Journal of Classical Analysis
Volume 5, Number 2 (2014), 91–105 doi:10.7153/jca-05-08

TRIGONOMETRIC APPROXIMATION OF PERIODIC SIGNALS BELONGING TO GENERALIZED WEIGHTED LIPSCHITZ $W^{'}(L_r,\xi(t)),(r\geqslant 1)$ — CLASS BY NÖRLUND–EULER $(N,p_n)(E,q)$ OPERATOR OF CONJUGATE SERIES OF ITS FOURIER SERIES

VISHNU NARAYAN MISHRA, KEJAL KHATRI,
LAKSHMI NARAYAN MISHRA AND DEEPMALA

Abstract. Approximation theory has been an established field of mathematics in the past century. Analysis of signals or time functions is of great importance, because it conveys information or attributes of some phenomenon. The engineers and scientists use properties of Fourier approximation for designing digital filters. In the present paper, an attempt is made to determine a theorem on the degree of approximation of a function $\tilde{f}$, conjugate to a 2π-periodic signal belonging to the generalized weighted Lipschitz $W'(L_r,\xi(t))$, $(r\geqslant 1)$-class by product $(N,p_n)(E,q)$ summability, which in turn generalizes the results of Mishra et al. [17]. In support of our theorem, we illustrated examples and deduced some corollaries from our main result of this paper.

1. Introduction

It is known that if $f\in L_r$ $(r>1)$ then the Fourier series of signal f converges in L_r-norm. Govil [3], Mohapatra and Chandra [19] study on the summability of a class of the derived conjugate series of a Fourier series and degree of approximation of functions in the Hölder metric respectively. Many authors, e.g., Khan ([6]–[9]), Mittal et al. ([10]–[12]), Mishra and Mishra [13], Mishra et al. ([14]–[17]) and Mishra et al. [18] studied the degree of approximation of a function belonging to various classes using different types of summability matrices. Recently, Canak and Erdem [2] studied very interesting results on Tauberian theorems for $A(C,\alpha)$ summability method. Recently, Mursaleen and Mohiuddine [20] discussed convergence methods for double sequences and its applications in various fields. Mishra et al. [17] have defined $(N,p_n)(E,q)$ summability means and determined the degree of approximation of a functions belonging to $Lip(\xi(t),r)$ class using $(N,p_n)(E,q)$ product summability method. In this paper, we continue the work in the same direction and extend the result of Mishra et al. [17] to obtain a theorem on the degree of approximation of a function $\tilde{f}$, conjugate to a 2π-periodic signal $f\in W'(L_r,\xi(t))$, $(r\geqslant 1)$-class by matrix $(N,p_n)(E,q)$ operator. More precisely, we prove a theorem in this paper which provide sharper estimates than those which were obtained in [17], while in a different setting

Mathematics subject classification (2010): 41A10, 42B05, 42B08, 40G05.
Keywords and phrases: Degree of approximation, generalized weighted Lipschitz class of functions, matrix operator, conjugate Fourier series, Lebesgue integral.

Paper JCA-05-08

2. Definitions and notations

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its n^{th} partial sums $\{s_n\}$. Let $\{p_n\}$ be a non-negative generating sequence of constants, real or complex, and let us write

$$P_n = \sum_{k=0}^{n} p_k \neq 0 \;\forall \quad n \geqslant 0, \quad p_{-1} = 0 = P_{-1} \text{ and } P_n \to \infty \text{ as } n \to \infty.$$

An infinite matrix is called regular if it maps every bounded convergent sequence into a convergent sequence with the same limit.

The conditions for regularity of Nörlund summability are easily seen to be

$$(1) \quad \lim_{n\to\infty} \frac{p_k}{P_n} \to 0, \;\forall k \text{ and}$$

$$(2) \quad \sum_{k=0}^{n} |p_k| = O(P_n).$$

The sequence-to-sequence transformation

$$t_n^N = \sum_{k=0}^{n} \frac{p_{n-k} s_k}{P_n} \tag{1}$$

defines the sequence $\{t_n^N\}$ of Nörlund means of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum_{n=0}^{\infty} u_n$ is said to be (N, p_n) summable to the sum s if $\lim_{n\to\infty} t_n^N$ exists and is equal to s.

In the special case in which

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)}; \quad (\alpha > 0).$$

The Nörlund summability N_p reduces to the familiar (C, α) summability.

An infinite series $\sum_{n=0}^{\infty} u_n$ is said to be $(C, 1)$ summable to s if

$$(C,1) = \frac{1}{n+1} \sum_{k=0}^{n} s_k \to s, \;\; as \;\; n \to \infty,$$

The (E, q) transform is defined as the n^{th} partial sum of (E, q), $q > 0$ summability and we denote it by E_n^q. If

$$E_n^q = \frac{1}{(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k \to s, \;\; as \;\; n \to \infty, \tag{2}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable (E, q) to a definite number s Hardy [5]. The (N, p_n) transform of the (E, q) transform defines $(N, p_n)(E, q)$ product transform and denotes it by t_n^{NE}. Thus if

$$t_n^{NE} = \frac{1}{P_n} \sum_{k=0}^{n} \frac{p_{n-k}}{(1+q)^k} \sum_{\nu=0}^{k} \binom{k}{\nu} q^{k-\nu} s_\nu \to s, \;\; as \;\; n \to \infty, \tag{3}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable $(N,p_n)(E,q)$ to the sum s.

$$s_n \to s \Rightarrow (E,q)(s_n) = E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}s_k \to s, \;\; as \;\; n\to\infty,$$

$$(E,q) \text{ method is regular,}$$

$$\Rightarrow ((N,p_n)(E,q)(s_n)) = t_n^{NE} \to s, \;\; as \;\; n\to\infty, \quad (N,p_n) \text{ method is regular}$$

$$\Rightarrow (N,p_n)(E,q) \text{ method is regular.}$$

A signal $f\in Lip\alpha$, if

$$f(x+t)-f(x) = O(|t^{\alpha}|) \;\; for \;\; 0<\alpha\leqslant 1, \;\; t>0$$

and $f\in Lip(\alpha,r), \; for\; 0\leqslant x\leqslant 2\pi$ [6], if

$$\left(\int_0^{2\pi}|f(x+t)-f(x)|^r dx\right)^{\frac{1}{r}} = O\Big(|t|^{\alpha}\Big), \;\; 0<\alpha\leqslant 1, \;\; r\geqslant 1, \;\; t>0.$$

Given a positive increasing function $\xi(t), f\in Lip(\xi(t),r)$ if

$$\left(\int_0^{2\pi}|f(x+t)-f(x)|^r dx\right)^{\frac{1}{r}} = O\Big(\xi(t)\Big), \;\; r\geqslant 1, \;\; t>0. \tag{4}$$

A signal $f\in W(L_r,\xi(t))$-class

$$\left(\int_0^{2\pi}|f(x+t)-f(x)|^r \sin^{\beta r}(x)dx\right)^{\frac{1}{r}} = O\Big(\xi(t)\Big), \;\; \beta\geqslant 0, \;\; r\geqslant \;\; t>0.$$

A signal $f\in W'(L_r,(\xi(t)))$-class ([8]–[9]), if

$$\left(\int_0^{2\pi}|f(x+t)-f(x)|^r \sin^{\beta r}\Big(x/2\Big)dx\right)^{\frac{1}{r}} = O\Big(\xi(t)\Big), \;\; \beta\geqslant 0, \;\; r\geqslant \;\; t>0. \tag{5}$$

where $\xi(t)$ is positive increasing function of t. If $\beta=0$, then the generalized weighted Lipschitz $W'(L_r,(\xi(t)))$, $(r\geqslant 1)$-class reduces to the class $Lip(\xi(t),r)$, if $\xi(t)=t^{\alpha}$, $(0\leqslant\alpha\leqslant 1)$ then $Lip(\xi(t),r)$ class coincides with the class $Lip(\alpha,r)$ and if $r\to\infty$ then $Lip(\alpha,r)$ reduces to the class $Lip\alpha$.

Suppose that $\omega(\delta;f)$, $\omega_r(\delta;f)$, $\omega_r^2(\delta;f)$ denote, respectively, stand for the modulus of continuity, integral modulus of continuity and integral modulus of smoothness which are non-negative and non-decreasing (see [[22], pp. 42 and 45]). In the case $0<\alpha\leqslant 1$ and $\omega(\delta;f)=O(\delta^{\alpha})$, we write $f\in Lip\alpha$ and if $\omega_r(\delta;f)=O(\delta^{\alpha})$ we write $f\in Lip(\alpha,r)$. Also if either

$$\omega_r(\delta;f)=O(\delta) \;\text{ or }\; \omega(\delta;f)=O(\delta), \;\text{ as }\; \delta\to 0,$$

holds then the function f turns out to be constant. Further, the class $Lip(\alpha,r)$ with $r=\infty$ will be taken as $f\in Lip\alpha$

$$\omega(\delta)=\omega(\delta;f)=\sup_{0\leqslant h\leqslant\delta}\|f(x+h)-f(x)\|_c$$

is called modulus of continuity.

$$\omega_p(\delta) = \omega_p(\delta; f) = \sup_{0 \leqslant h \leqslant \delta} \|f(x+h) - f(x)\|_p$$

is called the integral modulus of continuity.

$$\omega_p^{(2)}(\delta) = \omega_p^{(2)}(\delta; f) = \sup_{0 \leqslant h \leqslant \delta} \|f(x+h) + f(x-h) - 2f(x)\|_p$$

is called the integral modulus of smoothness.

The L_r-norm of a function $f : R \to R$ is defined by

$$\|f\|_r = \left(\int_0^{2\pi} \frac{1}{2\pi} |f(x)|^r dx \right)^{\frac{1}{r}}, \quad (1 \leqslant r < \infty). \tag{6}$$

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx \right), \quad (0 < r < 1).$$

Throughout the paper, $f \in L_r (r \geqslant 1)$ is taken to be non-constant so that (see [22], p. 45)

$$n^{-1} = O(1) w_p(n^{-1}; f), \text{ as } n \to \infty.$$

For the convenience in the working, we also write $\|f(x)\|_r$ for $\|f\|_r$ $(1 \leqslant r \leqslant \infty)$.

Let f be a real valued and measurable function defined on $[a,b]$. If

$$|f(x)| \leqslant M \text{ a.e. on } [a,b]$$

then f is said to be essential bounded on $[a,b]$ and M is an essential bound of it. The essential supremum of f on $[a,b]$ is defined by

$$ess\,sup|f(x)| = inf\{M : |f(x)| \leqslant M \text{ a.e. on } [a,b]\}.$$

If f doest not have any essential bound, then its essential supremum is defined to be ∞. Let us denote by $L^\infty[a,b]$ the class of all those (Lebesgue) measurable functions defined on $[a,b]$ which are essentially bounded on $[a,b]$, i.e.,

$$L^\infty[a,b] = \{f : [a,b] \to \mathbb{R} : ess\,sup|f| < \infty\}.$$

It is easy to verify that $L^\infty[a,b]$ is a linear space over $\mathbb{R}$. L_∞-norm of a function $f : R \to R$ is defined by $\|f\|_\infty = \|f\|_c = \sup\{|f(x)| : x \in R\}$. The space $L_r[0,2\pi]$ with $r = \infty$ includes the space $C_{2\pi}$ of all 2π-periodic continuous functions over $[0,2\pi]$, where $C_{2\pi}$ denote the Banach space of all 2π-periodic continuous functions defined on $[0,2\pi]$. The Lipschitz condition is given by

$$\sup_{0 \leqslant h \leqslant \delta} \frac{\|f(x+h) - f(x)\|_c}{|\delta|^\beta} \leqslant K \text{ (+ve constant) when } p = \infty$$

$$\sup_{0 \leqslant h \leqslant \delta} \frac{\|f(x+h) - f(x)\|_p}{|\delta|^\beta} \leqslant K \text{ (+ve constant) when } 0 < p < \infty.$$

A signal (function) f is approximated by trigonometric polynomials t_n of order n and the degree of approximation $E_n(f)$ is given by Zygmund [22] as follows:

$$E_n(f) = \min_n \|f(x) - t_n(f;x)\|_r \tag{7}$$

in terms of n, where $t_n(f;x)$ is a trigonometric polynomial of degree n. This method of approximation is called Trigonometric Fourier Approximation (TFA) [12].

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial t_n of order n under sup norm $\| \|_\infty$ is defined by

$$\|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\}.$$

Representation of a function over a certain interval by a linear combination of mutually orthogonal functions is called Fourier series representation. Let $f(x)$ be a 2π-periodic function and Lebesgue integrable. The Fourier series of $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x), \tag{8}$$

with n^{th} partial sums $s_n(f;x)$.

The conjugate series of Fourier series (8) is given by

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx -) \equiv \sum_{n=1}^{\infty} B_n(x). \tag{9}$$

We note that t_n^N and t_n^{NE} are also trigonometric polynomials of degree (or order) n.

Particular Cases:

(1) $(N,p_n)(E,q)$ means reduces to $(N,1/(n+1))(E,q)$ means if $p_n = 1/(n+1)$.

(2) $(N,p_n)(E,q)$ means reduces to $(N,1/(n+1))(E,1)$ means if $p_n = 1/(n+1)$ and $q_n = 1 \ \forall \ n$.

(3) $(N,p_n)(E,q)$ means reduces to $(N,p_n)(E,1)$ means if $q_n = 1 \ \forall \ n$.

(4) $(N,p_n)(E,q)$ means reduces to $(C,\delta)(E,q)$ means if $p_n = \binom{n+\delta-1}{\delta-1}$ $\delta > 0$.

(5) $(N,p_n)(E,q)$ means reduces to $(C,\delta)(E,1)$ means if $p_n = \binom{n+\delta-1}{\delta-1}$ $\delta > 0$ and $q_n = 1 \ \forall \ n$.

(6) $(N,p_n)(E,q)$ means reduces to $(C,1)(E,1)$ means if $p_n = 1$ and $q_n = 1 \ \forall \ n$.

We use the following notations throughout this paper:

$$\psi(t) = f(x+t) - f(x-t),$$

$$\widetilde{G}_n(t) = \frac{1}{2\pi P_n}\left[\sum_{k=0}^{n} \frac{p_{n-k}}{(1+q)^k} \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \frac{\cos\left(v+1/2\right)t}{\sin\left(t/2\right)}\right].$$

We note that the series, conjugate to a Fourier series, is not necessarily a Fourier series [1]. Hence a separate study of conjugate series is desirable and attracted the attention of researchers.

3. Known result

Approximation by trigonometric polynomials is at the heart of approximation theory. The most important trigonometric polynomials used in the approximation theory are obtained by linear summation methods of Fourier series of 2π- periodic signal (functions) on the real line (i.e. Euler means, Nörlund means and Product Nörlund-Euler means etc.). Much of the advance in the theory of trigonometric approximation is due to the periodicity of the functions. Very recently, Mishra et al. [17] have proved the following theorem:

THEOREM 3.1. *If $\widetilde{f}(x)$ is conjugate to a 2π-periodic function f belonging to $Lip(\xi(t),r)$-class, then the degree of approximation by $(N,p_n)(E,q)$ product summability means of conjugate series of Fourier series is given by*

$$\|\widetilde{t}_n^{NE}-\widetilde{f}\|_r = O\left((n+1)^{1/r}\xi(1/(n+1))\right), \tag{10}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left(\int_0^{\pi/(n+1)}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^r dt\right)^{\frac{1}{r}} = O((n+1)^{-1}), \tag{11}$$

and

$$\left(\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^r dt\right)^{\frac{1}{r}} = O((n+1)^{\delta}), \tag{12}$$

where δ is an arbitrary number such that $s(1-\delta)-1>0,\ r^{-1}+s^{-1}=1$ for $1\leqslant r\leqslant\infty$, conditions (11) and (12) hold uniformly in x and $\widetilde{t}_n^{NE}$ is $(N,p_n)(E,q)$ means of the series (9).

4. Main Theorem

The theory of approximation is a very extensive field and the study of theory of trigonometric approximation is of great mathematical interest and of great practical importance. It is well known that the theory of approximations i.e., TFA, which originated from a well-known theorem of Weierstrass, has become an exciting interdisciplinary field of study for the last 130 years. These approximations have assumed important new dimensions due to their wide applications in signal analysis in general and in Digital Signal Processing in particular, in view of the classical Shannon sampling theorem [14]. Broadly speaking, signals are treated as functions of one variable and images are represented by functions of two variables. Analysis of signals or time functions is of great importance, because it conveys information or attributes of some phenomenon. The engineers and scientists use properties of Fourier approximation for designing digital filters. Especially, Psarakis and Moustakides [21] presented a new L_2 based method for designing the Finite Impulse Response (FIR) digital filters and get corresponding optimum approximations having improved performance. In 2007 and 2012, one of the authors of present paper obtained a number of order estimates including those of "Jackson order" ([11], [13]). This has motivated us to proceed to obtain a general result and deduce from it some other order-estimates of Jackson order, as the degree of approximation of signal f by $(N,p_n)(E,q)$ in L_r-norm. But till now, nothing seems to have been done so far to obtain the degree of approximation of conjugate of a function using Nörlund-Euler $(N,p_n)(E,q)$ product summability method of its conjugate series of Fourier series on the generalized weighted Lipschitz $W^{'}(L^r,\xi(t))$ $(r\geqslant 1)$-class with a proper (new) set of conditions.

In the present paper, we extend the result of Mishra et al. [17] on the generalized weighted Lipschitz $W^{'}(L^r,\xi(t))$ $(r\geqslant 1)$-class. The main theorem in this direction as follows:

THEOREM 4.1. *If $\widetilde{f}(x)$ is conjugate to a 2π-periodic function $f\in W^{'}(L^r,\xi(t))$ $(r\geqslant 1)$-class, then its degree of approximation by matrix $(N,p_n)(E,q)$ operator of conjugate series of its Fourier series is given by*

$$\|\widetilde{t}_n^{NE}(f)-\widetilde{f}(x)\|_r=O\left\{(n+1)^{\frac{1}{r}+\beta}\right\}\xi\left(\frac{1}{n+1}\right), \tag{13}$$

provided $\xi(t)$ satisfies the following conditions:

$$\frac{\xi(t)}{t}\ \text{ is non-increasing in } {}'t', \tag{14}$$

$$\left(\int_0^{\frac{\pi}{n+1}}\left(\frac{|\psi(t)|}{\xi(t)}\right)^r\sin^{\beta r}(t/2)\,dt\right)^{1/r}=O(1), \tag{15}$$

$$\left(\int_{\frac{\pi}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^r dt\right)^{1/r}=O((n+1)^{\delta}), \tag{16}$$

where δ is an arbitrary number such that $s(\beta-\delta)-1>0$, $\underbrace{\frac{1}{r}+\frac{1}{s}}=1$, $1\leqslant r\leqslant\infty$, conditions (15) and (16) hold uniformly in x and $\widetilde{t}_n^{NE}$ is $(N,p_n)(E,q)$ means of the series (9) and the conjugate function $\widetilde{f}(x)$ is defined for almost every x by

$$\widetilde{f}(x)=-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cos(t/2)dt=\lim_{h\to 0}\left(-\frac{1}{2\pi}\int_h^{\pi}\psi(t)\cos(t/2)dt\right). \tag{17}$$

NOTE 4.2. $\xi\left(\frac{\pi}{n+1}\right)\leqslant\pi\xi\left(\frac{1}{n+1}\right)$, for $\left(\frac{\pi}{n+1}\right)\geqslant\left(\frac{1}{n+1}\right)$.

NOTE 4.3. The product transform $(N,p_n)(E,q)$ plays an important role in signal theory as a double digital filter and theory of machines in mechanical engineering ([14],[17]).

5. Lemmas

In order to prove our Main Theorem, the following lemmas are required.

LEMMA 5.1. [17] $|\widetilde{G}_n(t)| = O\left[\frac{1}{t}\right]$ *for* $0 < t \leqslant \frac{\pi}{n+1}$.

LEMMA 5.2. [17] $|\widetilde{G}_n(t)| = O\left[\frac{1}{t}\right]$ *for* $0 < t \leqslant \pi$ *and* n.

6. Proof of Theorem

Let $\tilde{s}_n(x)$ denotes the partial sum of series (9), we have

$$\widetilde{s}_n(x) - \widetilde{f}(x) = \frac{1}{2\pi}\int_0^\pi \psi(t)\frac{\cos\left(n+1/2\right)t}{\sin\left(t/2\right)}dt.$$

Therefore, using (2) the (E,q) transform E_n^q of $\widetilde{s}_n$ is given by

$$\widetilde{E}_n^q - \widetilde{f}(x) = \frac{1}{2\pi(1+q)^n}\int_0^\pi \frac{\psi(t)}{\sin\left(t/2\right)}\left\{\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\cos\left(k+1/2\right)t\right\}dt.$$

Now denoting $(N,\widetilde{p_n)(E},q)$ transform of $\widetilde{s}_n$ as $\widetilde{t}_n^{NE}$, we write

$$\begin{aligned}
\widetilde{t}_n^{NE} - \widetilde{f}(x) &= \frac{1}{2\pi P_n}\sum_{k=0}^{n}\left[\frac{p_{n-k}}{(1+q)^k}\int_0^\pi \frac{\psi(t)}{\sin\left(t/2\right)}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\cos\left(\nu+1/2\right)t\right\}dt\right] \\
&= \int_0^\pi \psi(t)\widetilde{G}_n(t)\,dt \\
&= \left[\int_0^{\pi/(n+1)} + \int_{\pi(n+1)}^{\pi}\right]\psi(t)\widetilde{G}_n(t)dt \\
&= I_1 + I_2, (say).
\end{aligned} \tag{18}$$

We consider,

$$|I_1| \leqslant \int_0^{\pi/(n+1)} |\psi(t)|\left|\widetilde{G}_n(t)\right| dt.$$

Using Hölder's inequality, equation (15), $(\sin t/2)^{-1} \leqslant \pi/t$, for $0 < t \leqslant \pi$ and Lemma (5.1), we have

$$\begin{aligned}
|I_1| &\leqslant \left[\int_0^{\pi/(n+1)} \left(\frac{|\psi(t)|}{\xi(t)}\right)^r \sin^{\beta r}(t/2)dt\right]^{1/r} \left[\int_0^{\pi/(n+1)} \left(\frac{\xi(t)|\widetilde{G}_n(t)|}{\sin^\beta(t/2)}\right)^s dt\right]^{1/s} \\
&= O(1) \underset{0<t\leqslant\pi/(n+1)}{ess\,sup} [(\xi(t))^s]^{1/s} \left[\int_0^{\pi/(n+1)} t^{-(1+\beta)s} dt\right]^{1/s} \\
&= O\left(\xi\left(\frac{\pi}{n+1}\right)\right) \underset{0<t\leqslant\pi/(n+1)}{ess\,sup} \left[\int_0^{\pi/(n+1)} t^{-(1+\beta)s} dt\right]^{1/s} \\
&= O\left[\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+1-1/s}\right] \\
&= O\left[\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+1/r}\right] \quad \therefore \frac{1}{r}+\frac{1}{s}=1, \ \ 1 \leqslant r \leqslant \infty.
\end{aligned}$$

Now, we consider,

$$|I_2| \leqslant \int_{\pi/(n+1)}^{\pi} |\psi(t)| \left|\widetilde{G}_n(t)\right| dt.$$

Using Hölder's inequality, equation (16), $(\sin t/2)^{-1} \leqslant \pi/t$, for $0 < t \leqslant \pi$, $|\sin t/2| \leqslant 1$ and Lemma (5.2), we have

$$\begin{aligned}
|I_2| &\leqslant \left[\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\delta}\sin^\beta(t/2)|\psi(t)|}{\xi(t)}\right)^r dt\right]^{1/r} \left[\int_{\pi/(n+1)}^{\pi} \left(\frac{\xi(t)|\widetilde{G}_n(t)|}{t^{-\delta}\sin^\beta(t/2)}\right)^s dt\right]^{1/s} \\
&\leqslant \left[\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^r dt\right]^{1/r} \left[\int_{\pi/(n+1)}^{\pi} \left(\frac{\xi(t)}{t^{1-\delta}\sin^\beta(t/2)}\right)^s dt\right]^{1/s} \\
&= O((n+1)^\delta) \left[\int_{\pi/(n+1)}^{\pi} \left(\frac{\xi(t)}{t^{1-\delta}\sin^\beta(t/2)}\right)^s dt\right]^{1/s} \\
&= O((n+1)^\delta) \left[\int_{\pi/(n+1)}^{\pi} \left(\frac{\xi(t)}{t^{-\delta+\beta+1}}\right)^s dt\right]^{1/s}.
\end{aligned}$$

Now putting $t = 1/y$,

$$|I_2| = O\left((n+1)^\delta\right) \left[\int_{1/\pi}^{(n+1)/\pi} \left(\left(\frac{\xi(1/y)}{y^{\delta-\beta-1}}\right)^s \frac{dy}{y^2}\right)^{1/s}\right].$$

Since $\xi(t)$ is a positive increasing function so $\frac{\xi(1/y)}{1/y}$ is also a positive increasing function and using second mean value theorem for integrals, we have

$$
\begin{aligned}
|I_2| &= O\left((n+1)^{\delta}\frac{\xi(\pi/n+1)}{\pi/n+1}\right)\left[\left(\int_{1/\pi}^{(n+1)/\pi}\frac{dy}{y^{-\beta s+\delta s+2}}\right)^{1/s}\right] \\
&= O\left((n+1)^{\delta+1}\xi\left(\frac{1}{n+1}\right)\right)\left\{\left[\frac{y^{-\delta s+\beta s-1}}{-\delta s+\beta s-1}\right]_{1/\pi}^{(n+1)/\pi}\right\}^{1/s} \\
&= O\left((n+1)^{\delta+1}\xi\left(\frac{1}{n+1}\right)\right)(n+1)^{-\delta-1/s+\beta} \\
&= O\left(\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+1/r}\right) \therefore \frac{1}{r}+\frac{1}{s}=1,\ \ 1\leqslant r\leqslant\infty. \qquad (19)
\end{aligned}
$$

Combining I_1 and I_2 yields

$$
|\widetilde{t}_n^{NE}(f)-\widetilde{f}(x)| = O\left\{(n+1)^{\beta+1/r}\xi\left(\frac{1}{n+1}\right)\right\}. \qquad (20)
$$

Now, using the L_r -norm of a function, we get

$$
\begin{aligned}
\|\widetilde{t}_n^{NE}(f)-\widetilde{f}(x)\|_r &= \left\{\int_0^{2\pi}|\widetilde{t}_n^{NE}(f)-\widetilde{f}(x)|^r dx\right\}^{1/r} \\
&= O\left\{\int_0^{2\pi}\left((n+1)^{\beta+1/r}\xi\left(\frac{1}{n+1}\right)\right)^r dx\right\}^{1/r} \\
&= O\left\{(n+1)^{\beta+1/r}\xi\left(\frac{1}{n+1}\right)\left(\int_0^{2\pi}dx\right)^{1/r}\right\} \\
&= O\left((n+1)^{\beta+1/r}\xi\left(\frac{1}{n+1}\right)\right).
\end{aligned}
$$

This completes the proof of Theorem 4.1.

7. Applications

Analysis and its applications have been major areas for research in mathematics and allied fields. The fast growing power of computation also made a significant and useful impact on these areas leading to computational analysis and emergence of fields like Bézier-Bernstein methods for computer aided geometric design, constructive and computational approximation, optimal control, and theory and its applications of function spaces and wavelets. The theory of approximation is a very extensive field, which has various applications. The L_p-space in general, L_2 and L_∞ in particular play an important role in the theory of signals and filters. From the point of view of the applications, sharper estimates of infinite matrices [3] are useful to get bounds for the lattice norms (which occur in solid state physics) of matrix valued functions, and enables to investigate perturbations of matrix valued functions and compare them. Approximation theory has widely influenced such other areas of mathematics as orthogonal polynomials, partial differential equations, harmonic analysis and wavelet analysis. Some modern applications include computer graphics, signal processing, economic forecasting and pattern recognition.

The following corollaries can be derived from Theorem 4.1.

COROLLARY 7.1. *If* $\beta = 0$, *then the generalized weighted Lipschitz class* $W'(L_r, \xi(t))$, $r \geqslant 1$ *reduces to the class* $Lip(\xi(t), r)$ *and the degree of approximation is given by*

$$\|\widetilde{t}_n^{NE}(f) - \widetilde{f}(x)\|_r = O\left((n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\right). \tag{21}$$

This is the result of Mishra et al. [17].

COROLLARY 7.2. *If* $\beta = 0$, $\xi(t) = t^{\alpha}$, $0 < \alpha \leqslant 1$, *then* $f \in Lip(\alpha, r)$, $\alpha > 1/r$,

$$\|\widetilde{t}_n^{NE}(f) - \widetilde{f}(x)\|_r = O\left(\frac{1}{(n+1)^{\alpha-1/r}}\right). \tag{22}$$

COROLLARY 7.3. *If* $r \to \infty$ *in Corollary 7.2 then* $f \in Lip\alpha$ $(0 < \alpha < 1)$, *(22) gives*

$$\|\widetilde{t}_n^{NE}(f) - \widetilde{f}(x)\|_\infty = O\left(\frac{1}{(n+1)^{\alpha}}\right). \tag{23}$$

8. Examples

(i) One should pick a sequence that it is summable (E,q). Then it will be automatically summable $(N,p)(E,q)$ summable for any regular Nörlund matrix (N,p).

Choose the sequence $\{s_n\}$ defined by $s_{2n} = 1$, $s_{2n+1} = 2$ and choose $q > 1/2$. (For the cases in which $q \leqslant 1/2$, one needs to choose a different bounded divergent sequence.) Then verify that

$$\begin{aligned}
t_{2n} :&= \frac{1}{q^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} q^k (1-q)^{2n-k} s_k \\
&= \frac{1}{q^{2n}} \left[\sum_{k=0}^{2n} \binom{2n}{2k} q^{2k} (1-q)^{2n-2k}(1) + \sum_{k=0}^{n-1} \binom{2n}{2k+1} q^{2k+1} (1-q)^{2n-(2k+1)}(2) \right] \\
&= \frac{1}{q^{2n}} \left[\sum_{j=0}^{2n} \binom{2n}{j} q^j (1-q)^{2n-j} + (2) \sum_{j=0}^{2n-1} \binom{2n}{j} q^j (1-q)^{2n-j} \right] \\
&= \frac{1}{q^{2n}} \left[(q+1+q)^{2n} + 2(q+1-q)^{2n} - q^{2n} \right] \\
&= \frac{1}{q^{2n}} \left[3 - q^{2n} \right] \to 0.
\end{aligned}$$

$$
\begin{aligned}
t_{2n+1} :&= \frac{1}{q^{2n+1}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} q^k (1-q)^{2n+1-k} s_k \\
&= \frac{1}{q^{2n+1}} \left[\sum_{k=0}^{n} \binom{2n+1}{2k} q^{2k} (1-q)^{2n+1-(2k)} (1) + \sum_{k=0}^{n} \binom{2n+1}{2k+1} q^{2k+1} (1-q)^{2n-2k} (2) \right] \\
&= \frac{1}{q^{2n+1}} \left[\sum_{j=0}^{2n} \binom{2n+1}{j} q^j (1-q)^{2n+2-j} + 2 \sum_{j=1}^{n} \binom{2n+1}{j} q^j (1-q)^{2n+1-j} \right] \\
&= \frac{1}{q^{2n+1}} \left[(1-q)(q+1+q)^{2n+1} + 2(q+1-q)^{2n+1} - 2(1-q)^{2n+1} \right] \\
&= \frac{1}{q^{2n+1}} \left[(1-q) + 2 - 2(1-q^{2n+1}) \right] \to 0.
\end{aligned}
$$

(ii) In the example, first we define *Cesàro* operator as follows: Let $x = (x_i) \in l_p$ where $1 < p < \infty$.

Define

$$(T(x))_n = \frac{x_1 + x_2 + \dots + x_n}{n}.$$

We know that $T \in \mathscr{L}(l_p)$ and that in fact, Hardy, Little wood and Polya also show that

$$\|T\| \leqslant \frac{p}{p-1}.$$

Now, we see how the sequence of averages (i.e. $\sigma_n^1(x)$-means or $(C,1)$ mean) and product N_p summability of partial sums of a Fourier series is better behaved than the sequence of partial sums $s_n(x)$ itself. Let

$$f(x) = \begin{cases} -1, & -\pi \leqslant x < 0, \\ 1, & 0 \leqslant x < \pi, \end{cases}$$

with $f(x+2\pi) = f(x)$ for all real x. Since the function is odd. Therefore, we have a sine series.

$$b_n = \begin{cases} 0, & n \text{ even} \\ -\frac{4}{n\pi}, & n \text{ odd.} \end{cases}$$

Fourier series of $f(x)$ is given by

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin nx, \quad -\pi \leqslant x \leqslant \pi. \tag{24}$$

Then n^{th} partial sum $s_n(x)$ of Fourier series (24) and n^{th} *Cesàro* sum for $\delta = 1$, i.e. $\sigma_n^1(x)$ for the series (24) are given by

$$s_n(x) = \frac{4}{\pi}(\sin x + (1/3)\sin 3x + \dots + (1/n)\sin nx), \tag{25}$$

$$\sigma_n^1(x) = \frac{2}{\pi} \sum_{k=1}^{n} (1 - \frac{k}{n}) \left(\frac{1-(-1)^k}{k} \right) \sin kx. \tag{26}$$

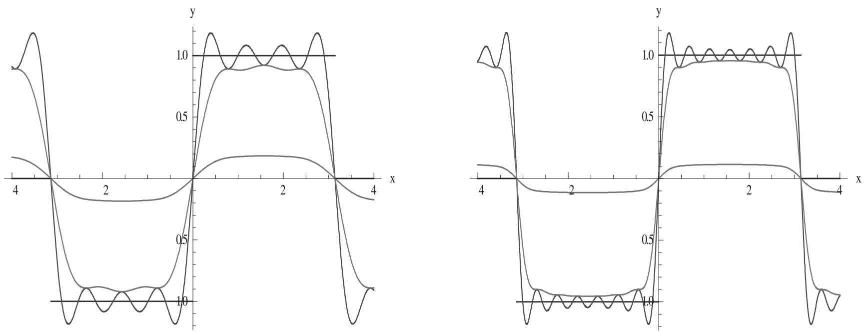

Figure 1: *Graph of* $f(x)$ *(blue),* $s_n(x)$ *(pink),* $\sigma_n^1(x)$ *(yellow),* $t_n^N(f;x)$ *(green),* $n=5$ *and* 10.

From Theorem 20 of Hardy's "Divergent Series", if a Nörlund method N_p has increasing weights p_n then it is stronger than $(C,1)$.

Now, take N_p to be the Nörlund matrix generated by $p_n = n+1$ then Nörlund means N_p is given by

$$t_n^N(f;x) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1) s_k(f;x). \tag{27}$$

In this graph (fig. 1), we observe that $\sigma_n^1(x)$, $t_n^N(f;x)$ converges to $f(x)$ faster than $s_n(x)$ in the interval $[-\pi,\pi]$. We further note that near the points of discontinuities i.e. $-\pi, 0$ and π the graph of s_5 and s_{10} show peaks and move closer the line passing through points of discontinuity as n increases (Gibbs Phenomenon), but in the graph of $\sigma_n^1(x)$, $t_n^N(f;x)$, $n=5,10$ the peaks become flatter. The Gibbs Phenomenon is an overshoot a peculiarity of the Fourier series and other eigen function series at a simple discontinuity i.e. the convergence of Fourier series is very slow at the point of discontinuity. Thus the product summability means of the Fourier series of $f(x)$ overshoot the Gibbs Phenomenon and show the smoothing effect of the method. Thus $\sigma_n^1(x)$, $t_n^N(f;x)$ is the better approximant than $s_n(x)$.

9. Conclusion

Various results pertaining to the degree of approximation of periodic signals (functions) belonging to the generalized weighted Lipschitz $W'(L_r,\xi(t))$ class by several matrix operators has been reviewed. Further, a proper set of conditions have been discussed in the Main Theorem 4.1 which is a generalization of Mishra et al. [17]. Theorem of this paper is an attempt to formulate the problem of approximation of function $f \in W'(L_r,\xi(t))$, $(r \geqslant 1)$ through trigonometric polynomials generated by the product summability $(N,p_n)(E,q)$ transform of the conjugate series of its Fourier series for signal f in a simpler manner. Some interesting application of the matrix $(N,p_n)(E,q)$ operator used in this paper pointed out in Note 4.3. Also, the result of our theorem is more general rather than the results of any other previous proved theorems, which will be enrich the literate of summability analysis of an infinite series. The researchers and professionals working or intend to work in areas of analysis and its applications will find this research article to be quite useful.

Acknowledgement. The authors are thankful to the anonymous learned referees for making valuable comments leading to the better presentation of the paper. Special thanks are due to Prof. Dr. Kristina Krulic Himmelreich, Journal of Classical Analysis Editorial office, for kind cooperation, smooth behavior during communication and for his efforts to send the reports of the manuscript timely. The authors are also thankful to all the Editorial board members and reviewers of prestigious journal i.e. Journal of Classical Analysis. The research of the fourth author i.e. Deepmala is supported by the Department of Science and Technology, Government of India under INSPIRE Fellowship Program No. DST/INSPIRE Fellowship/2013/ [182]. The second and third authors i.e. KK and LNM acknowledge the Ministry of Human Resource Development, New Delhi, India for supporting this research article. The first author VNM acknowledges that this project was supported by the Cumulative Professional Development Allowance (CPDA), SVNIT, Surat (Gujarat), India. All the authors carried out the proof. Each author contributed equally in the development of the manuscript. All the authors conceived of the study and participated in its design and coordination. All the authors read and approved the final manuscript. The authors declare that there is no conflict of interests regarding the publication of this research article.

REFERENCES

[1] G. BACHMAN, L. NARICI AND E. BECKENSTIEN, *Fourier and Wavelet Analysis*, Springer Verlag, New York, 2000.

[2] İ. CANAK AND Y. ERDEM, *On Tauberian theorems for $(A)(C,\alpha)$ summability method*, Appl. Math. Comp. **218** (2011) 2829–2836.

[3] M. I. GIL', *Estimates for entries of matrix valued functions of infinite matrices*, Math. Phys. Anal. Geom. **11** (2008) 175–186.

[4] N. K. GOVIL, *On the Summability of a Class of the Derived Conjugate Series of a Fourier series*, Canadian Mathematical Bulletin **8** (1965) 637–645.

[5] G. H. HARDY, *Divergent Series*, first ed., Oxford University Press, 70, 1949.

[6] H. H. KHAN, *On degree of approximation to a functions belonging to the class $Lip(\alpha,p)$*, Indian J. Pure Appl. Math. **5** (1974), 132–136.

[7] H. H. KHAN, *Approximation of classes of functions*, Ph. D. Thesis, AMU Aligarh, 1974.

[8] H. H. KHAN, *On the degree of approximation to a function belonging to weighted $W(L_p,\xi(t)$ class*, Aligarh Bull. Math. 3–4 (1973–1974) 83–88.

[9] H. H. KHAN, *A note on a theorem of Izumi*, Comm. Fac. Maths. Ankara, **31** (1982) 123–127.

[10] M. L. MITTAL, B. E. RHOADES, V. N. MISHRA, S. PRITI, AND S. S. MITTAL, *Approximation of functions belonging to $Lip(\xi(t),p)$, $(p \geqslant 1)$ Class by means of conjugate Fourier series using linear operators*, Indian Journal of Mathematics **47** (2005), 217–229.

[11] M. L. MITTAL, B. E. RHOADES, V. N. MISHRA, AND U. SINGH, *Using infinite matrices to approximate functions of class $Lip(\alpha,p)$ using trigonometric polynomials*, Journal of Mathematical Analysis and Applications **326** (2007), 667–676.

[12] M. L. MITTAL, B. E. RHOADES, AND V. N. MISHRA, *Approximation of signals (functions) belonging to the weighted $W(L_p,\xi(t)$, $(p \geqslant 1)$ Class by linear operators*, International Journal of Mathematics and Mathematical Sciences ID 53538 (2006), 1–10.

[13] V. N. MISHRA, L. N. MISHRA, *Trigonometric approximation in L_p $(p \geqslant 1)$ spaces*, Int. J. Contemp. Math. Sci. **7** (12) (2012) 909–918.
[14] V. N. MISHRA, K. KHATRI, AND L. N. MISHRA, *Product Summability Transform of Conjugate series of Fourier series*, International Journal of Mathematics and Mathematical Sciences Article ID 298923 (2012), 13 pages, doi: 10.1155/2012/298923.
[15] V. N. MISHRA, K. KHATRI, AND L. N. MISHRA, *Product $(N,p_n)(C,1)$ summability of sequence of Fourier coefficients*, Mathematical Sciences (Springer open access) 6:38 (2012), doi: 10.1186/2251 7456-6-38.
[16] V. N. MISHRA, K. KHATRI, AND L. N. MISHRA, *Using Linear Operators to Approximate Signals of $Lip(\alpha,p)$, $(p \geqslant 1)$-Class*, Filomat 27:2 (2013), 355–365.
[17] V. N. MISHRA, K. KHATRI, AND L. N. MISHRA, *Approximation of Functions belonging to $Lip(\xi(t),r)$ class by $(N,p_n)(E,q)$ summability of Conjugate Series of Fourier series*, Journal of Inequalities and Applications (2012), 2012:296. doi: 10.1186/1029-242X-2012-296.
[18] L. N. MISHRA, V. N. MISHRA, K. KHATRI, AND DEEPMALA, *On The Trigonometric Approximation of Signals Belonging to Generalized Weighted Lipschitz $W(L_r,\xi(t)$, $(r \geqslant 1)$*, Class by Matrix $(C^1.N_p)$ Operator of Conjugate Series of its Fourier series, Applied Mathematics and Computation, Vol. 237 (2014) 252–263. DOI: 10.1016/j.amc.2014.03.085.
[19] R. N. MOHAPATRA AND P. CHANDRA, *Degree of Approximation of Functions in the Hölder Metric*, Acta Math. Hungar. **41** (1983) 67–76.
[20] M. MURSALEEN, S. A. MOHIUDDINE, *Convergence Methods for Double Sequences and Applications*, Springer, New York, 2014.
[21] E. Z. PSARAKIS, G. V. MOUSTAKIDES, *An L_2-based method for the design of 1-D zero phase FIR digital filters*, IEEE Trans. Circuits Syst. I. Fundamental Theor. Appl. **44** (1997) 591–601.
[22] A. ZYGMUND, *Trigonometric Series*, second ed., Cambridge University Press, Cambridge, 1959.

(Received February 12, 2012)

Vishnu Narayan Mishra
Applied Mathematics and Humanities Department
Sardar Vallabhbhai National Institute of Technology
Ichchhanath Mahadev Dumas Road, Surat
Surat-395 007 (Gujarat), India
e-mail: vishnunarayanmishra@gmail.com;
vishnu_narayanmishra@yahoo.co.in

Kejal Khatri
Applied Mathematics and Humanities Department
Sardar Vallabhbhai National Institute of Technology
Ichchhanath Mahadev Dumas Road, Surat
Surat-395 007 (Gujarat), India
e-mail: kejal0909@gmail.com

Lakshmi Narayan Mishra
L. 1627 Awadh Puri Colony Beniganj
Phase-III, Opposite – Industrial Training Institute (I.T.I.)
Ayodhya Main Road, Faizabad-224 001, (Uttar Pradesh)
Department of Mathematics, National Institute of Technology
Silchar – 788 010, District – Cachar (Assam), India
e-mail: lakshminarayanmishra04@gmail.com;
l_n_mishra@yahoo.co.in

Deepmala
School of Studies in Mathematics
Pandit Ravishankar Shukla University
Raipur – 492 010, (Chhattisgarh), India
SQC & OR Unit, Indian Statistical Institute
203 Barrackpore Trunk Road
Kolkata – 700 108 (West Bengal), India
e-mail: dmrai23@gmail.com; deepmaladm23@gmail.com