Journal of Classical Analysis
Volume 5, Number 2 (2014), 137–145
doi:10.7153/jca-05-12

LIMITS OF LALESCU KIND SEQUENCES WITH p–HYPERFACTORIAL AND SUPERFACTORIAL

ARKADY ALT

Abstract. The subject of this article are limits of sequences that can be represented in the form $n^{\mu}\left((n+1)^{1-\mu}a_{n+1}-n^{1-\mu}a_n\right)$. In particular, we will derive limits of such sequences with p-hyperfactorial for $p\in\mathbb{R}^+$ and with superfiactorial.

1. Terminology and notations

• For any two almost everywhere nonzero sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ (only finite number of terms can be equal zero) in the case $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=1$, we will use a well known notation $a_n\sim b_n$ and say that such sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ are asymptotically equal (or asymptotically equivalent). It is easy to prove that "$\sim$" is an equivalence relation. In particular $\lim\limits_{n\to\infty}a_n=a\iff a_n\sim a$. From the definition of "$\sim$" immediately follows "multiplicative replacement" property:

If $a_n\sim b_n$ and $c_n\sim d_n$ then $a_nc_n\sim b_nd_n$, $\dfrac{a_n}{c_n}\sim\dfrac{b_n}{d_n}$ and for any fixed $p\neq 0$ we have $a_n\sim b_n\iff a_n^p\sim b_n^p$.

As examples of using notation "$\sim$", we present several well-known asymptotic equivalences that we will need later:

i) $\sqrt[n]{n!}\sim ne^{-1}$ (but $n!\sim\sqrt{2\pi n}n^ne^{-n}$), $\sqrt[n]{n}\sim 1$ (and even more $\sqrt[n]{a_nn^p}\sim 1$ for any fixed p and any $0<L\leqslant a_n\leqslant U$).

ii) If $\lim\limits_{n\to\infty}\alpha_n=0$ then $(1+\alpha_n)^{\frac{1}{\alpha_n}}\sim e$, $\ln(1+\alpha_n)\sim\alpha_n$, $e^{\alpha_n}-1\sim\alpha_n$, $(1+\alpha_n)^p-1\sim p\alpha_n$.

• Asymptotic notations:

i) If $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=0$ we will right down $a_n=o(b_n)$. Let $a_n=o(b_n)$ then $b_n\sim c_n\implies a_n=o(c_n)$;

ii) If $|a_n|\leqslant K|b_n|$, $n\in\mathbb{Z}$ for some positive constant K then we will right down $a_n=O(b_n)$.

Mathematics subject classification (2010): 40A05, 34C41.
Keywords and phrases: Lalescu sequence, hyperfactorial, superfactorial, asymptotic equivalence.

Paper JCA-05-12

• We will call a sequence of the form $n^{\mu}\left((n+1)^{1-\mu}a_{n+1}-n^{1-\mu}a_n\right)$ a sequence of Lalescu-kind, or an $\mathscr{L}$-sequence. The origin of this notation comes from the Lalescu sequence: $L_n=\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}$ (or the above sequence with $\mu=0$, and $a_n=\dfrac{\sqrt[n]{n!}}{n}$).

• p-Hyperfactorial is an expression of the form $H_p(n)=1^{1^p}2^{2^p}\ldots n^{n^p}$, $p\in\mathbb{R}^+$ and is the generalization of Hyperfactorial $H(n)=1^1 2^2\ldots n^n$.

2. Preliminary result

To prove the main theorem we need some auxiliary statements presented by the following three lemmas and one corollary:

LEMMA 1. *If* $a_n>0$ *for almost all* $n\in\mathbb{N}$ *and* $\lim\limits_{n\to\infty}\dfrac{a_{n+1}^{n+1}}{a_n^n}=a$ *then* $\lim\limits_{n\to\infty}a_n=a$ *and for any real* μ *holds* $\lim\limits_{n\to\infty}n^{\mu}\left((n+1)^{1-\mu}a_{n+1}-n^{1-\mu}a_n\right)=a(1-\mu)$.

Proof. If $a=\lim\limits_{n\to\infty}\dfrac{a_{n+1}^{n+1}}{a_n^n}$ then, assuming $a_0=1$, by Multiplicative Stolz Theorem we have $\dfrac{a_{n+1}^{n+1}}{a_n^n}\sim a\implies\sqrt[n]{\prod\limits_{k=1}^{n}\dfrac{a_k^k}{a_{k-1}^{k-1}}}\sim a\iff a_n\sim a$.

Hence, $\dfrac{a_{n+1}^n}{a_n^n}=\dfrac{a_{n+1}^{n+1}}{a_n^n}\cdot a_n^{-1}\sim a\cdot a^{-1}=1$.

Since $a_n\sim a$ then $\dfrac{(n+1)^{1-\mu}a_{n+1}}{n^{1-\mu}a_n}\sim 1$ and, therefore,

$$n\left(\frac{(n+1)^{1-\mu}a_{n+1}}{n^{1-\mu}a_n}-1\right)\sim n\ln\left(\frac{(n+1)^{1-\mu}a_{n+1}}{n^{1-\mu}a_n}\right).$$

Thus, we have

$$n^{\mu}\left((n+1)^{1-\mu}a_{n+1}-n^{1-\mu}a_n\right)=a_n\cdot n\left(\frac{(n+1)^{1-\mu}a_{n+1}}{n^{1-\mu}a_n}-1\right)$$
$$\sim a\cdot\ln\left(\frac{(n+1)^{n(1-\mu)}a_{n+1}^n}{n^{n(1-\mu)}a_n^n}\right).$$

On the other hand, since $\left(1+\dfrac{1}{n}\right)^n\sim e$ and $\dfrac{a_{n+1}^n}{a_n^n}\sim 1$ then

$$\frac{(n+1)^{n(1-\mu)}a_{n+1}^n}{n^{n(1-\mu)}a_n^n}=\left(1+\frac{1}{n}\right)^{n(1-\mu)}\cdot\frac{a_{n+1}^n}{a_n^n}\sim e^{1-\mu}$$

and, therefore, $\ln\left(\dfrac{(n+1)^{n(1-\mu)}a_{n+1}^n}{n^{n(1-\mu)}a_n^n}\right)\sim 1-\mu$.

Thus,

$$n^{\mu}\left((n+1)^{1-\mu}a_{n+1}-n^{1-\mu}a_n\right)\sim a(1-\mu)$$
$$\iff \lim_{n\to\infty}\left(n^{\mu}\left((n+1)^{1-\mu}a_{n+1}-n^{1-\mu}a_n\right)\right)=a(1-\mu). \quad \square$$

LEMMA 2. *Let $f:[1,\infty)\to(0,\infty)$ be twice differentiable on $(1,\infty)$ function such that f'' preserve sign on $(1,\infty)$ and let $F(x)$ be primitive function for $f(x)$ on $(1,\infty)$. Then for any positive integer n holds inequality*

$$\left|\sum_{k=1}^{n}f(k)-\frac{f(n)}{2}-F(n)\right|\leqslant\left|\frac{f(1)}{2}-F(1)\right|+\frac{|f'(n)-f'(1)|}{4}.$$

Proof. Note that for any $g(t)\in C^2((1,\infty))$ an easily verified identity

$$g(t)-\frac{1}{2}\left(t-\frac{1}{2}\right)^2g''(t)=\left(\left(t-\frac{1}{2}\right)g(t)\right)'-\frac{1}{2}\left(\left(t-\frac{1}{2}\right)^2g'(t)\right)'$$

holds.

Substitution $t=x-k$ and $g(t)=f(t+k)$ gives us

$$\int_k^{k+1}f(x)\,dx-\frac{1}{2}\int_k^{k+1}\left(x-k-\frac{1}{2}\right)^2f''(x)\,dx=\frac{f(k+1)+f(k)}{2}-\frac{f'(k+1)-f'(k)}{8}. \tag{1}$$

Then, applying to (1) summation by $k=1,2,\ldots,n-1$ we obtain

$$\int_1^n f(x)\,dx-\frac{1}{2}\sum_{k=1}^{n-1}\int_k^{k+1}\left(x-k-\frac{1}{2}\right)^2f''(x)\,dx$$
$$=\sum_{k=1}^{n}f(k)-\frac{f(n)+f(1)}{2}-\frac{f'(n)-f'(1)}{8}\implies$$
$$\left|\sum_{k=1}^{n}f(k)-\int_1^n f(x)\,dx-\frac{f(n)+f(1)}{2}-\frac{f'(n)-f'(1)}{8}\right|$$
$$=\frac{1}{2}\left|\sum_{k=1}^{n-1}\int_k^{k+1}\left(x-k-\frac{1}{2}\right)^2f''(x)\right|.$$

Since $\max\limits_{[k,k+1]}\left(x-k-\dfrac{1}{2}\right)^2=\dfrac{1}{4}$ and $f''(x)$ preserves it's sign on $(0,\infty)$ then

$$\left|\sum_{k=1}^{n-1}\int_k^{k+1}\left(x-k-\frac{1}{2}\right)^2f''(x)\right|$$

$$= \sum_{k=1}^{n-1} \int_k^{k+1} \left(x-k-\frac{1}{2}\right)^2 \left|f''(x)\right| dx \leqslant \frac{1}{4} \sum_{k=1}^{n-1} \int_k^{k+1} \left|f''(x)\right| dx$$
$$= \frac{1}{4} \left| \int_1^n f''(x)\, dx \right| = \frac{|f'(n) - f'(1)|}{4}$$

and, therefore,

$$\left| \sum_{k=1}^{n} f(k) - \frac{f(n)}{2} - F(n) - \left(\frac{f(1)}{2} - F(1)\right) - \frac{f'(n) - f'(1)}{8} \right| \leqslant \frac{|f'(n) - f'(1)|}{8}.$$

Hence,

$$\left| \sum_{k=1}^{n} f(k) - \frac{f(n)}{2} - F(n) \right| - \left| \frac{f(1)}{2} - F(1) \right| - \frac{|f'(n) - f'(1)|}{8} \leqslant \frac{|f'(n) - f'(1)|}{8}$$
$$\Longleftrightarrow \left| \sum_{k=1}^{n} f(k) - \frac{f(n)}{2} - F(n) \right| \leqslant \left| \frac{f(1)}{2} - F(1) \right| + \frac{|f'(n) - f'(1)|}{4}. \quad \square$$

COROLLARY 1. *For real* $p > 0$ *let* $S_p(n) := \sum_{k=1}^{n} k^p$. *Then:*

i) $\left| S_p(n) - \frac{n^p}{2} - \frac{n^{p+1}}{p+1} \right| \leqslant \left| \frac{1}{2} - \frac{1}{p+1} \right| + \frac{p\left|n^{p-1} - 1\right|}{4}$;

ii) $\frac{S_p(n)}{n^p} = \frac{1}{2} + \frac{n}{p+1} + O\left(\frac{1}{n^{\min\{1,p\}}}\right)$;

iii) $n^{\frac{S_p(n)}{n^p}} \sim n^{\frac{1}{2} + \frac{n}{p+1}}$.

Proof. For $f(x) = x^p$, $x \in (0, \infty)$, $p > 0$, $p \neq 1$ we have $F(x) = \dfrac{x^{p+1}}{p+1}$, $f'(x) = px^{p-1}$, $f''(x) = p(p-1)x^{p-2}$. Since $\operatorname{sign} f''(x) = \operatorname{sign}(p-1)$ then applying inequality in Lemma 2 we obtain

$$\left| S_p(n) - \frac{n^p}{2} - \frac{n^{p+1}}{p+1} \right| \leqslant \left| \frac{1}{2} - \frac{1}{p+1} \right| + \frac{p\left|n^{p-1} - 1\right|}{4}.$$

But since in the case $p = 1$ we have $\left| \dfrac{1}{2} - \dfrac{1}{p+1} \right| = \dfrac{p\left|n^{p-1} - 1\right|}{4} = 0$ and

$$\left| S_p(n) - \frac{n^p}{2} - \frac{n^{p+1}}{p+1} \right| = \left| \sum_{k=1}^{n} k - \frac{n}{2} - \frac{n^2}{2} \right| = 0$$

then inequality (i) holds for any $p > 0$.

Also, $\frac{1}{n^p}\left|\frac{1}{2}-\frac{1}{p+1}\right|+\frac{p\left|n^{p-1}-1\right|}{4n^p}=O\left(\frac{1}{n^{\min\{1,p\}}}\right)$ yields

$$\frac{S_p(n)}{n^p}=\frac{1}{2}+\frac{n}{p+1}+O\left(\frac{1}{n^{\min\{1,p\}}}\right).$$

Since for any $\alpha>0$ we have $\lim_{n\to\infty} n^{O\left(\frac{1}{n^\alpha}\right)}=1$ then

$$n^{\frac{S_p(n)}{n^p}}=n^{\frac{1}{2}+\frac{n}{p+1}+O\left(\frac{1}{n^{\min\{1,p\}}}\right)}=n^{\frac{1}{2}+\frac{n}{p+1}}\cdot n^{O\left(\frac{1}{n^{\min\{1,p\}}}\right)}\sim n^{\frac{1}{2}+\frac{n}{p+1}}. \quad \square$$

LEMMA 3. $(H_p(n))^{\frac{1}{n^p}}\sim e^{-\frac{n}{(p+1)^2}}n^{\frac{S_p(n)}{n^p}}\sim e^{-\frac{n}{(p+1)^2}}n^{\frac{1}{2}+\frac{n}{p+1}}$ *for any real* $p>0$.

Proof. We will prove $\lim_{n\to\infty}\ln\left((H_p(n))^{\frac{1}{n^p}}e^{\frac{n}{(p+1)^2}}n^{-\frac{S_p(n)}{n^p}}\right)=0$, that is $\lim_{n\to\infty}\frac{c_n}{n^p}=0$, where

$$c_n:=n^p\ln\left((H_p(n))^{\frac{1}{n^p}}e^{\frac{n}{(p+1)^2}}n^{-\frac{S_p(n)}{n^p}}\right)=\frac{n^{p+1}}{(p+1)^2}+\ln H_p(n)-S_p(n)\ln n.$$

Noting that

$$\begin{aligned}
c_{n+1}-c_n &= \frac{(n+1)^{p+1}-n^{p+1}}{(p+1)^2}+(n+1)^p\ln(n+1)\\
&\quad -(S_p(n)+(n+1))\ln(n+1)+S_p(n)\ln n\\
&= \frac{(n+1)^{p+1}-n^{p+1}}{(p+1)^2}-\ln\left(1+\frac{1}{n}\right)S_p(n),\\
\ln\left(1+\frac{1}{n}\right) &= \frac{1}{n}-\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right),\\
\left(1+\frac{1}{n}\right)^\alpha-1 &= \frac{\alpha}{n}+\frac{\alpha(\alpha-1)}{2n^2}+o\left(\frac{1}{n^2}\right),
\end{aligned}$$

$\alpha=p,p+1$ and $\frac{S_p(n)}{n^p}=\frac{1}{2}+\frac{n}{p+1}+O\left(\frac{1}{n^{\min\{1,p\}}}\right)$ we obtain

$$\begin{aligned}
&\frac{c_{n+1}-c_n}{(n+1)^p-n^p}\\
&=\frac{(n+1)^{p+1}-n^{p+1}}{(p+1)^2((n+1)^p-n^p)}-\frac{\ln\left(1+\frac{1}{n}\right)S_p(n)}{(n+1)^p-n^p}\\
&=\frac{(n+1)^{p+1}-n^{p+1}}{n((n+1)^p-n^p)}\left(\frac{n}{(p+1)^2}-\frac{n\ln\left(1+\frac{1}{n}\right)S_p(n)}{(n+1)^{p+1}-n^{p+1}}\right)
\end{aligned}$$

$$= \frac{\left(1+\frac{1}{n}\right)^{p+1}-1}{\left(1+\frac{1}{n}\right)^{p}-1}\left(\frac{n}{(p+1)^2}-\frac{n\ln\left(1+\frac{1}{n}\right)\frac{S_p(n)}{n^p}}{n\left(\left(1+\frac{1}{n}\right)^{p+1}-1\right)}\right)$$

$$= \frac{\left(1+\frac{1}{n}\right)^{p+1}-1}{\left(1+\frac{1}{n}\right)^{p}-1}\left(\frac{n}{(p+1)^2}-\frac{\left(1-\frac{1}{2n}+o\left(\frac{1}{n}\right)\right)\left(\frac{1}{2}+\frac{n}{p+1}+O\left(\frac{1}{n^{\min\{1,p\}}}\right)\right)}{p+1+\frac{p(p+1)}{2n}+o\left(\frac{1}{n}\right)}\right).$$

Since

$$\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{p+1}-1}{\left(1+\frac{1}{n}\right)^{p}-1}=\frac{p+1}{p},$$

$$\lim_{n\to\infty}\frac{o\left(\frac{1}{n}\right)\left(\frac{1}{2}+\frac{n}{p+1}+O\left(\frac{1}{n^{\min\{1,p\}}}\right)\right)}{p+1+\frac{p(p+1)}{2n}+o\left(\frac{1}{n}\right)}=0,$$

$$\lim_{n\to\infty}\frac{\left(1-\frac{1}{2n}\right)O\left(\frac{1}{n^{\min\{1,p\}}}\right)}{p+1+\frac{p(p+1)}{2n}+o\left(\frac{1}{n}\right)}=0$$

and

$$\lim_{n\to\infty}\left(\frac{n}{(p+1)^2}-\frac{\left(1-\frac{1}{2n}\right)\left(\frac{1}{2}+\frac{n}{p+1}\right)}{p+1+\frac{p(p+1)}{2n}+o\left(\frac{1}{n}\right)}\right)=0$$

$$\iff \lim_{n\to\infty}\left(n\left(p+1+\frac{p(p+1)}{2n}+o\left(\frac{1}{n}\right)\right)-(p+1)^2\left(1-\frac{1}{2n}\right)\left(\frac{1}{2}+\frac{n}{p+1}\right)\right)$$

$$\iff \lim_{n\to\infty}\left(n\left(p+1+\frac{p(p+1)}{2n}\right)-(p+1)^2\left(1-\frac{1}{2n}\right)\left(\frac{1}{2}+\frac{n}{p+1}\right)\right)$$

$$=\lim_{n\to\infty}\frac{(p+1)^2}{4n}=0$$

then $\lim_{n\to\infty}\frac{c_{n+1}-c_n}{(n+1)^p-n^p}=0$ and by Stolz Theorem

$$\lim_{n\to\infty}\frac{c_n}{n^p}=0 \implies \lim_{n\to\infty}(H_p(n))^{\frac{1}{n^p}}e^{\frac{n}{(p+1)^2}}n^{-\frac{S_p(n)}{n^p}}=1$$

$$\iff (H_p(n))^{\frac{1}{n^p}}\sim e^{-\frac{n}{(p+1)^2}}n^{\frac{S_p(n)}{n^p}}\sim e^{-\frac{n}{(p+1)^2}}n^{\frac{1}{2}+\frac{n}{p+1}}. \quad \square$$

REMARK 1. In particular, for $p = 1$ we obtain well known asymptotic equivalence for hyperfactorial $H(n)$, namely $H(n) = H_1(n) \sim e^{-\frac{n}{4}} n^{\frac{n+1}{2}}$ [1].

3. Main result

THEOREM 1. *Let* α, β, p *be real numbers such that* $p > 0$ *and* $\alpha + \beta = \dfrac{p+2}{p+1}$. *Then*

$$\lim_{n\to\infty}\left(n^{\alpha}\left((n+1)^{\beta}\cdot(H_p(n+1))^{-\frac{1}{(n+1)^{p+1}}} - n^{\beta}\cdot(H_p(n))^{-\frac{1}{n^{p+1}}}\right)\right)$$
$$= (1-\alpha)\lim_{n\to\infty}\left((H_p(n))^{-\frac{1}{n^{p+1}}}\cdot n^{\frac{1}{p+1}}\right) = (1-\alpha)e^{\frac{1}{(p+1)^2}}.$$

Proof. Let $a_n := (H_p(n))^{-\frac{1}{n^{p+1}}}\cdot n^{\frac{1}{p+1}}$. Then

$$\frac{a_{n+1}^{n+1}}{a_n^n} = \left(1+\frac{1}{n}\right)^{\frac{n}{p+1}}\cdot\frac{(H_p(n))^{\frac{1}{n^p}}}{(H_p(n+1))^{\frac{1}{(n+1)^p}}}\cdot(n+1)^{\frac{1}{p+1}}$$
$$\sim e^{\frac{1}{p+1}}\cdot(n+1)^{\frac{1}{p+1}}\cdot\frac{e^{-\frac{n}{(p+1)^2}}n^{\frac{1}{2}+\frac{n}{p+1}}}{e^{-\frac{n+1}{(p+1)^2}}(n+1)^{\frac{1}{2}+\frac{n+1}{p+1}}}$$
$$= e^{\frac{p+2}{(p+1)^2}}\cdot\frac{n^{\frac{1}{2}}}{(n+1)^{\frac{1}{2}}}\cdot\left(\frac{n}{n+1}\right)^{\frac{n}{p+1}} \sim e^{\frac{p+2}{(p+1)^2}}\cdot e^{-\frac{1}{p+1}} = e^{\frac{1}{(p+1)^2}}.$$

Since

$$n^{\beta}\cdot(H_p(n))^{-\frac{1}{n^{p+1}}} = n^{\beta-\frac{1}{p+1}}\cdot(H_p(n))^{-\frac{1}{n^{p+1}}}\cdot n^{\frac{1}{p+1}} = n^{\beta-\frac{1}{p+1}}a_n,$$

$\alpha + \left(\beta - \dfrac{1}{p+1}\right) = 1$ and $\lim_{n\to\infty} a_n = e^{\frac{1}{(p+1)^2}}$ then, applying Lemma1 to $\mu = \alpha$, we obtain

$$\lim_{n\to\infty} n^{\alpha}\left((n+1)^{\beta}\cdot(H_p(n+1))^{-\frac{1}{(n+1)^{p+1}}} - n^{\beta}\cdot(H_p(n))^{-\frac{1}{n^{p+1}}}\right)$$
$$= \lim_{n\to\infty} n^{\alpha}\left((n+1)^{1-\alpha}a_{n+1} - n^{1-\alpha}a_n\right) = (1-\alpha)e^{\frac{1}{(p+1)^2}}. \quad \square$$

In particular, for $\alpha = 0,\ \frac{1}{p+1},\ \frac{p+2}{p+1},\ \frac{p}{p+1}$ we obtain, respectively,

$$(LH1)\quad \lim_{n\to\infty}\left((n+1)^{\frac{p+2}{p+1}}\cdot(H_p(n+1))^{-\frac{1}{(n+1)^{p+1}}}-n^{\frac{p+2}{p+1}}\cdot(H_p(n))^{-\frac{1}{n^{p+1}}}\right) = \lim_{n\to\infty}\left((H_p(n))^{-\frac{1}{n^{p+1}}}\cdot n^{\frac{1}{p+1}}\right)=e^{\frac{1}{(p+1)^2}},$$

$$(LH2)\quad \lim_{n\to\infty} n^{\frac{p+2}{p+1}}\left((H_p(n+1))^{-\frac{1}{(n+1)^{p+1}}}-(H_p(n))^{-\frac{1}{n^{p+1}}}\right)=-\frac{e^{\frac{1}{(p+1)^2}}}{p+1},$$

$$(LH3)\quad \lim_{n\to\infty} n^{\frac{p}{p+1}}\left((n+1)^{\frac{2}{p+1}}\cdot(H_p(n+1))^{-\frac{1}{(n+1)^{p+1}}}-n^{\frac{2}{p+1}}\cdot(H_p(n))^{-\frac{1}{n^{p+1}}}\right) = \frac{e^{\frac{1}{(p+1)^2}}}{p+1},$$

$$(LH4)\quad \lim_{n\to\infty} n^{\frac{1}{p+1}}\left((n+1)\cdot(H_p(n+1))^{-\frac{1}{(n+1)^{p+1}}}-n\cdot(H_p(n))^{-\frac{1}{n^{p+1}}}\right)=\frac{pe^{\frac{1}{(p+1)^2}}}{p+1}.$$

REMARK 2. When $p=1$ then for hyperfactorial $H(n)=H_1(n)=1^1 2^2\dots n^n$ the limits (LH1), (LH2), (LH3), respectively, becomes:

$$\lim_{n\to\infty}\left(\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{H(n+1)}}-\frac{n\sqrt{n}}{\sqrt[n^2]{H(n)}}\right)=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt[n^2]{H(n)}}=\sqrt[4]{e},$$

$$\lim_{n\to\infty} n^{\frac{3}{2}}\left(\frac{1}{\sqrt[(n+1)^2]{H(n+1)}}-\frac{1}{\sqrt[n^2]{H(n)}}\right)=-\frac{\sqrt[4]{e}}{2},$$

$$\lim_{n\to\infty}\sqrt{n}\left(\frac{n+1}{\sqrt[(n+1)^2]{H(n+1)}}-\frac{n}{\sqrt[n^2]{H(n)}}\right)=\frac{\sqrt[4]{e}}{2}.$$

4. More applications of Lemma 1

THEOREM 2. *Let $sf(n) := 1!2!\dots n!$ (superfactorial). Then*

$$\lim_{n\to\infty}\left(\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}}-\frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}\right)=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt[n^2]{sf(n)}}=e^{\frac{3}{4}}.$$

Proof. Noting that $sf(n) = \dfrac{(n!)^{n+1}}{H(n)}$ we obtain

$$\frac{\sqrt{n}}{\sqrt[n^2]{sf(n)}} = \frac{\sqrt{n}\sqrt[n^2]{H(n)}}{(n!)^{\frac{n+1}{n^2}}} = \frac{\sqrt[n^2]{H(n)}}{\sqrt{n}} \cdot \frac{n}{(n!)^{\frac{1}{n}}} \cdot \frac{1}{(n!)^{\frac{1}{n^2}}} \sim e^{-\frac{1}{4}} \cdot e \cdot \frac{1}{\left(\sqrt[n]{n!}\right)^{\frac{1}{n}}} \sim e^{\frac{3}{4}}$$

(because $\sqrt[n]{n!} \sim ne^{-1}$) and, therefore,

$$\left(\sqrt[n]{n!}\right)^{\frac{1}{n}} \sim 1 \iff \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt[n^2]{sf(n)}} = e^{\frac{3}{4}}.$$

Let $a_n := \dfrac{\sqrt{n}}{\sqrt[n^2]{sf(n)}}$. Since $\sqrt[n]{H(n)} \sim e^{-\frac{n}{4}} n^{\frac{n+1}{2}}$ and $(n!)^{\frac{n+1}{n}} = n!\sqrt[n]{n!} \sim n! \cdot ne^{-1}$ then

$$a_n^n = \frac{n^{\frac{n}{2}}}{\sqrt[n]{sf(n)}} = \frac{n^{\frac{n}{2}}\sqrt[n]{H(n)}}{(n!)^{\frac{n+1}{n}}} \sim \frac{n^{\frac{n}{2}} \cdot e^{-\frac{n}{4}} n^{\frac{n+1}{2}}}{nn!e^{-1}} = \frac{n^n e^{-\frac{n}{4}}}{\sqrt{n}n!e^{-1}}$$

and, therefore,

$$\frac{a_{n+1}^{n+1}}{a_n^n} \sim \frac{(n+1)^{n+1} e^{-\frac{n+1}{4}}}{\sqrt{n+1}\,(n+1)!} \cdot \frac{\sqrt{n}n!}{n^n e^{-\frac{n}{4}}} = \frac{(n+1)^n}{n^n} \cdot \frac{e^{-\frac{1}{4}}\sqrt{n}}{\sqrt{n+1}} \sim e^{\frac{3}{4}}.$$

Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{a_{n+1}^{n+1}}{a_n^n} = e^{\frac{3}{4}}$ then by Lemma 1 we obtain

$$\lim_{n\to\infty} \left((n+1)a_{n+1} - na_n\right) = e^{\frac{3}{4}}. \quad \square$$

REFERENCES

[1] G. POLYA, G. SZEGO, *Problems and Theorems in Analysis*, Part 1, Problem 15, p. 50.

(Received August 31, 2014)

Arkady Alt
San Jose, California
USA
e-mail: arkady.alt@gmail.com