GROWTH OF THE MAXIMUM MODULUS OF POLYNOMIALS WITH PRESCRIBED ZEROS

M. S. PUKHTA

Abstract. If \( p(z) = \sum_{j=0}^{n} a_j^j \) is a polynomial of degree \( n \) satisfying \( p(z) \neq 0 \) in \( |z| < 1 \), then for \( R \geq 1 \). Ankeny and Rivlin [1] proved that \( M(p,R) \leq \left( \frac{R^n+1}{2} \right) M(p,1) \). In this paper we obtain some results in this direction by considering polynomials of degree \( n \geq 2 \), having all its zeros on \( |z| = k, k \leq 1 \) which is an improvement of the result recently proved by M. S. Pukhta (2013) [Progress in Applied Mathematics, 6 (2), 50–58].

1. Introduction and Statement of Result

For an arbitrary entire function \( p(z) \), let \( M(f,r) = \max_{|z|=r} |f(z)| \). Then for a polynomial \( p(z) \) of degree \( n \), it is a simple consequence of maximum modulus principle (for reference see [4, Vol. I, p. 137, Problem III, 269]) that

\[
M(p,R) \leq R^n M(p,1), \quad \text{for } R \geq 1.
\]

The result is best possible and equality holds for \( p(z) = \lambda z^n \), where \( |\lambda| = 1, R \geq 1 \).

If we restrict ourselves to the class of polynomials having no zeros in \( |z| < 1 \), then inequality (1.1) can be sharpened. In fact it was shown by Ankeny and Rivlin [1] that if \( p(z) \neq 0 \) in \( |z| < 1 \), then (1.1) can be replaced by

\[
M(p,R) \leq \left( \frac{R^n+1}{2} \right) M(p,1), \quad \text{for } R \geq 1
\]

The result is sharp and equality holds for \( p(z) = \alpha + \beta z^n \), where \( |\alpha| = |\beta| \).

While trying to obtain inequality analogous to (1.2) for polynomials not vanishing in \( |z| < k, k \leq 1 \), K.K. Dewan and Arty Ahuja [2] proved the following result.

THEOREM A. If \( p(z) = \sum_{j=0}^{n} a_j^j \) is a polynomial of degree \( n \) having all its zeros on \( |z| = k, k \leq 1 \), then for every positive integer \( s \)

\[
\{M(p,R)\}^s \leq \left( \frac{k^n-1(1+k)+(R^{ns}-1)}{k^{n-1}+k^n} \right) \{M(p,1)\}^s, \quad R \geq 1
\]


Keywords and phrases: Derivative, Polynomial, Inequality, Zeros.
If \( p(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) having all its zeros on \( |z| = k, \ k \leq 1 \), then for every positive integer \( s \)

\[
\{M(p,R)\}^s \leq \frac{1}{k^n} \left[ \frac{n|a_n| \{k^n(1 + k^2) + k^2(R^{ns} - 1) + |a_{n-1}| \{2k^n + R^{ns} - 1\} \}}{2|a_{n-1}| + n|a_n|(1 + k^2)} \right] \\
\times \{M(p,1)\}^s, \quad R \geq 1.
\] (1.4)

In this paper, we not only improve Theorem A and Theorem B but also improve the results recently proved by M.S. Pukhta [7]. More precisely, we prove

**Theorem 1.** If \( p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{\nu-v} z^{n-v}, \ 1 \leq \mu < n \) is a polynomial of degree \( n \) having all its zeros on \( |z| = k, \ k \leq 1 \), then for every positive integer \( s \) and \( R \geq 1 \),

\[
\{M(p,R)\}^s \leq \frac{k^{n-2\mu+1}(1 + k^\mu) + (R^{ns} - 1)}{k^{n-2\mu+1} + k^{n-\mu+1}}\{M(p,1)\}^s \\
- s|a_1| \left( \frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns-2} \right) \{M(p,1)\}^{s-1}, \quad \text{if } n > 2 \quad (1.5)
\]

and

\[
\{M(p,R)\}^s \leq \frac{k^{n-2\mu+1}(1 + k^\mu) + (R^{ns} - 1)}{k^{n-2\mu+1} + k^{n-\mu+1}}\{M(p,1)\}^s \\
- s|a_1| \left( \frac{R^{ns} - 1}{ns} - \frac{R^{ns-1} - 1}{ns-1} \right) \{M(p,1)\}^{s-1}, \quad \text{if } n = 2 \quad (1.6)
\]

If we choose \( \mu = 1 \) in Theorem 1, we get the following result recently proved by M. S. Pukhta [7].

**Corollary 1.** If \( p(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \geq 2 \) having all its zeros on \( |z| = k, \ k \leq 1 \), then for \( R \geq 1 \),

\[
\{M(p,R)\}^s \leq \frac{k^{n-1}(1 + k)}{k^{n-1} + k^n} \{M(p,1)\}^s \\
- s|a_1| \left( \frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns-2} \right) \{M(p,1)\}^{s-1}, \quad \text{if } n > 2 \quad (1.7)
\]

and

\[
\{M(p,R)\}^s \leq \frac{k^{n-1}(1 + k)}{k^{n-1} + k^n} \{M(p,1)\}^s \\
- s|a_1| \left( \frac{R^{ns} - 1}{ns} - \frac{R^{ns-1} - 1}{ns-1} \right) \{M(p,1)\}^{s-1}, \quad \text{if } n = 2 \quad (1.8)
\]

Next we prove the following result which is a refinement of Theorem 1.
THEOREM 2. If \( p(z) = c_n z^n + \sum_{v=\mu}^{n} c_{n-v} z^{n-v}, 1 \leq \mu < n \) is a polynomial of degree \( n \) having all its zeros on \( |z| = k, k \leq 1 \), then for every positive integer \( s \) and \( R \geq 1 \),

\[
\{ M(p, R) \}^s \leq \frac{1}{k^{n-\mu+1}} \left[ \frac{n|c_n|\{ k^n(1+k^2) + k^2(R^{ns}_{\mu} - 1) \}}{\mu|c_{n-\mu}|(k^{\mu-1} + 1) + n|c_n|(k^{\mu-1} + k^2)} \right] \times \{ M(p, 1) \}^s - s|a_1| \left( \frac{R^{ns}_{\mu} - 1}{ns} - \frac{R^{ns}_{\mu-1} - 1}{ns - 1} \right) \{ M(p, 1) \}^{s-1},
\]

if \( n > 2 \) \hspace{1cm} (1.9)

and

\[
\{ M(p, R) \}^s \leq \frac{1}{k^{n-\mu+1}} \left[ \frac{n|c_n|\{ k^n(1+k^2) + k^2(R^{ns}_{\mu} - 1) \}}{\mu|c_{n-\mu}|(k^{\mu-1} + 1) + n|c_n|(k^{\mu-1} + k^2)} \right] \times \{ M(p, 1) \}^s - s|a_1| \left( \frac{R^{ns}_{\mu} - 1}{ns} - \frac{R^{ns}_{\mu-1} - 1}{ns - 1} \right) \{ M(p, 1) \}^{s-1}, \text{ if } n = 2 \hspace{1cm} (1.10)
\]

If we choose \( \mu = 1 \) in Theorem 2, we get the following result.

COROLLARY 2. If \( p(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \geq 2 \) having all its zeros on \( |z| = k, k \leq 1 \), then for every \( R \geq 1 \)

\[
\{ M(p, R) \}^s \leq \frac{1}{k^n} \left[ \frac{n|c_n|\{ k^n(1+k^2) + k^2(R^{ns}_{\mu} - 1) \}}{2|c_{n-1}| + n|c_n|(1+k^2)} \right] \{ M(p, 1) \}^s - s|a_1| \left( \frac{R^{ns}_{\mu} - 1}{ns} - \frac{R^{ns}_{\mu-1} - 1}{ns - 1} \right) \{ M(p, 1) \}^{s-1}, \text{ if } n > 2 \hspace{1cm} (1.11)
\]

and

\[
\{ M(p, R) \}^s \leq \frac{1}{k^n} \left[ \frac{n|c_n|\{ k^n(1+k^2) + k^2(R^{ns}_{\mu} - 1) \}}{2|c_{n-1}| + n|c_n|(1+k^2)} \right] \{ M(p, 1) \}^s - s|a_1| \left( \frac{R^{ns}_{\mu} - 1}{ns} - \frac{R^{ns}_{\mu-1} - 1}{ns - 1} \right) \{ M(p, 1) \}^{s-1}, \text{ if } n = 2 \hspace{1cm} (1.12)
\]
2. Lemmas

For the proof of these theorems, we need the following lemmas.

**Lemma 1.** If \( p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu} \), \( 1 \leq \mu < n \) is a polynomial of degree \( n \) having all its zeros on \( |z| = k, k \leq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{k^{n-\mu-1} + k^n - \mu + 1} \max_{|z|=1} |p(z)|. \tag{2.1}
\]

The above lemma is due to Govil [3].

**Lemma 2.** If \( p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu} \), \( 1 \leq \mu < n \) is a polynomial of degree \( n \) having all its zeros on \( |z| = k, k \leq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{k^{n-\mu} + k^{n-\mu-1}} \max_{|z|=1} |p(z)| \tag{2.2}
\]

The above lemma is due to Dewan and Mir [5].

**Lemma 3.** If \( p(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree, then for all \( R \geq 1 \)

\[
\max_{|z|=R} |p(z)| \leq R^n M(p, 1) - (R^n - R^{n-2})|p(0)|, \text{ if } n > 1 \tag{2.3}
\]

and

\[
\max_{|z|=R} |p(z)| \leq R M(p, 1) - (R - 1)|p(0)|, \text{ if } n = 1 \tag{2.4}
\]

The above lemma is due to Frappier, Rahman and Ruscheweyh [6].

3. Proof of the theorems

**Proof of Theorem 1.** We first consider the case when polynomial \( p(z) \) is of degree \( n > 2 \). Since \( p(z) \) is a polynomial of degree \( n \) having all its zeros on \( |z| = k, k \leq 1 \), therefore, by Lemma 1, we have

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{k^{n-\mu-1} + k^n - \mu + 1} M(p, 1) \tag{3.1}
\]

Now applying inequality (1.1) to the polynomial \( p'(z) \) which is of degree \( n - 1 \) and noting (3.1), it follows that for all \( r \geq 1 \) and \( 0 \leq \theta < 2\pi \)

\[
|p'(re^{i\theta})| \leq \frac{nr^{n-1}}{k^{n-\mu} + k^n - \mu + 1} M(p, 1) \tag{3.2}
\]
Also for each $\theta$, $0 \leq \theta < \pi$ and $R \geq 1$, we obtain
\[
\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s = \int_1^R \frac{d}{dt}\{p(te^{i\theta})\}^s dt = \int_1^R s\{p(te^{i\theta})\}^{s-1} p'(te^{i\theta})e^{i\theta} dt.
\]
This implies
\[
|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leq s \int_1^R |p(te^{i\theta})|^{s-1}|p'(te^{i\theta})| dt \tag{3.3}
\]
Since $p(z)$ is a polynomial of degree $n > 2$, the polynomial $p'(z)$ which is of degree $n - 1 \geq 2$, hence applying inequality (2.3) of Lemma 3 to $p'(z)$, we have for $r \geq 1$ and $0 \leq \theta < 2\pi$
\[
|p'(re^{i\theta})| \leq r^{n-1} M(p', 1) - (r^{n-1} - r^{n-3})|p'(0)| \tag{3.4}
\]
Inequality (3.4) in conjunction with inequalities (3.3) and (1.1), yields for $n > 2$ and for $R \geq 1$
\[
|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \\
\leq s \int_1^R (t^n M(p, 1))^{s-1} [t^{n-1} M(p', 1) - (t^{n-1} - t^{n-3})|p'(0)|] dt \\
= s \int_1^R t^{ns-1} [M(p, 1)]^{s-1} M(p', 1) - (t^{ns-1} - t^{ns-3}) [M(p, 1)]^{s-1} |p'(0)| dt \\
= s \left[ \frac{R^{ns} - 1}{ns} \{M(p, 1)\}^{s-1} M(p', 1) \\
- \left( \frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns - 2} \right) \{M(p, 1)\}^{s-1} |p'(0)| \right] \tag{3.5}
\]
On applying Lemma 1 to inequality (3.5), we get for $n > 2$,
\[
|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leq \frac{R^{ns} - 1}{kn^{2\mu + 1} + kn^{-\mu + 1}} \{M(p, 1)\}^s \\
- s \left( \frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns - 2} \right) \{M(p, 1)\}^{s-1} |p'(0)|
\]
This gives
\[
\{M(p, R)\}^s \leq \frac{kn^{2\mu + 1}(1 + k\mu) + (R^{ns} - 1)}{kn^{2\mu + 1} + kn^{-\mu + 1}} \{M(p, 1)\}^s \\
- s|a_1| \left( \frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns - 2} \right) \{M(p, 1)\}^{s-1}
\]
This completes the proof of inequality (1.5). \(\square\)
The proof of inequality (1.6) follows on the same lines as that of inequality (1.5), but instead of using inequality (2.3) of Lemma 3 we use inequality (2.4) of Lemma 3.

**Proof of Theorem 2.** The proof of Theorem 2 follows on the same lines as that of Theorem 1. But for the sake of completeness we give a brief outline of the proof. We first consider the case when polynomial $p(z)$ is of degree $n > 2$, then the polynomial $p'(z)$ is of degree $(n - 1) \geq 2$, hence applying inequality (2.3) of Lemma 3 to $p'(z)$, we have for $r \geq 1$ and $0 \leq \theta < 2\pi$

$$|p'(re^{i\theta})| \leq r^{n-1}M(p',1) - (r^{n-1} - r^{n-3})|p'(0)|$$

(3.6)

Also for each $\theta$, $0 \leq \theta < 2\pi$ and $R \geq 1$, we obtain

$$\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s = \int_1^R \frac{dt}{t^{n+1}} \{p(te^{i\theta})\}^s dt$$

$$= \int_1^R t^{n-1} \{p(te^{i\theta})\}^{s-1} p'(te^{i\theta})e^{i\theta} dt.$$

This implies

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leq s \int_1^R |p(te^{i\theta})|^{s-1} |p'(te^{i\theta})| dt$$

(3.7)

Inequality (3.6) in conjunction with inequalities (3.6) and (1.1), yields for $n > 2$

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s|$$

$$\leq s \int_1^R (t^{n}M(p,1))^{s-1}[t^{n-1}M(p',1) - (t^{n-1} - t^{n-3})|p'(0)|] dt$$

$$= s \int_1^R t^{ns-1} \{M(p,1)\}^{s-1}M(p',1) - (t^{ns-1} - t^{ns-3})\{M(p,1)\}^{s-1}|p'(0)| dt$$

$$= s \left[ \frac{R^{ns} - 1}{ns} \{M(p,1)\}^{s-1}M(p',1) - \left( \frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns - 2} \right) \{M(p,1)\}^{s-1}|p'(0)| \right]$$

Which on combining with Lemma 2, yields for $n > 2$

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s|$$

$$\leq \frac{R^{ns} - 1}{kn^{\mu+1}} \left[ \frac{n}{c_n} |k^{2\mu} + k^{2\mu+1} + \mu | c_n - \mu | (k^{\mu-1} + 1) \right] \{M(p,1)\}^s$$

$$- s \left( \frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns - 2} \right) \{M(p,1)\}^{s-1}|p'(0)|$$

from which we get the desired result. □

The proof of inequality (1.10) follows on the same lines as that of inequality (1.9), but instead of using inequality (2.3) of Lemma 3 we use inequality (2.4) of Lemma 3.
REFERENCES


(Received January 31, 2014)

M. S. Pukhta
Division of Agri. Statistics
Sher-e-Kashmir University of Agricultural Sciences and Technology of Kashmir
Srinagar 191121, India
e-mail: mspukhta57@yahoo.co.in