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THE NECESSARY AND SUFFICIENT CONDITIONS FOR GENERAL HADAMARD PRODUCT OF CLASSES OF ANALYTIC FUNCTIONS

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Abstract. Let $P_a(A,B)$ be the classes of analytic functions $f(z)$, where $f(z) \prec \frac{a+Az}{1-Bz}$, $A+aB \neq 0$ and $|B| \leqslant 1$. For classes $\mathscr{H}_1, \mathscr{H}_2, \cdots, \mathscr{H}_n$ of analytic functions, we define the general hadamard product of the form $\mathscr{H}_1 *_{m_1} \mathscr{H}_2 *_{m_2} \mathscr{H}_3 * \cdots *_{m_{n-1}} \mathscr{H}_n(z) = \{f_1 *_{m_1} f_2 *_{m_2} f_3 * \cdots *_{m_{n-1}} f_n(z) : f_i \in \mathscr{H}_i, i = 1,2,...,n, n \in \mathbb{Z}^+, m_i \in \mathbb{C}\}$. In this paper, we discuss the conditions for equality $P_{a_1}(A_1,B_1) *_{m_1} P_{a_2}(A_2,B_2) * \cdots *_{m_{n-1}} P_{a_n}(A_n,B_n) = P_c(X,Y)$. Some consequences of the main results for known classes of analytic functions are also pointed out.

1. Introduction

Let $\mathscr{A}$ be the class of function analytic in the unit disc $\mathscr{U} = \{z : |z| < 1\}$. Let Ω denote the subclass of $\mathscr{A}$ consisting of functions such that $w(0) = 0$ and $|w(z)| < 1$. Suppose that functions $g \in \mathscr{A}$ and $F \in \mathscr{A}$, then the function g is said to be subordinate to F, written $g \prec F$, if there exists a function $w(z) \in \Omega$ and such that $g(z) = F(w(z))$, $z \in \mathscr{U}$. Many classes of functions studied in geometric function theory can be described in terms of subordination. If $f_i(z) \in \mathscr{A}$ are given by

$$f_i(z) = \sum_{k=0}^{\infty} a_{k,i} z^k,\ i = 1,2,..,n,\ z \in \mathscr{U},$$

then the Hadamard product $f_1 * f_2 * \ldots * f_n(z)$ of $f_1, f_2, \ldots, f_n$ is defined by

$$f_1 * f_2 * \ldots * f_n(z) = \sum_{k=0}^{\infty} a_{k,1} a_{k,2} \ldots a_{k,n} z^k,\ z \in \mathscr{U}. \tag{1.1}$$

Kinds of Hamadard product problems were studied in [3, 4, 8, 9, 11, 12, 13, 16, 17]. Moreover, if m_i $(i = 1,2,\ldots,n-1)$ are the fixed complex numbers, then we can also give the general finite Hadamard product as

$$f_1 *_{m_1} f_2 * \ldots *_{m_{n-1}} f_n(z) = a_{0,1} a_{0,2} \ldots a_{0,n} + \prod_{j=1}^{n-1} m_j \sum_{k=1}^{\infty} a_{k,1} a_{k,2} \ldots a_{k,n} z^k. \tag{1.2}$$

In fact, $f_1 *_1 f_2 * \ldots *_1 f_n(z) = f_1 * f_2 * \ldots * f_n(z)$. In [13], Piejko defined and studied the following class

$$P_a(A,B) = \left\{ f \in \mathscr{A} : f(z) \prec \frac{a+Az}{1-Bz},\ z \in \mathscr{U} \right\}, \tag{1.3}$$

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where a,A,B are given complex numbers such that $A+aB\neq 0$ and $|B|\leqslant 1$. The class $P_a(A,B)$ is the natural generalization of the class $P=P_1(1,1)$ of functions $\mathscr{P}\in\mathscr{A}$ with positive real part in $\mathscr{U}$. W. Janowski [7] introduced and considered the $P(A,B)=P_1(A,B)$ with some real A,B. Subsequently, several properties of various subclasses concerning $P(A,B)$ were obtained, see [1, 2, 14, 15].

Let $\mathscr{H}_1,\mathscr{H}_2,\cdots,\mathscr{H}_n$ be the subclasses of $\mathscr{A}$. For certain complex numbers $m_1,m_2,\ldots,m_{n-1}$, we denote $\mathscr{H}_1 *_{m_1} \mathscr{H}_2 *_{m_2} \mathscr{H}_3 * \cdots *_{m_{n-1}} \mathscr{H}_n(z)$ be the general Hadamard product of $\mathscr{H}_1,\mathscr{H}_2,\cdots,\mathscr{H}_n$, where

$$\mathscr{H}_1 *_{m_1} \mathscr{H}_2 *_{m_2} \mathscr{H}_3 * \cdots *_{m_{n-1}} \mathscr{H}_n(z) = \{f_1 *_{m_1} f_2 *_{m_2} f_3 * \cdots *_{m_{n-1}} f_n(z) : f_i\in\mathscr{H}_i, i=1,2,..,n, n\in\mathbb{Z}^+\}.$$

Komatu [8] and Nehari, Netanyahu [10] gave the function $f_{\frac{1}{2}} * g(z)\in P$ if $f\in P$ and $g\in P$. As the more general results, Goel and Merhok [5] obtained $f_{\frac{1}{2}} * g(z)\in P(A,B)$ for $f\in P(A,B)$ and $g\in P(A,B)$. In 1996, London [9] completed solution of the problem with the inclusion: $P(A,B)*P(C,D)\subset P(X,Y)$. The problem of equality of the $P(A,B)*P(C,D)=P(X,Y)$ was also solved by Piejko [11]. Furthermore, Piejko [12] studied the problem of inclusion of the classes $P_a(A,B) *_m P_b(C,D)$ and $P_c(X,Y)$ and the problem of inverse inclusion was also solved by Piejko [13]. In 2014, Liangpeng Xiong, Xiaoli Liu [18] extended these studies and obtained the general conditions for equality $P(A_1,B_1)*P(A_2,B_2)*\ldots*P(A_n,B_n)=P(X,Y)$.

In the present paper we establish some interesting on the problem of equality of classes $P_{a_1}(A_1,B_1) *_{m_1} P_{a_2}(A_2,B_2) * \cdots *_{m_{n-1}} P_{a_n}(A_n,B_n)=P_c(X,Y)$.

2. Main result

To discuss our problem, we have to recall here the following lemmas due to Piejko [13] and Janowski [7]:

LEMMA 1. ([13]) *Let m,a,b,A,B,C,D are given complex numbers such that $m\neq 0$, $A+aB\neq 0$, $C+bD\neq 0$, $|B|\leqslant 1$, $|D|\leqslant 1$, then*

$$P_a(A,B) *_m P_b(C,D)\subset P_{ab}(m(A+aB)(C+bD)-abBD, BD).$$

LEMMA 2. ([7]) *If $f\prec g$, $f(z)=\sum\limits_{n=0}^{\infty} a_n z^n$, $g(z)=\sum\limits_{n=0}^{\infty} b_n z^n$, then*

$$\sum_{n=0}^{\infty}|a_n|^2\leqslant\sum_{n=0}^{\infty}|b_n|^2.$$

THEOREM 1. *Let a_i, A_i, B_i, $(i=1,2,\ldots,n, n\in\mathbb{Z}^+)$ and m_j, $(j=1,2,\ldots,n-1$, $n\in\mathbb{Z}^+)$ are given complex numbers such that $m_j\neq 0$, $A_i+a_iB_i\neq 0$, $|B_i|\leqslant 1$, then*

$$P_{a_1}(A_1,B_1) *_{m_1} P_{a_2}(A_2,B_2) * \cdots *_{m_{n-1}} P_{a_n}(A_n,B_n)\subset P_c(X,Y),$$

where

$$c=\prod_{i=1}^{n}a_i,\ X=\prod_{j=1}^{n-1}m_j\prod_{i=1}^{n}(A_i+a_iB_i)-\prod_{i=1}^{n}a_iB_i,\ Y=\prod_{i=1}^{n}B_i.$$

Proof. Proof of the Theorem is based on mathematical induction.

(i) We first consider the case $n=2$. In view of the Lemma 1, it is easily seen that

$$\begin{aligned}P_{a_1}(A_1,B_1)*_{m_1}P_{a_2}(A_2,B_2) &\subset P_{a_1a_2}(m_1(A_1+a_1B_1)(A_2+a_2B_2)-a_1a_2B_1B_2,B_1B_2)\\ &=P_{a_1a_2}(\prod_{j=1}^{1}m_j\prod_{i=1}^{2}(A_i+a_iB_i)-\prod_{i=1}^{2}a_iB_i,\prod_{i=1}^{2}B_i).\end{aligned}$$

(ii) Suppose that the inclusion

$$\begin{aligned}&P_{a_1}(A_1,B_1)*_{m_1}P_{a_2}(A_2,B_2)\ *\ _{m_2}P_{a_3}(A_3,B_3)*\cdots*_{m_{n-2}}P_{a_{n-1}}(A_{n-1},B_{n-1})\\ &\subset P_{\prod_{i=1}^{n-1}a_i}(\prod_{j=1}^{n-2}m_j\prod_{i=1}^{n-1}(A_i+a_iB_i)-\prod_{i=1}^{n-1}a_iB_i,\prod_{i=1}^{n-1}B_i)\end{aligned}$$

is true. Now, we want to prove that

$$\begin{aligned}&P_{a_1}(A_1,B_1)*_{m_1}P_{a_2}(A_2,B_2)\ *\ _{m_2}P_{a_3}(A_3,B_3)*\cdots*_{m_{n-1}}P_{a_n}(A_n,B_n)\\ &\subset P_{\prod_{i=1}^{n}a_i}(\prod_{j=1}^{n-1}m_j\prod_{i=1}^{n}(A_i+a_iB_i)-\prod_{i=1}^{n}a_iB_i,\prod_{i=1}^{n}B_i)\end{aligned}$$

is also true.

In fact, for any function

$$\mathscr{H}^{\star}(z)\in P_{a_1}(A_1,B_1)*_{m_1}P_{a_2}(A_2,B_2)*_{m_2}P_{a_3}(A_3,B_3)*\cdots*_{m_{n-1}}P_{a_n}(A_n,B_n),$$

then there exists $f_i\in P_{a_i}(A_i,B_i)$, $i=1,2,3,\ldots,n$ such that $\mathscr{H}^{\star}(z)=f_1*_{m_1}f_2*_{m_2}f_3*\cdots*_{m_{n-1}}f_n(z)$.

Let $\mathscr{H}=f_1*_{m_1}f_2*_{m_2}f_3*\cdots*_{m_{n-2}}f_{n-1}(z)$. From the above assumption we have

$$\mathscr{H}\in P_{\prod_{i=1}^{n-1}a_i}(\prod_{j=1}^{n-2}m_j\prod_{i=1}^{n-1}(A_i+a_iB_i)-\prod_{i=1}^{n-1}a_iB_i,\prod_{i=1}^{n-1}B_i).$$

On the one hand, since $A_i+a_iB_i\neq 0$, $m_j\neq 0$ and $|B_i|\leqslant 1(i=1,2,\ldots,n,n\in\mathbb{Z}^+)$, it makes sure that

$$\prod_{i=1}^{n-2}m_j\prod_{i=1}^{n-1}(A_i+a_iB_i)-\prod_{i=1}^{n-1}a_iB_i+\prod_{i=1}^{n-1}a_iB_i\neq 0,|\prod_{i=1}^{n-1}B_i|=\prod_{i=1}^{n-1}|B_i|\leqslant 1.$$

On the other hand, we note that $f_n \in P_{a_n}(A_n, B_n)$ and $A_n + a_nB_n \neq 0, |B_n| \leqslant 1$. Hence by Lemma 1 we have

$$\begin{aligned}
\mathscr{H}^\star(z) &= \mathscr{H} *_{m_{n-1}} f_n(z) \\
&\in P_{\prod\limits_{i=1}^{n-1} a_i}\Big(\prod_{j=1}^{n-2} m_j \prod_{i=1}^{n-1}(A_i + a_iB_i) - \prod_{i=1}^{n-1} a_iB_i, \prod_{i=1}^{n-1} B_i\Big) *_{m_{n-1}} P_{a_n}(A_n, B_n) \\
&\subset P_{\prod\limits_{i=1}^{n} a_i}\Big(m_{n-1}\prod_{j=1}^{n-2} m_j \prod_{i=1}^{n-1}(A_i + a_iB_i)(A_n + B_n) - \prod_{i=1}^{n} a_iB_i, \prod_{i=1}^{n-1} B_i.B_n\Big) \\
&= P_{\prod\limits_{i=1}^{n} a_i}\Big(\prod_{j=1}^{n-1} m_j \prod_{i=1}^{n}(A_i + a_iB_i) - \prod_{i=1}^{n} a_iB_i, \prod_{i=1}^{n} B_i\Big).
\end{aligned}$$

Therefore we prove that the Theorem 1 is true for any $n \in \mathbb{Z}^+$, which complete the proof. □

THEOREM 2. *Let a_i, A_i, B_i $(i = 1,2,\ldots,n,\ n \in \mathbb{Z}^+)$ and m_j $(j = 1,2,\ldots,n-1, n \in \mathbb{Z}^+)$ are given complex numbers such that $m_j \neq 0$, $A_i + B_i \neq 0$, $|B_i| \leqslant 1$. If moreover $|B_{s_n}| = 1$ or $|B_{s_1}B_{s_2}\cdots B_{s_{n-1}}| = 1$, where $s_1, s_2, \ldots, s_n \in \{1,2,\ldots,n\}$, then*

$$P_{a_1}(A_1,B_1) *_{m_1} P_{a_2}(A_2,B_2) * \cdots *_{m_{n-1}} P_{a_n}(A_n,B_n) = P_c(X,Y),$$

where

$$c = \prod_{i=1}^{n} a_i,\ X = \prod_{j=1}^{n-1} m_j \prod_{i=1}^{n}(A_i + a_iB_i) - \prod_{i=1}^{n} a_iB_i,\ Y = \prod_{i=1}^{n} B_i.$$

Proof. As in Theorem 1, it is sufficient to show that if $|B_{s_n}| = 1$ or $|B_{s_1}B_{s_2}\cdots B_{s_{n-1}}| = 1$, then

$$P_{\prod\limits_{i=1}^{n} a_i}\Big(\prod_{j=1}^{n-1} m_j \prod_{i=1}^{n}(A_i + a_iB_i) - \prod_{i=1}^{n} a_iB_i, \prod_{i=1}^{n} B_i\Big) \subset P_{a_1}(A_1,B_1) *_{m_1} P_{a_2}(A_2,B_2) * \cdots *_{m_{n-1}} P_{a_n}(A_n,B_n).$$

Without loss of generality, we assume that $|B_1B_2\cdots B_{n-1}| = 1$ and

$$\mathscr{G}(z) \in P_{\prod\limits_{i=1}^{n} a_i}\Big(\prod_{j=1}^{n-1} m_j \prod_{i=1}^{n}(A_i + a_iB_i) - \prod_{i=1}^{n} a_iB_i, \prod_{i=1}^{n} B_i\Big),$$

now we will prove that $\mathscr{G}(z) \in P_{a_1}(A_1,B_1) *_{m_1} P_{a_2}(A_2,B_2) * \cdots *_{m_{n-1}} P_{a_n}(A_n,B_n)$. Let

$$f_i(z) \in P_{a_i}(A_i,B_i), i = 1,2,\ldots,n.$$

From Theorem 1, we know that

$$\mathscr{H}(z) = f_1 *_{m_1} f_2 *_{m_2} f_3 * \ldots *_{m_{n-2}} f_{n-1}(z) \in P_{\prod\limits_{i=1}^{n-1} a_i}\Big(\prod_{j=1}^{n-2} m_j \prod_{i=1}^{n-1}(A_i + a_iB_i) - \prod_{i=1}^{n-1} a_iB_i, \prod_{i=1}^{n-1} B_i\Big).$$

Since $|B_1 B_2 \cdots B_{n-1}| = 1$, so $|\frac{1}{B_1 B_2 \cdots B_{n-1}}| = 1$. Let

$$\mathscr{X}(z) = \frac{\frac{1}{\prod_{i=1}^{n-1} a_i} + \frac{\prod_{i=1}^{n-1} a_i B_i - m_{n-1} \prod_{i=1}^{n-1} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i)}{m_{n-1} \prod_{i=1}^{n-1} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i) \prod_{i=1}^{n-1} a_i B_i}}{1 - \frac{1}{\prod_{i=1}^{n-1} B_i} z}$$

$$\in P_{\frac{1}{\prod_{i=1}^{n-1} a_i}} \left(\frac{\prod_{i=1}^{n-1} a_i B_i - m_{n-1} \prod_{i=1}^{n-1} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i)}{m_{n-1} \prod_{i=1}^{n-1} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i) \prod_{i=1}^{n-1} a_i B_i}, \frac{1}{\prod_{i=1}^{n-1} B_i} \right). \tag{2.1}$$

In view of Lemma 1, it is easy to obtain

$$\mathscr{G}(z) *_{m_{n-1}} \mathscr{X}(z) \in P_{\prod_{i=1}^{n} a_i} (\prod_{j=1}^{n-1} m_j \prod_{i=1}^{n} (A_i + a_i B_i) - \prod_{i=1}^{n} a_i B_i, \prod_{i=1}^{n} B_i)$$
$$*_{m_{n-1}} P_{\frac{1}{\prod_{i=1}^{n-1} a_i}} \left(\frac{\prod_{i=1}^{n-1} a_i B_i - m_{n-1} \prod_{i=1}^{n-1} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i)}{m_{n-1} \prod_{i=1}^{n-1} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i) \prod_{i=1}^{n-1} a_i B_i}, \frac{1}{\prod_{i=1}^{n-1} B_i} \right) \subset P(A_n, B_n). \tag{2.2}$$

Furthermore, from Lemma 1,

$$\mathscr{H}(z) *_{m_{n-1}} \mathscr{X}(z) \in P_{\prod_{i=1}^{n} a_i} (\prod_{j=1}^{n-1} m_j \prod_{i=1}^{n} (A_i + a_i B_i) - \prod_{i=1}^{n} a_i B_i, \prod_{i=1}^{n} B_i)$$
$$*_{m_{n-1}} P_{\frac{1}{\prod_{i=1}^{n-1} a_i}} \left(\frac{\prod_{i=1}^{n-1} a_i B_i - m_{n-1} \prod_{i=1}^{n-1} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i)}{m_{n-1} \prod_{i=1}^{n-1} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i) \prod_{i=1}^{n-1} a_i B_i}, \frac{1}{\prod_{i=1}^{n-1} B_i} \right) \subset P(0,1) \tag{2.3}$$

and $\mathscr{H}(z) *_{m_{n-1}} \mathscr{X}(z) = 1 + \frac{1}{m_{n-1}}(z + z^2 + z^3 + \ldots + z^n + \ldots)$. Obviously,

$$\begin{aligned} \mathscr{G}(z) &= \mathscr{G}(z) *_{m_{n-1}} (\mathscr{H}(z) *_{m_{n-1}} \mathscr{X}(z)) = \mathscr{H}(z) *_{m_{n-1}} (\mathscr{G}(z) *_{m_{n-1}} \mathscr{X}(z)) \\ &\in P_{a_1}(A_1, B_1) *_{m_1} P_{a_2}(A_2, B_2) * \ldots *_{m_{n-1}} P_{a_n}(A_n, B_n). \end{aligned}$$

The same proof works for $|B_n| = 1$. This is the end of the proof. □

REMARK 1. (1) Putting $n = 2$, $a_1 = a_2 = 1$, $m_1 = 1$ in Theorem 2, we can obtain the results proved by J. Stankiewicz, Z. Stankiewicz [16].

(2) Putting $n = 2$, $a_1 = a$, $a_2 = b$, $m_1 = m$, $A_1 = A$, $B_1 = B$, $A_2 = C$, $A_2 = D$ in Theorem 2, we can obtain the results proved by Piejko ([13], Theorem 1).

(3) Putting $a_1 = a_2 = \ldots = a_n = 1$ and $m_1 = m_2 = \ldots = m_{n-1} = 1$ in Theorem 2, we can obtain the results proved by L. Xiong, X. Liu ([18], Theorem 2.2).

THEOREM 3. *Let a_i, A_i, B_i, $(i=1,2,\ldots,n,\ n\in\mathbb{Z}^+)$, X, Y and m_j $(j=1,2,\ldots,n-1, n\in\mathbb{Z}^+)$ are given complex numbers such that $m_j\neq 0$, $A_i+a_iB_i\neq 0$, $|B_i|\leqslant 1$, $|Y|\leqslant 1$. Then*

$$P_{a_1}(A_1,B_1)*_{m_1}P_{a_3}(A_2,B_2)\cdots *_{m_{n-1}}P_{a_n}(A_n,B_n)=P_c(X,Y) \tag{2.4}$$

if and only if $|\prod_{i=1}^{n}B_i|=|Y|$, $\Big(\prod_{j=1}^{n-1}m_j\prod_{i=1}^{n}(A_i+a_iB_i)-\prod_{i=1}^{n}a_iB_i\Big)Y=(\prod_{i=1}^{n}B_i)X, c=\prod_{i=1}^{n}a_i$, $|B_{s_1}B_{s_2}\cdots B_{s_{n-1}}|=1$ *or* $|B_{s_n}|=1$, *where* $s_1,s_2,\cdots s_n\in\{1,2,\ldots,n\}$.

Proof. Assume the formula (2.4) holds for some complex $m_j\neq 0$ $(j=1,2,\ldots,n-1)$, $A_i+B_i\neq 0$ $(i=1,2,\ldots,n)$, $|B_i|\leqslant 1$, $|Y|\leqslant 1$. Now we prove that $|B_{s_1}B_{s_2}\cdots B_{s_{n-1}}|=1$ or $|B_{s_n}|=1$, where $s_1,s_2,\cdots,s_n\in\{1,2,\ldots,n\}$. To obtain the contradiction, suppose that $|B_{s_1}B_{s_2}\cdots B_{s_{n-1}}|<1$ and $|B_{s_n}|<1$. Without loss of generality, we can set $0\leqslant B_1B_2\cdots B_{n-1}<1$, $0\leqslant B_n<1$, $0\leqslant Y<1$. Using the method of Piejko [12] we show that there exists a sequence of functions $w_\nu(z)$ of the class Ω such that for all positive integers ν,

$$w_\nu(z)=\sum_{n=1}^{\infty}r_{\nu,n}z^n \tag{2.5}$$

and that coefficients $r_{\nu,n}$ of power series have the following properties: $r_{\nu,n}>0$ for all ν and $n\in\{1,2,\ldots,\nu+1\}$ and $S_\nu\to\infty$ as $\nu\to\infty$, where $S_\nu=r_{\nu,1}+r_{\nu,2}+\ldots+r_{\nu,\nu+1}$. Consider the sequence of functions T_ν, where

$$T_\nu(z)=\frac{\prod_{i=1}^{n}a_i+Xw_\nu(z)}{1-Yw_\nu(z)} \tag{2.6}$$

and $w_\nu(z)\in\Omega$ is given by (2.5). For all $\nu\in\mathbb{Z}^+$, we have $T_\nu(z)\in P_{\prod_{i=1}^{n}a_i}(X,Y)$. Obviously, from the above assumption, there exists $f_i\in P_{a_i}(A_i,B_i)$, $i=1,2,\cdots,n$, such that $f_1*_{m_1}f_2*_{m_2}f_3(z)*\cdots *_{m_{n-1}}f_n(z)=T_\nu(z)$. Let

$$\mathscr{H}=f_1*_{m_1}f_2*_{m_2}f_3(z)*\cdots *_{m_{n-2}}f_{n-1}(z),$$

then from Theorem 1, we have

$$\mathscr{H}\in P_{\prod_{i=1}^{n-1}a_i}\Big(\prod_{j=1}^{n-2}m_j\prod_{i=1}^{n-1}(A_i+a_iB_i)-\prod_{i=1}^{n-1}a_iB_i,\prod_{i=1}^{n-1}B_i\Big)$$

and $\mathscr{H}*_{m_{n-1}}f_n(z)=T_\nu(z)$. Let the functions $\mathscr{H}(z), f_n(z)$ and $T_\nu(z)$ have the following forms:

$$\mathscr{H}(z)=\prod_{i=1}^{n-1}a_i+\prod_{j=1}^{n-2}m_j\prod_{i=1}^{n-1}(A_i+a_iB_i)\overline{\mathscr{H}},$$

$$f_n(z)=a_n+(A_n+a_nB_n)\overline{f_n},\quad T_\nu(z)=\prod_{i=1}^{n}a_i+(X+Y)\overline{T_\nu}.$$

where

$$\overline{\mathscr{H}} = \frac{w_1(z)}{1-(\prod_{i=1}^{n-1} B_i)w_1(z)} = \sum_{n=1}^{\infty} a_n z^n,$$

$$\overline{f}_n(z) = \frac{w_2(z)}{1-B_n w_2(z)} = \sum_{n=1}^{\infty} b_n z^n, \quad \overline{T}_v = \frac{w_v(z)}{1-Y w_v(z)} = \sum_{n=1}^{\infty} c_{v,n} z^n, \tag{2.7}$$

Since $\mathscr{H} *_{m_{n-1}} f_n(z) = T_v(z)$, using the notations we can rewrite as

$$\overline{\mathscr{H}} * \overline{f_n}(z) = \frac{X+Y}{\prod_{j=1}^{n-1} m_j \prod_{i=1}^{n} (A_i + a_i B_i)} \overline{T}_v,$$

So

$$a_n b_n = \frac{X+Y}{\prod_{j=1}^{n-1} m_j \prod_{i=1}^{n} (A_i + a_i B_i)} c_{v,n}, \tag{2.8}$$

On the one hand, from (2.7) we have

$$\begin{aligned}\sum_{n=1}^{\infty} c_{v,n} z^n &= \sum_{n=1}^{\infty} r_{v,n} z^n + Y(\sum_{n=1}^{\infty} r_{v,n} z^n)^2 + Y^3(\sum_{n=1}^{\infty} r_{v,n} z^n)^3 + \ldots \\ &= r_{v,1} z + (r_{v,2} + Y(r_{v,1})^2) z^2 + (r_{v,3} + 2Y r_{v,1} r_{v,2} + Y^2 (r_{v,1})^3) z^3 + \ldots \end{aligned} \tag{2.9}$$

Thus, since $0 \leqslant Y < 1$ and $r_{v,n} > 0$ $(n = 1, 2, \ldots, v+1)$, so (2.9) makes sure that

$$\sum_{n=1}^{\infty} |c_{v,n}| \geqslant \sum_{n=1}^{v+1} |c_{v,n}| \geqslant r_{v,1} + r_{v,2} + r_{v,3} + \ldots + r_{v,v} + r_{v,v+1} = S_v. \tag{2.10}$$

On the other hand, we have

$$\overline{\mathscr{H}} \prec \frac{z}{1-\left(\prod_{i=1}^{n-1} B_i\right) z} = z + \left(\prod_{i=1}^{n-1} B_i\right) z^2 + \left(\prod_{i=1}^{n-1} B_i\right)^2 z^3 + \ldots$$

and $0 \leqslant B_1 B_2 \cdots B_{n-1} < 1$, $0 \leqslant B_n < 1$, then from Lemma 2, we obtain:

$$\sum_{n=1}^{\infty} |a_n|^2 \leqslant \frac{1}{1-\left(\prod_{i=1}^{n-1} B_i\right)^2}$$

and

$$\sum_{n=1}^{\infty} |b_n|^2 \leqslant \frac{1}{1-(B_n)^2}. \tag{2.11}$$

Combining (2.8), (2.10) and (2.11), it gives:

$$0 \leqslant \sum_{n=1}^{\infty}(|a_n|-|b_n|)^2 \leqslant \frac{1}{1-\left(\prod\limits_{i=1}^{n-1} B_i\right)^2}+\frac{1}{1-B_n^2}-\left|\frac{2(X+Y)}{\prod\limits_{j=1}^{n-1} m_j \prod\limits_{i=1}^{n}(A_i+a_iB_i)}\right|\left|\sum_{n=1}^{\infty}|c_{v,n}\right|$$
$$\leqslant \frac{1}{1-\left(\prod\limits_{i=1}^{n-1} B_i\right)^2}+\frac{1}{1-B_n^2}-\left|\frac{2(X+Y)}{\prod\limits_{j=1}^{n-1} m_j \prod\limits_{i=1}^{n}(A_i+a_iB_i)}\right|S_v. \quad (2.12)$$

As $S_v \to +\infty$ when $v \to \infty$, it follows from (2.5) that we are able to choose a suitable v such that the right side of (2.12) is negative. In this way, (2.12) follows the contradiction and so we prove that $|B_1B_2\cdots B_{n-1}|=1$ or $|B_n|=1$. In fact, the progress of the proof implies that $|B_{s_1}B_{s_2}\cdots B_{s_{n-1}}|=1$ or $|B_{s_n}|=1$, where $s_1,s_2,\cdots s_n \in \{1,2,\ldots,n\}$.

From Theorem 2, if $|B_{s_1}B_{s_2}\cdots B_{s_{n-1}}|=1$ or $|B_{s_n}|=1$, we have

$$P_{a_1}(A_1,B_1) \ast_{m_1} P_{a_2}(A_2,B_2)\ast\cdots\ast_{m_{n-1}} P_{a_n}(A_n,B_n)$$
$$= P_{\prod\limits_{i=1}^{n} a_i}\Big(\prod_{j=1}^{n-1} m_j\prod_{i=1}^{n}(A_i+a_iB_i)-\prod_{i=1}^{n} a_iB_i,\prod_{i=1}^{n} B_i\Big).$$

However, following the assumption, it gives

$$P_{a_1}(A_1,B_1)\ast_{m_1} P_{a_2}(A_2,B_2)\ast\cdots\ast_{m_{n-1}} P_{a_n}(A_n,B_n)=P_c(X,Y).$$

So $c=\prod\limits_{i=1}^{n} a_i$ and

$$P_c\Big(\prod_{j=1}^{n-1} m_j\prod_{i=1}^{n}(A_i+a_iB_i)-\prod_{i=1}^{n} a_iB_i,\prod_{i=1}^{n} B_i\Big)=P_c(X,Y).$$

Piejko [13] gave $P_a(A,B)=P_a(X,Y)$ if and only if $X=Ae^{i\theta}$, $Y=Be^{i\theta}$, where θ is a real number. Thus $|Y|=|\prod\limits_{i=1}^{n} B_ie^{i\theta}|=|\prod\limits_{i=1}^{n} B_i|$, $(\prod\limits_{j=1}^{n-1} m_j\prod\limits_{i=1}^{n}(A_i+a_iB_i)-\prod\limits_{i=1}^{n} a_iB_i)Y = Xe^{-i\theta}\prod\limits_{i=1}^{n} B_ie^{i\theta}=(\prod\limits_{i=1}^{n} B_i)X$. This ends the first part of the proof.

Conversely, if $|B_{s_1}B_{s_2}\cdots B_{s_{n-1}}|=1$ or $|B_{s_n}|=1$, where $s_1,s_2,\cdots s_n\in\{1,2,\ldots,n\}$, then from the Theorem 2, it has

$$P_{a_1}(A_1,B_1)\ast_{m_1} P_{a_2}(A_2,B_2) \ast \cdots\ast_{m_{n-1}} P_{a_n}(A_n,B_n)$$
$$= P_{\prod\limits_{i=1}^{n} a_i}\Big(\prod_{j=1}^{n-1} m_j\prod_{i=1}^{n}(A_i+a_iB_i)-\prod_{i=1}^{n} a_iB_i,\prod_{i=1}^{n} B_i\Big).$$

Since $|\prod\limits_{i=1}^{n} B_i|=|Y|$, so $B_1B_2B_3\cdots B_ne^{i\theta}=Y$, where θ is a real number. Moreover, it notes

$$\Big(\prod_{j=1}^{n-1} m_j\prod_{i=1}^{n}(A_i+a_iB_i)-\prod_{i=1}^{n} a_iB_i\Big)Y=(\prod_{i=1}^{n} B_i)X=Ye^{-i\theta}X,$$

thus $X = \left(\prod_{j=1}^{n-1} m_j \prod_{i=1}^{n} (A_i + a_i B_i) - \prod_{i=1}^{n} a_i B_i\right) e^{i\theta}$. This is the end of the proof. □

REMARK 2. (1) Putting $n = 2$, $a_1 = a_2 = 1$, $m_1 = 1$, $A_1 = A$, $B_1 = B$, $A_2 = C$, $B_2 = D$ in Theorem 3, we can obtain the results proved by Piejko ([12], Theorem 1).

(2) Putting $n = 2$, $a_1 = a$, $a_2 = b$, $m_1 = m$, $A_1 = A$, $B_1 = B$, $A_2 = C$, $B_2 = D$ in Theorem 3, we can obtain the results proved by Piejko ([13], Corollary 1).

(3) Putting $a_1 = a_2 = \ldots = a_n = 1$, $m_1 = m_2 = \ldots = m_{n-1} = 1$ in Theorem 3, we can obtain the results proved by L. Xiong, X. Liu ([18], Theorem 2.3).

(4) Theorem 1 and Theorem 2 imply that we can choose some suitable complex numbers X, Y such that

$$P_{a_1}(A_1, B_1) *_{m_1} P_{a_2}(A_2, B_2) *_{m_3} P_{a_3}(A_3, B_3) * \cdots *_{m_{n-1}} P_{a_n}(A_n, B_n) \subset P_c(X, Y)$$

but the problem of inverse inclusion can hold unless A_i, B_i, m_j, X, Y satisfy for some conditions. Moreover, following the proof of Theorem 3, we can know that there exists some functions belonging to $P_a(A, B)$ which can not be represented as the finite Hadamard product of the form (2.4). In fact, It is clear that there do not exist complex m_i, a_i, A_i, B_i, X, Y with $A_i + a_i B_i \neq 0$, $X + cY \neq 0$, $|B_i| < 1$, $|Y| < 1$ such that $P_c(X, Y) = P_{a_1}(A_1, B_1) *_{m_1} P_{a_2}(A_2, B_2) *_{m_3} P_{a_3}(A_3, B_3) * \cdots *_{m_{n-1}} P_{a_n}(A_n, B_n)$.

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