THE NECESSARY AND SUFFICIENT CONDITIONS FOR GENERAL 
HADAMARD PRODUCT OF CLASSES OF ANALYTIC FUNCTIONS

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Abstract. Let $P_a(A,B)$ be the classes of analytic functions $f(z)$, where $f(z) \sim \frac{a+Az}{1-Bz}, A+eB \neq 0$ and $|B| \leq 1$. For classes $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ of analytic functions, we define the general hadamard product of the form $\mathcal{H}_1 *_{m_1} \mathcal{H}_2 *_{m_2} \mathcal{H}_3 *_{m_3} \cdots *_{m_{n-1}} \mathcal{H}_n (z) = \{ f_1 *_{m_1} f_2 *_{m_2} f_3 *_{m_3} \cdots *_{m_{n-1}} f_n(z) : f_i \in \mathcal{H}_i, i = 1, 2, \ldots, n \in \mathbb{Z}^+, m_i \in \mathbb{C} \}$. In this paper, we discuss the conditions for equality $P_{a_1}(A_1, B_1) *_{m_1} P_{a_2}(A_2, B_2) *_{m_2} *_{m_3} \cdots *_{m_{n-1}} P_{a_n}(A_n, B_n) = P_a(X, Y)$. Some consequences of the main results for known classes of analytic functions are also pointed out.

1. Introduction

Let $\mathcal{A}$ be the class of function analytic in the unit disc $\mathcal{U} = \{ z : |z| < 1 \}$. Let $\Omega$ denote the subclass of $\mathcal{A}$ consisting of functions such that $w(0) = 0$ and $|w(z)| < 1$. Suppose that functions $g \in \mathcal{A}$ and $F \in \mathcal{A}$, then the function $g$ is said to be subordinate to $F$, written $g \prec F$, if there exists a function $w(z) \in \Omega$ and such that $g(z) = F(w(z))$, $z \in \mathcal{U}$. Many classes of functions studied in geometric function theory can be described in terms of subordination. If $f_i(z) \in \mathcal{A}$ are given by

$$f_i(z) = \sum_{k=0}^{\infty} a_{k,i} z^k, \quad i = 1, 2, \ldots, n, \quad z \in \mathcal{U},$$

then the Hadamard product $f_1 * f_2 * \ldots * f_n(z)$ of $f_1, f_2, \ldots, f_n$ is defined by

$$f_1 * f_2 * \ldots * f_n(z) = \sum_{k=0}^{\infty} a_{k,1} a_{k,2} \cdots a_{k,n} z^k, \quad z \in \mathcal{U}. \quad (1.1)$$

Kinds of Hadamard product problems were studied in [3, 4, 8, 9, 11, 12, 13, 16, 17]. Moreover, if $m_i$ $(i = 1, 2, \ldots, n - 1)$ are the fixed complex numbers, then we can also give the general finite Hadamard product as

$$f_1 *_{m_1} f_2 *_{m_2} *_{m_3} *_{m_{n-1}} f_n(z) = a_{0,1} a_{0,2} \cdots a_{0,n} + \prod_{j=1}^{n-1} m_j \sum_{k=1}^{\infty} a_{k,1} a_{k,2} \cdots a_{k,n} z^k. \quad (1.2)$$

In fact, $f_1 *_{1} f_2 *_{1} *_{1} f_n(z) = f_1 *_{2} f_2 *_{2} *_{2} f_n(z)$. In [13], Piejko defined and studied the following class

$$P_{a}(A,B) = \left\{ f \in \mathcal{A} : f(z) \sim \frac{a + Az}{1 - Bz}, \quad z \in \mathcal{U} \right\}, \quad (1.3)$$

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where $a, A, B$ are given complex numbers such that $A + aB \neq 0$ and $|B| \leq 1$. The class $P_a(A, B)$ is the natural generalization of the class $P = P_1(1, 1)$ of functions $f \in A$ with positive real part in $A$. W. Janowski [7] introduced and considered the $P(A, B) = P_1(A, B)$ with some real $A, B$. Subsequently, several properties of various subclasses concerning $P(A, B)$ were obtained, see [1, 2, 14, 15].

Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ be the subclasses of $A$. For certain complex numbers $m_1, m_2, \ldots, m_{n-1}$, we denote $\mathcal{H}_1 \ast m_1 \mathcal{H}_2 \ast m_2 \mathcal{H}_3 \ast \cdots \ast m_{n-1} \mathcal{H}_n(z)$ be the general Hadamard product of $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$, where

$$\mathcal{H}_1 \ast m_1 \mathcal{H}_2 \ast m_2 \mathcal{H}_3 \ast \cdots \ast m_{n-1} \mathcal{H}_n(z) = \{f_1 \ast m_1 f_2 \ast m_2 f_3 \ast \cdots \ast m_{n-1} f_n(z) : f_i \in \mathcal{H}_i, i = 1, 2, \ldots, n, n \in \mathbb{Z}^+\}.$$ 

Komatu [8] and Nehari, Netanyahu [10] gave the function $f_1^g g(z) \in P$ if $f \in P$ and $g \in P$. As the more general results, Goel and Merhok [5] obtained $f_1^g g(z) \in P(A, B)$ for $f \in P(A, B)$ and $g \in P(A, B)$. In 1996, London [9] completed solution of the problem with the inclusion: $P(A, B) \ast P(C, D) \subset P(X, Y)$. The problem of equality of the $P(A, B) \ast P(C, D) = P(X, Y)$ was also solved by Piejko [11]. Furthermore, Piejko [12] studied the problem of inclusion of the classes $P_a(A, B) \ast m P_b(C, D)$ and $P_c(X, Y)$ and the problem of inverse inclusion was also solved by Piejko [13]. In 2014, Liangpeng Xiong, Xiaoli Liu [18] extended these studies and obtained the general conditions for equality $P(A_1, B_1) \ast P(A_2, B_2) \ast \cdots \ast P(A_n, B_n) = P(X, Y)$.

In the present paper we establish some interesting on the problem of equality of classes $P_{a_1}(A_1, B_1) \ast m_1 P_{a_2}(A_2, B_2) \ast \cdots \ast m_{n-1} P_{a_n}(A_n, B_n) = P_c(X, Y)$.

2. Main result

To discuss our problem, we have to recall here the following lemmas due to Piejko [13] and Janowski [7]:

**Lemma 1.** ([13]) Let $m, a, b, A, B, C, D$ be given complex numbers such that $m \neq 0$, $A + aB \neq 0$, $C + bD \neq 0$, $|B| \leq 1$, $|D| \leq 1$, then

$$P_a(A, B) \ast m P_b(C, D) \subset P_{ab}(m(A + aB)(C + bD) – abBD, BD).$$

**Lemma 2.** ([7]) If $f \prec g$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\sum_{n=0}^{\infty} |a_n|^2 \leq \sum_{n=0}^{\infty} |b_n|^2.$$

**Theorem 1.** Let $a_i, A_i, B_i$, $(i = 1, 2, \ldots, n, n \in \mathbb{Z}^+)$ and $m_j$, $(j = 1, 2, \ldots, n-1, n \in \mathbb{Z}^+)$ are given complex numbers such that $m_j \neq 0$, $A_i + a_i B_i \neq 0$, $|B_i| \leq 1$, then

$$P_{a_1}(A_1, B_1) \ast m_1 P_{a_2}(A_2, B_2) \ast \cdots \ast m_{n-1} P_{a_n}(A_n, B_n) \subset P_c(X, Y).$$
then there exists

\[ c = \prod_{i=1}^{n} a_i, X = \prod_{j=1}^{n-1} m_j \prod_{i=1}^{n} (A_i + a_i B_i) - \prod_{i=1}^{n} a_i B_i, Y = \prod_{i=1}^{n} B_i. \]

**Proof.** Proof of the Theorem is based on mathematical induction.

(i) We first consider the case \( n = 2 \). In view of the Lemma 1, it is easily seen that

\[
P_{a_1}(A_1, B_1) \ast_{m_1} P_{a_2}(A_2, B_2) \subset P_{a_1,a_2}(m_1(A_1 + a_1 B_1)(A_2 + a_2 B_2) - a_1 a_2 B_1 B_2)
= P_{a_1,a_2}\left(\prod_{j=1}^{1} m_j \prod_{i=1}^{2} (A_i + a_i B_i) - \prod_{i=1}^{2} a_i B_i, \prod_{i=1}^{2} B_i\right).
\]

(ii) Suppose that the inclusion

\[
P_{a_1}(A_1, B_1) \ast_{m_1} P_{a_2}(A_2, B_2) \ast_{m_2} P_{a_3}(A_3, B_3) \ast \cdots \ast_{m_{n-2}} P_{a_{n-1}}(A_{n-1}, B_{n-1})
\subset P_{n-1}\left(\prod_{i=1}^{n-1} m_i \prod_{i=1}^{n-1} (A_i + a_i B_i) - \prod_{i=1}^{n-1} a_i B_i, \prod_{i=1}^{n-1} B_i\right)
\]

is true. Now, we want to prove that

\[
P_{a_1}(A_1, B_1) \ast_{m_1} P_{a_2}(A_2, B_2) \ast_{m_2} P_{a_3}(A_3, B_3) \ast \cdots \ast_{m_{n-1}} P_{a_n}(A_n, B_n)
\subset P_n\left(\prod_{i=1}^{n-1} m_i \prod_{i=1}^{n} (A_i + a_i B_i) - \prod_{i=1}^{n} a_i B_i, \prod_{i=1}^{n} B_i\right)
\]

is also true.

In fact, for any function

\[ \mathcal{H}^*(z) \in P_{a_1}(A_1, B_1) \ast_{m_1} P_{a_2}(A_2, B_2) \ast_{m_2} P_{a_3}(A_3, B_3) \ast \cdots \ast_{m_{n-1}} P_{a_n}(A_n, B_n), \]

then there exists \( f_i \in P_{a_i}(A_i, B_i) \), \( i = 1, 2, 3, \ldots, n \) such that \( \mathcal{H}^*(z) = f_1 \ast_{m_1} f_2 \ast_{m_2} f_3 \ast \cdots \ast_{m_{n-1}} f_n(z) \).

Let \( \mathcal{H} = f_1 \ast_{m_1} f_2 \ast_{m_2} f_3 \ast \cdots \ast_{m_{n-1}} f_n(z) \). From the above assumption we have

\[ \mathcal{H} \in P_{n-1}\left(\prod_{i=1}^{n-1} m_i \prod_{i=1}^{n} (A_i + a_i B_i) - \prod_{i=1}^{n} a_i B_i, \prod_{i=1}^{n} B_i\right). \]

On the one hand, since \( A_i + a_i B_i \neq 0, m_j \neq 0 \) and \( |B_i| \leq 1(i = 1, 2, \ldots, n, n \in \mathbb{Z}^+) \), it makes sure that

\[
\prod_{i=1}^{n-1} m_i \prod_{i=1}^{n} (A_i + a_i B_i) - \prod_{i=1}^{n} a_i B_i + \prod_{i=1}^{n} a_i B_i \neq 0, \prod_{i=1}^{n-1} B_i = \prod_{i=1}^{n} |B_i| \leq 1.
\]
On the other hand, we note that \( f_n \in P_{a_n}(A_n, B_n) \) and \( A_n + a_n B_n \neq 0, |B_n| \leq 1 \). Hence by Lemma 1 we have

\[
\mathcal{H}^*(z) = \mathcal{H}^*_{m_{n-1}} f_n(z) = \frac{P_n}{\prod_{i=1}^{n-1} a_i} \left( \prod_{j=1}^{n-2} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i) - \prod_{i=1}^{n-1} a_i B_i, \prod_{i=1}^{n-1} B_i \right) \ast P_{a_n}(A_n, B_n)
\]

\[
\subset \frac{P_n}{\prod_{i=1}^{n-1} a_i} \left( m_{n-1} \prod_{j=1}^{n-2} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i)(A_n + B_n) - \prod_{i=1}^{n-1} a_i B_i, \prod_{i=1}^{n-1} B_i, B_n \right)
\]

\[
= \frac{P_n}{\prod_{i=1}^{n-1} a_i} \left( \prod_{j=1}^{n-1} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i) - \prod_{i=1}^{n-1} a_i B_i, \prod_{i=1}^{n-1} B_i \right).
\]

Therefore we prove that Theorem 1 is true for any \( n \in \mathbb{Z}^+ \), which complete the proof. \( \square \)

**Theorem 2.** Let \( a_i, A_i, B_i (i = 1, 2, \ldots, n, n \in \mathbb{Z}^+) \) and \( m_j (j = 1, 2, \ldots, n-1, n \in \mathbb{Z}^+) \) are given complex numbers such that \( m_j \neq 0, A_i + B_i \neq 0, |B_i| \leq 1 \). If moreover \( |B_{s_n}| = 1 \) or \( |B_{s_1}B_{s_2} \cdots B_{s_{n-1}}| = 1 \), where \( s_1, s_2, \ldots, s_n \in \{1, 2, \ldots, n\} \), then

\[
P_{a_1}(A_1, B_1) \ast_{m_1} P_{a_2}(A_2, B_2) \ast \cdots \ast_{m_{n-1}} P_{a_n}(A_n, B_n) = P_c(X, Y),
\]

where

\[
c = \prod_{i=1}^{n} a_i, \quad X = \prod_{j=1}^{n-1} m_j \prod_{i=1}^{n} (A_i + a_i B_i) - \prod_{i=1}^{n} a_i B_i, \quad Y = \prod_{i=1}^{n} B_i.
\]

**Proof.** As in Theorem 1, it is sufficient to show that if \( |B_{s_n}| = 1 \) or \( |B_{s_1}B_{s_2} \cdots B_{s_{n-1}}| = 1 \), then

\[
P_n \prod_{i=1}^{n-1} a_i \left( \prod_{j=1}^{n-1} m_j \prod_{i=1}^{n} (A_i + a_i B_i) - \prod_{i=1}^{n} a_i B_i, \prod_{i=1}^{n} B_i \right)
\]

\[
\subset P_{a_1}(A_1, B_1) \ast_{m_1} P_{a_2}(A_2, B_2) \ast \cdots \ast_{m_{n-1}} P_{a_n}(A_n, B_n).
\]

Without loss of generality, we assume that \( |B_1 B_2 \cdots B_{n-1}| = 1 \) and

\[
G(z) \in P_n \prod_{i=1}^{n-1} a_i \left( \prod_{j=1}^{n-1} m_j \prod_{i=1}^{n} (A_i + a_i B_i) - \prod_{i=1}^{n} a_i B_i, \prod_{i=1}^{n} B_i \right),
\]

now we will prove that \( G(z) \in P_{a_1}(A_1, B_1) \ast_{m_1} P_{a_2}(A_2, B_2) \ast \cdots \ast_{m_{n-1}} P_{a_n}(A_n, B_n) \). Let

\[
f_i(z) \in P_{a_i}(A_i, B_i), i = 1, 2, \ldots, n.
\]

From Theorem 1, we know that

\[
H(z) = f_1 \ast_{m_1} f_2 \ast_{m_2} f_3 \ast \cdots \ast_{m_{n-2}} f_{n-1}(z) \in P_n \prod_{i=1}^{n-1} a_i \left( \prod_{j=1}^{n-2} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i) - \prod_{i=1}^{n-1} a_i B_i, \prod_{i=1}^{n-1} B_i \right).
\]
The same proof works for the results proved by J. Stankiewicz, Z. Stankiewicz [16].

we can obtain the results proved by L. Xiong, X. Liu ([18], Theorem 2.2).

Since $|B_1 B_2 \cdots B_{n-1}| = 1$, so $|B_1 B_2 \cdots B_{n-1}| = 1$. Let

$$
\mathcal{X}(z) = \frac{\prod_{i=1}^{n-1} a_i B_i - m_{n-1} \prod_{i=1}^{n-1} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i) + \prod_{i=1}^{n-1} a_i B_i - m_{n-1} \prod_{i=1}^{n-1} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i) \prod_{i=1}^{n-1} a_i B_i}{\prod_{i=1}^{n-1} m_{n-1} \prod_{i=1}^{n-1} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i) \prod_{i=1}^{n-1} a_i B_i} \cdot \frac{1}{\prod_{i=1}^{n-1} B_i}.
$$

(2.1)

In view of Lemma 1, it is easy to obtain

$$
\mathcal{G}(z) * m_{n-1} \mathcal{X}(z) \in P_{P_{a_i}} \left( \prod_{i=1}^{n-1} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i) - \prod_{i=1}^{n-1} a_i B_i \prod_{i=1}^{n-1} B_i \right) \subset P(A_n, B_n). \quad (2.2)
$$

Furthermore, from Lemma 1,

$$
\mathcal{H}(z) * m_{n-1} \mathcal{X}(z) \in P_{P_{a_i}} \left( \prod_{i=1}^{n-1} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i) - \prod_{i=1}^{n-1} a_i B_i \prod_{i=1}^{n-1} B_i \right) \subset P(0, 1) \quad (2.3)
$$

and $\mathcal{H}(z) * m_{n-1} \mathcal{X}(z) = 1 + \frac{1}{m_{n-1}}(z + z^2 + z^3 + \cdots + z^n + \cdots)$. Obviously,

$$
\mathcal{G}(z) = \mathcal{G}(z) * m_{n-1} \left( \mathcal{H}(z) * m_{n-1} \mathcal{X}(z) \right) = \mathcal{H}(z) * m_{n-1} \left( \mathcal{G}(z) * m_{n-1} \mathcal{X}(z) \right)
$$

$$
\in P_{a_1}(A_1, B_1) * m_{1} P_{a_2}(A_2, B_2) * \cdots * m_{n-1} P_{a_n}(A_n, B_n).
$$

The same proof works for $|B_n| = 1$. This is the end of the proof. □

**Remark 1.** (1) Putting $n = 2$, $a_1 = a_2 = 1$, $m_1 = 1$ in Theorem 2, we can obtain the results proved by J. Stankiewicz, Z. Stankiewicz [16].

(2) Putting $n = 2$, $a_1 = a$, $a_2 = b$, $m_1 = m$, $A_1 = A$, $B_1 = B$, $A_2 = C$, $A_2 = D$ in Theorem 2, we can obtain the results proved by Piejko ([13], Theorem 1).

(3) Putting $a_1 = a_2 = \cdots = a_n = 1$ and $m_1 = m_2 = \cdots = m_{n-1} = 1$ in Theorem 2, we can obtain the results proved by L. Xiong, X. Liu ([18], Theorem 2.2).
**Theorem 3.** Let $a_i, A_i, B_i, \ (i=1,2,\ldots,n, \ n \in \mathbb{Z}^+)$, $X, Y$ and $m_j \ (j=1,2,\ldots,\ n-1, n \in \mathbb{Z}^+)$ are given complex numbers such that $m_j \neq 0$, $A_i + a_i B_i \neq 0$, $|B_i| \leq 1$, $|Y| \leq 1$. Then

$$P_{a_1}(A_1, B_1) \ast m_1 P_{a_2}(A_2, B_2) \cdots \ast m_{n-1} P_{a_n}(A_n, B_n) = P_c(X, Y) \quad (2.4)$$

if and only if $|\prod_{i=1}^n B_i| = |Y|, \ \left(\prod_{j=1}^{n-1} m_j \prod_{i=1}^n (A_i + a_i B_i) - \prod_{i=1}^n a_i B_i\right) = (\prod_{i=1}^n B_i) X, c = \prod_{i=1}^n a_i, |B_{s_1}B_{s_2}B_{s_{n-1}}| = 1$ or $|B_{s_n}| = 1$, where $s_1, s_2, \ldots, s_n \in \{1,2,\ldots,n\}$.

**Proof.** Assume the formula (2.4) holds for some complex $m_j \neq 0 \ (j=1,2,\ldots,n-1)$, $A_i + B_i \neq 0 \ (i=1,2,\ldots,n)$, $|B_i| \leq 1$, $|Y| \leq 1$. Now we prove that $|B_{s_1}B_{s_2}\cdots B_{s_{n-1}}| = 1$ or $|B_{s_n}| = 1$, where $s_1, s_2, \ldots, s_n \in \{1,2,\ldots,n\}$. To obtain the contradiction, suppose that $|B_{s_1}B_{s_2}\cdots B_{s_{n-1}}| < 1$ and $|B_{s_n}| < 1$. Without loss of generality, we can set $0 \leq B_1 B_2 \cdots B_{n-1} < 1$, $0 \leq B_n < 1$, $0 \leq Y < 1$. Using the method of Piejko [12] we show that there exists a sequence of functions $w_\nu(z)$ of the class $\Omega$ such that for all positive integers $\nu$,

$$w_\nu(z) = \sum_{n=1}^{\infty} r_{\nu,n} z^n \quad (2.5)$$

and that coefficients $r_{\nu,n}$ of power series have the following properties: $r_{\nu,n} > 0$ for all $\nu$ and $n \in \{1,2,\ldots,n+1\}$ and $S_\nu \to \infty$ as $\nu \to \infty$, where $S_\nu = r_{\nu,1} + r_{\nu,2} + \cdots + r_{\nu,n+1}$. Consider the sequence of functions $T_\nu$, where

$$T_\nu(z) = \frac{\prod_{i=1}^n a_i + X w_\nu(z)}{1 - Y w_\nu(z)} \quad (2.6)$$

and $w_\nu(z) \in \Omega$ is given by (2.5). For all $\nu \in \mathbb{Z}^+$, we have $T_\nu(z) \in P_n \left(\prod_{i=1}^n a_i, X, Y\right)$. Obviously, from the above assumption, there exists $f_i \in P_{a_i}(A_i, B_i), \ i = 1,2,\cdots,n$, such that $f_1 \ast m_1 f_2 \ast m_2 f_3(z) \ast \cdots \ast m_{n-1} f_n(z) = T_\nu(z)$. Let

$$\mathcal{H} = f_1 \ast m_1 f_2 \ast m_2 f_3(z) \ast \cdots \ast m_{n-2} f_{n-1}(z),$$

then from Theorem 1, we have

$$\mathcal{H} \in P_{n-1} \prod_{i=1}^{n-1} (\prod_{j=1}^{m_j} \prod_{i=1}^n (A_i + a_i B_i) - \prod_{i=1}^n a_i B_i, \prod_{i=1}^n B_i)$$

and $\mathcal{H} \ast m_{n-1} f_n(z) = T_\nu(z)$. Let the functions $\mathcal{H}(z), f_n(z)$ and $T_\nu(z)$ have the following forms:

$$\mathcal{H}(z) = \prod_{i=1}^{n-1} a_i + \prod_{j=1}^{n-2} m_j \prod_{i=1}^{n-1} (A_i + a_i B_i) \mathcal{H},$$

$$f_n(z) = a_n + (A_n + a_n B_n) f_n, \quad T_\nu(z) = \prod_{i=1}^n a_i + (X + Y T_\nu(z).$$
where
\[
\mathcal{H} = \frac{w_1(z)}{1 - (\prod_{i=1}^{n-1} B_i)w_1(z)} = \sum_{n=1}^{\infty} a_n z^n,
\]
\[
\mathcal{f}_n(z) = \frac{w_2(z)}{1 - B_n w_2(z)} = \sum_{n=1}^{\infty} b_n z^n,
\]
\[
\mathcal{T}_v = \frac{w_v(z)}{1 - Yw_v(z)} = \sum_{n=1}^{\infty} c_{v,n} z^n,
\] (2.7)

Since \(\mathcal{H} * m_{n-1} f_n(z) = T_v(z)\), using the notations we can rewrite as
\[
\mathcal{H} * f_n(z) = X + Y \prod_{j=1}^{n} m_j \prod_{i=1}^{n} (A_i + a_i B_i),
\]

So
\[
a_n b_n = \frac{X + Y}{\prod_{j=1}^{n} m_j \prod_{i=1}^{n} (A_i + a_i B_i)} c_{v,n},
\] (2.8)

On the one hand, from (2.7) we have
\[
\sum_{n=1}^{\infty} c_{v,n} z^n = \sum_{n=1}^{\infty} r_{v,n} z^n + Y(\sum_{n=1}^{\infty} r_{v,n} z^n)^2 + Y^3(\sum_{n=1}^{\infty} r_{v,n} z^n)^3 + \ldots
\]
\[= r_{v,1} z + (r_{v,2} + Y(r_{v,1})^2) z^2 + (r_{v,3} + 2Y r_{v,1} r_{v,2} + Y^2(r_{v,1})^3) z^3 + \ldots (2.9)\]

Thus, since \(0 \leq Y < 1\) and \(r_{v,n} > 0\) \((n = 1, 2, \ldots, v + 1)\), so (2.9) makes sure that
\[
\sum_{n=1}^{\infty} |c_{v,n}| \geq \sum_{n=1}^{v+1} |c_{v,n}| \geq r_{v,1} + r_{v,2} + r_{v,3} + \ldots + r_{v,v} + r_{v,v+1} = S_v. \] (2.10)

On the other hand, we have
\[
\mathcal{H} \prec \frac{z}{1 - (\prod_{i=1}^{n-1} B_i)z} = z + (\prod_{i=1}^{n-1} B_i) z^2 + (\prod_{i=1}^{n-1} B_i)^2 z^3 + \ldots
\]
and \(0 \leq B_1 B_2 \cdots B_{n-1} < 1\), \(0 \leq B_n < 1\), then from Lemma 2, we obtain:
\[
\sum_{n=1}^{\infty} |a_n|^2 \leq \frac{1}{1 - (\prod_{i=1}^{n-1} B_i)^2}
\]
and
\[
\sum_{n=1}^{\infty} |b_n|^2 \leq \frac{1}{1 - (B_n)^2}. \] (2.11)
Combining (2.8), (2.10) and (2.11), it gives:

$$0 \leq \sum_{n=1}^{\infty} (|a_n| - |b_n|)^2 \leq \frac{1}{1 - \left(\prod_{i=1}^{n-1} B_i\right)^2} + \frac{1}{1 - B_n^2} - \left|\frac{2(X + Y)}{\prod_{j=1}^{n-1} m_j \prod_{i=1}^{n} (A_i + a_i B_i)}\right| \sum_{n=1}^{\infty} |c_{v,n}|$$

As $S_v \to +\infty$ when $v \to \infty$, it follows from (2.5) that we are able to choose a suitable $v$ such that the right side of (2.12) is negative. In this way, (2.12) follows the contradiction and so we prove that $|B_1 B_2 \cdots B_{n-1}| = 1$ or $|B_n| = 1$. In fact, the progress of the proof implies that $|B_{s_1} B_{s_2} \cdots B_{s_{n-1}}| = 1$ or $|B_{s_n}| = 1$, where $s_1, s_2, \cdots, s_n \in \{1, 2, \ldots, n\}$.

From Theorem 2, if $|B_{s_1} B_{s_2} \cdots B_{s_{n-1}}| = 1$ or $|B_{s_n}| = 1$, we have

$$P_{a_1}(A_1, B_1) * P_{a_2}(A_2, B_2) * \cdots * P_{a_n}(A_n, B_n) = P_c(X, Y).$$

However, following the assumption, it gives

$$P_{a_1}(A_1, B_1) * P_{a_2}(A_2, B_2) * \cdots * P_{a_n}(A_n, B_n) = P_c(X, Y).$$

So $c = \prod_{i=1}^{n} a_i$ and

$$P_c\left(\prod_{j=1}^{n-1} m_j \prod_{i=1}^{n} (A_i + a_i B_i) - \prod_{i=1}^{n} a_i B_i, \prod_{i=1}^{n} B_i\right) = P_c(X, Y).$$

Piejko [13] gave $P_a(A, B) = P_a(X, Y)$ if and only if $X = Ae^{i\theta}$, $Y = Be^{i\theta}$, where $\theta$ is a real number. Thus $|Y| = \left|\prod_{i=1}^{n} B_i e^{i\theta}\right| = \prod_{i=1}^{n} |B_i|$, $(\prod_{j=1}^{n-1} m_j \prod_{i=1}^{n} (A_i + a_i B_i) - \prod_{i=1}^{n} a_i B_i)Y = X e^{-i\theta} \prod_{i=1}^{n} B_i e^{i\theta} = (\prod_{i=1}^{n} B_i)X$. This ends the first part of the proof.

Conversely, if $|B_{s_1} B_{s_2} \cdots B_{s_{n-1}}| = 1$ or $|B_{s_n}| = 1$, where $s_1, s_2, \cdots, s_n \in \{1, 2, \ldots, n\}$, then from the Theorem 2, it has

$$P_{a_1}(A_1, B_1) * P_{a_2}(A_2, B_2) * \cdots * P_{a_n}(A_n, B_n)$$

$$= P_n \prod_{i=1}^{n} a_i \left(\prod_{j=1}^{n-1} m_j \prod_{i=1}^{n} (A_i + a_i B_i) - \prod_{i=1}^{n} a_i B_i, \prod_{i=1}^{n} B_i\right).$$

Since $\left|\prod_{i=1}^{n} B_i\right| = |Y|$, so $B_1 B_2 B_3 \cdots B_n e^{i\theta} = Y$, where $\theta$ is a real number. Moreover, it notes

$$\left(\prod_{j=1}^{n-1} m_j \prod_{i=1}^{n} (A_i + a_i B_i) - \prod_{i=1}^{n} a_i B_i\right)Y = \left(\prod_{i=1}^{n} B_i\right)X = Ye^{-i\theta}X,$$
thus $X = \left( \prod_{j=1}^{n-1} m_j \prod_{i=1}^{n} (A_i + a_i B_i) - \prod_{i=1}^{n} a_i B_i \right) e^{i\theta}$. This is the end of the proof. \qed

**Remark 2.**
(1) Putting $n = 2$, $a_1 = a_2 = 1$, $m_1 = 1$, $A_1 = A$, $B_1 = B$, $A_2 = C$, $B_2 = D$ in Theorem 3, we can obtain the results proved by Piejko ([12], Theorem 1).

(2) Putting $n = 2$, $a_1 = a$, $a_2 = b$, $m_1 = m$, $A_1 = A$, $B_1 = B$, $A_2 = C$, $B_2 = D$ in Theorem 3, we can obtain the results proved by Piejko ([13], Corollary 1).

(3) Putting $a_1 = a_2 = \ldots = a_n = 1$, $m_1 = m_2 = \ldots = m_{n-1} = 1$ in Theorem 3, we can obtain the results proved by L. Xiong, X. Liu ([18], Theorem 2.3).

(4) Theorem 1 and Theorem 2 imply that we can choose some suitable complex numbers $X, Y$ such that

$$P_{a_1}(A_1, B_1) * P_{a_2}(A_2, B_2) * P_{a_3}(A_3, B_3) * \cdots * P_{a_{n-1}}(A_{n-1}, B_{n-1}) \subset P_c(X, Y)$$

but the problem of inverse inclusion can hold unless $A_1, B_i, m_i, X, Y$ satisfy for some conditions. Moreover, following the proof of Theorem 3, we can know that there exists some functions belonging to $P_c(A, B)$ which can not be represented as the finite Hadamard product of the form (2.4). In fact, it is clear that there do not exist complex $m_i, a_i, A_i, B_i, X, Y$ with $A_i + a_i B_i \neq 0$, $X + c Y \neq 0$, $|B_i| < 1$, $|Y| < 1$ such that $P_c(X, Y) = P_{a_1}(A_1, B_1) * P_{a_2}(A_2, B_2) * P_{a_3}(A_3, B_3) * \cdots * P_{a_{n-1}}(A_{n-1}, B_{n-1}).$

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