac{\ell}{2}\right]\right)_n n!}\\
&\quad-\frac{2a\alpha}{\beta}\sum_{n=0}^{\infty} E_\ell(n)\frac{(\alpha+1)_{2n}(a+1)_n\left(a-b+\frac{3}{2}-\left[\frac{\ell+1}{2}\right]\right)_n}{(\beta+1)_{2n}\left(b+\frac{1}{2}+\left[\frac{\ell+1}{2}\right]\right)_n n!}\\
&\qquad\cdot\,{}_2F_1\left[\begin{matrix}\alpha+\beta-1-2a, \;\; \alpha+1+2n; \\ \beta+1+2n;\end{matrix}\;-1\right],
\end{aligned}\tag{5.8}$$

where $\ell = 0, \pm 1, \pm 2$.

Furthermore, if we make the following substitutions $\ell = 1$ and $a \mapsto \beta - 1$ with $\Re(\alpha) < \Re(\beta)$ in the last result (5.8) and next, if we apply again the Kummer's summation formula (5.7), we obtain

$$\begin{aligned}
&\frac{2^{1-\beta}}{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}\beta-\frac{1}{2}\alpha+\frac{1}{2}\right)}\,{}_3F_2\left[\begin{matrix}\beta-1, & b, & \alpha; \\ 2b+1, & \frac{1}{2}\alpha+\frac{1}{2}\beta; & \end{matrix}\;1\right]\\
&=\frac{1}{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\beta-\frac{1}{2}\alpha\right)}\,{}_3F_2\left[\begin{matrix}\frac{1}{2}\alpha, & \beta-1, \; \beta-b-\frac{1}{2}; \\ \beta-\frac{1}{2}\alpha, & b+\frac{1}{2};\end{matrix}\;1\right]\\
&\quad-\frac{(\beta-1)\alpha}{(2b+1)\Gamma\left(1+\frac{1}{2}\alpha\right)\Gamma\left(\beta-\frac{1}{2}\alpha+\frac{1}{2}\right)}\,{}_3F_2\left[\begin{matrix}\frac{1}{2}\alpha+\frac{1}{2}, & \beta, & \beta-b-\frac{1}{2}; \\ \beta-\frac{1}{2}\alpha+\frac{1}{2}, & b+\frac{3}{2}; & \end{matrix}\;1\right].
\end{aligned}\tag{5.9}$$

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(Received April 27, 2014)

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