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DOUBLE SUBORDINATION PRESERVING PROPERTIES FOR GENERALIZED FRACTIONAL DIFFER–INTEGRAL OPERATOR

JUGAL K. PRAJAPAT AND TEODOR BULBOACĂ

Abstract. We obtain subordination and superordination preserving properties for the Saigo type generalized fractional differ-integral operator, defined for multivalent functions in the open unit disk. A differential sandwich-type theorem for these multivalent function, and some consequences are also presented.

1. Introduction

Let $\mathscr{H}(\mathrm{U})$ represent the space of analytic functions in the open unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let

$$\mathscr{H}[a,n] = \{f \in \mathscr{H}(\mathrm{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots\}.$$

We denote by $\mathscr{A}$ the subclass of the functions $f \in \mathscr{H}[a,1]$ normalized with the conditions $f(0) = f'(0) - 1 = 0$. Denote also by $\mathscr{K}$ the subclass of $\mathscr{A}$ consisting of all those functions that are *convex (univalent) and normalized in* U, i.e.

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ \ z \in \mathrm{U}.$$

If f and g are two members of $\mathscr{H}(\mathrm{U})$, then the function f is said to be *subordinate to* g, and write $f(z) \prec g(z)$, if there exists a function w analytic in U with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathrm{U}$, such that $f(z) = g(w(z))$, $z \in \mathrm{U}$. Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow [f(0) = g(0) \quad \text{and} \quad f(\mathrm{U}) \subset g(\mathrm{U})]. \tag{1.1}$$

DEFINITION 1.1. [15] Let $\psi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the following differential subordination

$$\psi(p(z), zp'(z)) \prec h(z), \tag{1.2}$$

then p is called a *solution of the differential subordination* (1.2). A univalent function q is called a *dominant of the solutions of the differential subordination* (1.2) or, more simply, a dominant if $p(z) \prec q(z)$ for all p satisfying (1.2). A dominant $\widetilde{q}$ that satisfies $\widetilde{q}(z) \prec q(z)$ for all dominants q of (1.2) is said to be the *best dominant of* (1.2).

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Recently, Miller and Mocanu [16] introduced the notion of differential superordinations, as the dual concept of differential subordinations.

DEFINITION 1.2. [16] Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be analytic in U. If p and $\phi(p(z), zp'(z))$ are univalent in U and satisfy the differential superordination

$$h(z) \prec \phi(p(z), zp'(z)), \tag{1.3}$$

then p is called a *solution of the differential superordination* (1.3). An analytic function q is called a *subordinant of the solutions of the differential superordination* (1.3) or, more simply, a subordinant if $q(z) \prec p(z)$ for all p satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all subordinants q of (1.3) is said to be the *best subordinant of* (1.3).

DEFINITION 1.3. [15, Definition 2.2b, p. 21] We denote by $\mathscr{Q}$ the class of functions f that are analytic and injective on $\overline{\mathrm{U}} \setminus \mathrm{E}(f)$, where

$$\mathrm{E}(f) = \left\{ \zeta \in \partial\mathrm{U} : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathrm{U} \setminus \mathrm{E}(f)$.

Now we recall the following generalized fractional integral and generalized fractional derivative operators due to Srivastava et al. [27] (see also [21]).

DEFINITION 1.4. For the numbers $\mu, \eta, \lambda \in \mathbb{R}$ with $\lambda > 0$, the *Saigo hypergeometric fractional integral operator* $\mathrm{I}_{0,z}^{\lambda,\mu,\eta}$ is defined by

$$\mathrm{I}_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} \,{}_2F_1\left(\lambda+\mu, -\eta; \lambda; 1-\frac{t}{z}\right) f(t)\,\mathrm{d}t,$$

where the function f is analytic in a simply-connected region of the complex z–plane containing the origin, with the order

$$f(z) = \mathrm{O}\left(|z|^{\varepsilon}\right) \qquad \left(z \to 0,\ \ \varepsilon > \max\{0; \mu-\eta\} - 1\right),$$

and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

The function ${}_2F_1$ is the well-known *Gauss hypergeometric function* defined by

$$\begin{aligned} {}_2F_1(a,b;c;z) &= 1 + \frac{a \cdot b}{c}\frac{z}{1!} + \frac{a(a+1) \cdot b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots \\ &= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k}\frac{z^k}{k!}, \quad a,b \in \mathbb{C},\ \ c \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}, \end{aligned} \tag{1.4}$$

where $(d)_k = d(d+1)\ldots(d+k-1)$, $k \geqslant 1$ and $(d)_0 = 1$. The series (1.4) converges absolutely for $z \in \mathrm{U}$, hence it represents an analytic function in U (see [30, Chapter 14]).

DEFINITION 1.5. Under the assumptions of Definition 1.4, the *Saigo hypergeometric fractional derivative operator* $\mathrm{J}_{0,z}^{\lambda,\mu,\eta}$ is defined by

$$\mathrm{J}_{0,z}^{\lambda,\mu,\eta} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{\mathrm{d}}{\mathrm{d}z} z^{\lambda-\mu} \int_0^z (z-t)^{-\lambda} \ {}_2F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\frac{t}{z}\right) f(t)\,\mathrm{d}t, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad 0 \leqslant \lambda < 1, \\ \frac{\mathrm{d}^n}{\mathrm{d}z^n} \mathrm{I}_{0,z}^{\lambda,\mu,\eta} f(z), \quad \text{if} \quad n \leqslant \lambda < n+1, \ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \end{cases}$$

where the multiplicity of $(z-t)^{-\lambda}$ is removed as in Definition 1.4.

It may be remarked that

$$\mathrm{I}_{0,z}^{\lambda,-\lambda,\eta} f(z) = \mathrm{D}_z^{-\lambda} f(z) \qquad (\lambda > 0)$$

and

$$\mathrm{J}_{0,z}^{\lambda,\lambda,\eta} f(z) = \mathrm{D}_z^{\lambda} f(z) \qquad (0 \leqslant \lambda < 1),$$

where $\mathrm{D}_z^{-\lambda}$ $(\lambda > 0)$ denotes the fractional integral operator, while D_z^{λ} $(0 \leqslant \lambda < 1)$ denotes the fractional derivative operator considered by Owa [17].

Let $\mathscr{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N}), \tag{1.5}$$

which are analytic and p–valent U. Recently, Goyal and Prajapat [7] (see also [22, 24]) introduced the *generalized fractional differ-integral operator* $\mathscr{S}_{0,z}^{\lambda,\mu,\eta} : \mathscr{A}_p \to \mathscr{A}_p$, by

$$\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z) = \begin{cases} \frac{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)} \ z^{\mu} \mathrm{J}_{0,z}^{\lambda,\mu,\eta} f(z), \text{ if } 0 \leqslant \lambda < \eta+p+1 \\ \\ \frac{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)} \ z^{\mu} \mathrm{I}_{0,z}^{\lambda,\mu,\eta} f(z), \text{ if } -\infty < \lambda < 0, \end{cases} \tag{1.6}$$

$$(\mu, \eta \in \mathbb{R}, \ \mu < p+1, \ \lambda < \eta+p+1).$$

From (1.6) it is easy to see that if the function f has the form (1.5), then

$$\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z) = z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n(1+p+\eta-\mu)_n}{(1+p-\mu)_n(1+p+\eta-\lambda)_n} a_{p+n} z^{p+n}. \tag{1.7}$$

From (1.7) it is easy to prove that the operator $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}$ satisfies the recurrence relation

$$z\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)' = (p+\eta-\lambda) \ \mathscr{S}_{0,z}^{\lambda+1,\mu,\eta} f(z) - (\eta-\lambda) \ \mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z). \tag{1.8}$$

Also note that

$$\mathscr{S}_{0,z}^{0,0,0} f(z) = f(z), \qquad \mathscr{S}_{0,z}^{1,1,1} f(z) = \mathscr{S}_{0,z}^{1,0,0} f(z) = \frac{zf'(z)}{p},$$

$$\mathscr{S}_{0,z}^{2,1,1} f(z) = \frac{z f'(z) + z^2 f''(z)}{p^2},$$

and

$$\mathscr{S}_{0,z}^{\lambda,\lambda,\eta} f(z) = \mathscr{S}_{0,z}^{\lambda,\mu,0} f(z) = \Omega_z^{\lambda,p} f(z),$$

where $\Omega_z^{\lambda,p}$ is the *extended fractional differ-integral operator* studied recently by Patel and Mishra [19] and $\Omega_z^{\lambda,1}$ is the *fractional differ-integral operator* (see [23]). Setting

$$\lambda = -\alpha, \quad \mu = 0 \quad \text{and} \quad \eta = \beta - 1,$$

in (1.7), we obtain the following p–valent generalization of *multiplier transformation operator* [9], i.e.

$$\begin{aligned}
\mathscr{S}_{0,z}^{-\alpha,0,\beta-1} f(z) &= \mathscr{Q}_{\beta}^{\alpha,p} f(z) \\
&= \binom{p+\alpha+\beta-1}{p+\beta-1} \frac{\alpha}{z^{\beta}} \int_0^z t^{\beta-1} \left(1 - \frac{t}{z}\right)^{\alpha-1} f(t)\,\mathrm{d}t \\
&= z^p + \sum_{n=1}^{\infty} \frac{\Gamma(p+\beta+n)\ \Gamma(p+\alpha+\beta)}{\Gamma(p+\alpha+\beta+n)\ \Gamma(p+\beta)}\ a_{p+n} z^{p+n},
\end{aligned}$$

$$(\beta > -p,\ \ \alpha+\beta > -p).$$

On the other hand, if we set

$$\lambda = -1, \quad \mu = 0 \quad \text{and} \quad \eta = \beta - 1,$$

in (1.7), we obtain the *generalized Libera operator* $\mathscr{F}_{\beta,p} : \mathscr{A}_p \to \mathscr{A}_p$ $(\beta > -p)$ defined by (cf. [3, 6, 11, 18])

$$\begin{aligned}
\mathscr{S}_{0,z}^{-1,0,\beta-1} f(z) &= \mathscr{F}_{\beta,p} f(z) = \frac{p+\beta}{z^{\beta}} \int_0^z t^{\beta-1} f(t)\,\mathrm{d}t \\
&= z^p + \sum_{n=1}^{\infty} \frac{p+\beta}{p+\beta+n} a_{p+n} z^{p+n} \qquad (\beta > -p). \qquad (1.9)
\end{aligned}$$

For the choice $p = 1$, where $\beta \in \mathbb{N}$, the operator defined by (1.9) reduces to the well-known *Bernardi integral operator* [3]. In particular, Bernardi [3] showed that the function $\mathscr{F}_{\beta,1}(f)$ belongs to the classes $\mathscr{S}^*$ and $\mathscr{K}$, whenever f belongs to the classes $\mathscr{S}^*$ and $\mathscr{K}$, respectively, which include the results earlier by Libera [11] (here, $\mathscr{S}^*$ represents the subclass of $\mathscr{A}$ consisting of all the *starlike (univalent) and normalized functions in* U). Moreover, for the choice

$$\lambda = -1, \quad \mu = 0, \quad \text{and} \quad \eta = -1$$

in (1.7), we have

$$\mathscr{S}_{0,z}^{-1,0,-1} f(z) = \mathscr{J}_p f(z) = p \int_0^z \frac{f(t)}{t}\,\mathrm{d}t = z^p + \sum_{n=1}^{\infty} \frac{p}{p+n} a_{p+n} z^{p+n},$$

and the operator in $\mathscr{J}_p$ is the *generalization of the well-known Alexander integral operator* (cf. [25, 26]).

Using of the principle of subordination, Miller and Mocanu [13] obtained different subordination-preserving theorems for certain integral operators for analytic functions in U. Moreover, in [4, 5] the author investigated the subordination and superordination preserving properties of integral operators, while some other interesting developments involving subordination and superordination were considered in [1, 2, 28, 29]. In the present paper, by a sandwich-type theorem we obtained subordination-and superordination-preserving properties of the differintegral operator $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}$ defined by (1.6).

The following lemmas will be required in our present investigation.

LEMMA 1.1. [15, Theorem 2.3i, p. 35] *Suppose that the function $H:\mathbb{C}^2\to\mathbb{C}$ satisfies the condition*

$$\operatorname{Re} H(is,t)\leqslant 0,$$

for all $s,t\in\mathbb{R}$ with $t\leqslant -n(1+s^2)/2$, where n is a positive integer. If the function $p(z)=1+p_nz^n+\dots$ is analytic in U and

$$\operatorname{Re} H(p(z),zp'(z))>0,\ \ z\in\mathrm{U},$$

then $\operatorname{Re} p(z)>0,\ z\in\mathrm{U}$.

LEMMA 1.2. [12] *Let $\beta,\gamma\in\mathbb{C}$ with $\beta\neq 0$, and let $h\in\mathscr{H}(\mathrm{U})$ with $h(0)=c$. If $\operatorname{Re}[\beta h(z)+\gamma]>0$ for $z\in\mathrm{U}$, then the differential equation*

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=h(z),\quad q(0)=c,$$

has an analytic solution in U, that satisfies $\operatorname{Re}[\beta q(z)+\gamma]>0,\ z\in\mathrm{U}$.

LEMMA 1.3. [15, Lemma 2.2d, p. 24] *Let $q\in\mathscr{Q}$ with $q(0)=a$, and let $p(z)=a+a_nz^n+\dots$ be analytic in U with $p(z)\not\equiv a$ and $n\geqslant 1$. If p is not subordinate to q, then there exist the points $z_0=r_0e^{i\theta}\in\mathrm{U}$ and $\zeta_0\in\partial\mathrm{U}\setminus E(f)$, and an $m\geqslant n\geqslant 1$ for which $p\left(\mathrm{U}_{r_0}\right)\subset q(\mathrm{U})$,*

$$p(z_0)=q(\zeta_0),\quad \textit{and}\quad z_0p'(z_0)=m\zeta_0q'(\zeta_0),$$

where $\mathrm{U}_{r_0}=\{z\in\mathbb{C}:|z|<r_0\}$.

A function $L(z,t):\mathrm{U}\times[0,+\infty)\to\mathbb{C}$ is called *a subordination (or a Loewner) chain* if $L(\cdot,t)$ is analytic and univalent in U for all $t\geqslant 0$, and $L(z,s)\prec L(z,t)$ when $0\leqslant s\leqslant t$.

LEMMA 1.4. [16, Theorem 7, p. 822] *Let $q\in\mathscr{H}[a,1]$, let $\phi:\mathbb{C}^2\to\mathbb{C}$, and set $\phi(q(z),zq'(z))\equiv h(z)$. If $L(z,t)=\phi(q(z),tzq'(z))$ is a subordination chain and $p\in\mathscr{H}[a,1]\cap\mathscr{Q}$, then*

$$h(z)\prec\phi(p(z),zp'(z))$$

implies that

$$q(z) \prec p(z).$$

Furthermore, if the differential equation $\phi(q(z), zq'(z)) = h(z)$ *has a univalent solution* $q \in \mathscr{Q}$, *then* q *is the best subordinant.*

The next well-known lemma gives a sufficient condition so that the $L(z,t)$ function will be a subordination chain.

LEMMA 1.5. [20, p. 159] *Let* $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$, *with* $a_1(t) \neq 0$ *for all* $t \geqslant 0$ *and* $\lim_{t\to+\infty} |a_1(t)| = +\infty$. *Suppose that* $L(\cdot,t)$ *is analytic in* U *for all* $t \geqslant 0$, $L(z,\cdot)$ *is continuously differentiable on* $[0,+\infty)$ *for all* $z \in \mathrm{U}$. *If* $L(z,t)$ *satisfies*

$$\operatorname{Re}\left[z\frac{\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right] > 0, \quad z \in \mathrm{U}, \quad t \geqslant 0.$$

and

$$|L(z,t)| \leqslant K_0 |a_1(t)|, \quad |z| < r_0 < 1, \quad t \geqslant 0$$

for some positive constants K_0 *and* r_0, *then* $L(z,t)$ *is a subordination chain.*

2. Main results

We first prove the following subordination theorem involving the operator $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}$.

THEOREM 2.1. *Let* $\lambda, \mu, \eta, \nu \in \mathbb{R}$ *and* $p \in \mathbb{N}$, *with* $\mu < p+1$, $\lambda < \eta + p + 1$, $\nu < p$, *and*

$$\alpha \equiv \frac{p(p+\eta-\lambda)}{p-\nu} \geqslant 1. \tag{2.1}$$

Let $g \in \mathscr{A}_p$, *and set*

$$\varphi(z) \equiv \frac{p-\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta}g(z)}{z^{p-1}} + \frac{\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}g(z)}{z^{p-1}}. \tag{2.2}$$

Suppose that

$$\operatorname{Re}\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\rho, \quad z \in \mathrm{U}, \tag{2.3}$$

where $\rho = 0$ *if* $\alpha = 1$, *and*

$$\rho = \begin{cases} \dfrac{\alpha-1}{2}, & \text{if } 1 < \alpha \leqslant 2, \\ \dfrac{1}{2(\alpha-1)}, & \text{if } \alpha > 2. \end{cases} \tag{2.4}$$

If $f \in \mathscr{A}_p$, *then the subordination condition*

$$\frac{p-\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta}f(z)}{z^{p-1}} + \frac{\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)}{z^{p-1}} \prec \varphi(z) \tag{2.5}$$

implies

$$\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^{p-1}} \prec \frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} g(z)}{z^{p-1}}. \tag{2.6}$$

Moreover, the function $\mathscr{S}_{0,z}^{\lambda,\mu,\eta} g(z)/z^{p-1}$ *is the best dominant of* (2.5).

Proof. If we define the functions F and G by

$$F(z) = \frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^{p-1}} \quad \text{and} \quad G(z) = \frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} g(z)}{z^{p-1}}, \tag{2.7}$$

then $F, G \in \mathscr{A}$. We first show that, if the function q is defined by

$$q(z) = 1 + \frac{zG''(z)}{G'(z)}, \tag{2.8}$$

then

$$\operatorname{Re} q(z) > 0, \quad z \in \mathrm{U}.$$

Taking logarithmic differentiation of both sides of the second equation in (2.7) and using (1.8) for $g \in \mathscr{A}_p$, we have

$$\alpha\varphi(z) = (\alpha - 1)G(z) + zG'(z), \tag{2.9}$$

where α is defined by (2.1). Differentiating both sides of (2.9) and using the definition formula (2.2) we get

$$1 + \frac{z\varphi''(z)}{\varphi'(z)} = q(z) + \frac{zq'(z)}{q(z) + \alpha - 1} \equiv h(z). \tag{2.10}$$

From (2.3) we have

$$\operatorname{Re}\left[h(z) + \alpha - 1\right] > 0, \quad z \in \mathrm{U},$$

and by using Lemma 1.2 we deduce that the differential equation (2.10) has a solution $q \in \mathscr{H}(\mathrm{U})$, with $q(0) = h(0) = 1$.

Let define the function

$$H(u,v) = u + \frac{v}{u + \alpha - 1} + \rho \tag{2.11}$$

where ρ is given by (2.4). From (2.3), (2.10), and (2.11) we obtain

$$\operatorname{Re} H\left(q(z), zq'(z)\right) > 0, \quad z \in \mathrm{U}.$$

Now we will show that $\operatorname{Re} H(is,t) \leqslant 0$ for all $s \in \mathbb{R}$ and $t \leqslant -(1+s^2)/2$. From (2.11) we have

$$\begin{aligned} \operatorname{Re} H(is,t) &= \operatorname{Re}\left(is + \frac{t}{is + \alpha - 1} + \rho\right) \\ &= \frac{(\alpha - 1)t}{|\alpha - 1 + is|^2} + \rho \leqslant -\frac{E_\rho(s)}{2\,|\alpha - 1 + is|^2}, \end{aligned} \tag{2.12}$$

where

$$E_\rho(s) = (\alpha - 1 - 2\rho)s^2 + (\alpha - 1)[1 - 2\rho(\alpha - 1)]. \tag{2.13}$$

For ρ given by (2.4), the coefficient of s^2 in the $E_\rho(s)$ given by (2.13) is positive or equal to zero, and $E_\rho(0) \geqslant 0$, hence we deduce that $E_\rho(s) \geqslant 0$ for all $s \in \mathbb{R}$. Now, from (2.12) we see that $\operatorname{Re} H(is,t) \leqslant 0$ for all $s \in \mathbb{R}$ and $t \leqslant -(1+s^2)/2$. Thus, by using Lemma 1.1 we conclude that $\operatorname{Re} q(z) > 0$ for all $z \in \mathrm{U}$, i.e. the function G defined by (2.7) is convex (univalent) in U.

Next we will prove that the subordination condition (2.5) implies

$$F(z) \prec G(z), \tag{2.14}$$

for the functions F and G defined by (2.7). Without loss of generality, we can assume that G is analytic and univalent on $\overline{\mathrm{U}}$ and $G'(\zeta) \neq 0$ for $|\zeta| = 1$. Otherwise, we replace F and G by $F_r(z) = F(rz)$ and $G_r(z) = G(rz)$ respectively, where $r \in (0,1)$. These functions satisfy the conditions of the theorem on $\overline{\mathrm{U}}$, and we need to prove that $F_r(z) \prec G_r(z)$ for all $r \in (0,1)$, which enables us to obtain (2.14) by letting $r \to 1^-$.

Let define the function $L(z,t)$ by

$$L(z,t) \equiv \left(1 - \frac{1}{\alpha}\right) G(z) + \frac{1+t}{\alpha} zG'(z), \quad z \in \mathrm{U}, \quad t \geqslant 0. \tag{2.15}$$

Then,

$$\left.\frac{\partial L(z,t))}{\partial z}\right|_{z=0} = G'(0)\left(1 + \frac{t}{\alpha}\right) = 1 + \frac{t}{\alpha} \neq 0, \quad t \geqslant 0,$$

and this shows that the function $L(z,t) = a_1(t)z + \ldots$ satisfies the conditions $a_1(t) \neq 0$ for all $t \geqslant 0$ and $\lim\limits_{t \to +\infty} |a_1(t)| = +\infty$.

From the definition (2.15) and using the assumption (2.1), for all $t \geqslant 0$ we have that

$$\frac{|L(z,t)|}{|a_1(t)|} = \frac{\left|\left(1 - \frac{1}{\alpha}\right) G(z) + \frac{1+t}{\alpha} zG'(z)\right|}{1 + \frac{t}{\alpha}} \leqslant \frac{\left(1 - \frac{1}{\alpha}\right) |G(z)| + \frac{1+t}{\alpha} |zG'(z)|}{1 + \frac{t}{\alpha}}. \tag{2.16}$$

Since the function G is convex and normalized in the unit disk, i.e. $G \in \mathscr{K}$, hence the following well-known growth and distortion sharp inequalities (see [8]) are true:

$$\frac{r}{1+r} \leqslant |G(z)| \leqslant \frac{r}{1-r}, \quad \text{if } |z| \leqslant r, \tag{2.17}$$

$$\frac{1}{(1+r)^2} \leqslant |G'(z)| \leqslant \frac{1}{(1-r)^2}, \quad \text{if } |z| \leqslant r. \tag{2.18}$$

Using the right-hand sides of these inequalities in (2.16), we deduce that

$$\frac{|L(z,t)|}{|a_1(t)|} \leqslant \frac{r}{(1-r)^2} \, \frac{t + 1 + (\alpha - 1)(1-r)}{\alpha + t} \leqslant \frac{r}{(1-r)^2}, \quad |z| \leqslant r, \ t \geqslant 0,$$

and thus, the second assumption of Lemma 1.5 holds.

Furthermore,

$$\operatorname{Re}\left[z\frac{\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right] = \alpha - 1 + (1+t)\operatorname{Re}\left(1+\frac{zG''(z)}{G'(z)}\right) > 0,\ z\in \mathrm{U},\ t\geqslant 0,$$

and according to Lemma 1.5 the function $L(z,t)$ is a subordination chain. From the definition of the subordination chain combined with (1.1), we obtain

$$L(\zeta,t)\notin L(\mathrm{U},0)=\varphi(\mathrm{U}) \quad \text{whenever} \quad \zeta\in\partial\mathrm{U},\ t\geqslant 0.$$

Supposing that F is not subordinate to G, then by Lemma 1.3 there exist the points $z_0\in\mathrm{U}$ and $\zeta_0\in\partial\mathrm{U}$, and the number $t\geqslant 0$, such that

$$F(z_0)=G(\zeta_0) \quad \text{and} \quad z_0F'(z_0)=(1+t)\zeta_0G'(\zeta_0).$$

From these two relations, and by virtue of the subordination condition (2.5), we deduce that

$$\begin{aligned} L(\zeta_0,t) &= \left(1-\frac{1}{\alpha}\right)G(\zeta_0)+\frac{1+t}{\alpha}\zeta_0G'(\zeta_0) \\ &= \left(1-\frac{1}{\alpha}\right)F(z_0)+\frac{1}{\alpha}z_0F'(z_0) \\ &= \frac{p-\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta}f(z_0)}{z_0^{p-1}}+\frac{\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z_0)}{z_0^{p-1}}\in\varphi(\mathrm{U}), \end{aligned}$$

which contradicts the above observation $L(\zeta_0,t)\notin\varphi(\mathrm{U})$. Therefore, the subordination condition (2.5) must imply the subordination given by (2.14). Considering $F(z)=G(z)$, we see that the function G is the best dominant, which completes the proof of the theorem. □

We next prove the dual result of Theorem 2.1, in the sense that subordinations are replaced by superordinations.

THEOREM 2.2. *Let $\lambda,\mu,\eta,\nu\in\mathbb{R}$ and $p\in\mathbb{N}$, with $\mu<p+1$, $\lambda<\eta+p+1$, $\nu<p$, and*

$$\alpha\equiv\frac{p(p+\eta-\lambda)}{p-\nu}>1. \tag{2.19}$$

Let $g\in\mathscr{A}_p$, and set

$$\varphi(z)\equiv\frac{p-\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta}g(z)}{z^{p-1}}+\frac{\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}g(z)}{z^{p-1}}.$$

Suppose that

$$\operatorname{Re}\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right)>-\rho,\ z\in\mathrm{U},$$

where ρ is given by (2.4). If $f \in \mathscr{A}_p$, suppose that the function

$$\frac{p-\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta}f(z)}{z^{p-1}}+\frac{\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)}{z^{p-1}}$$

is univalent in U, *and* $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)/z^{p-1} \in \mathscr{H}[1,1]\cap\mathscr{Q}$. *Then the superordination condition*

$$\varphi(z) \prec \frac{p-\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta}f(z)}{z^{p-1}}+\frac{\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)}{z^{p-1}} \tag{2.20}$$

implies

$$\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}g(z)}{z^{p-1}} \prec \frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)}{z^{p-1}}. \tag{2.21}$$

Moreover, the function $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}g(z)/z^{p-1}$ *is the best subordinate of* (2.20).

Proof. The first part of the proof is similar to that of the previous theorem, and so we will use the same notations as in the proof of Theorem 2.1.

Let F and G be two functions defined by (2.7). If the function q is defined by (2.8), similarly as in the proof of Theorem 2.1 we obtain that

$$\varphi(z)=\left(1-\frac{1}{\alpha}\right)G(z)+\frac{1}{\alpha}zG'(z)\equiv\phi(G(z),zG'(z)). \tag{2.22}$$

Using the same method as in the proof of the above theorem, we may prove that $\operatorname{Re} q(z)>0$ for all $z\in$ U, i.e. the function G defined by (2.7) is convex (univalent) in U.

Next we prove that the subordination condition (2.20) implies

$$G(z)\prec F(z). \tag{2.23}$$

Considering the function $L(z,t)$ defined by

$$L(z,t)\equiv\left(1-\frac{1}{\alpha}\right)G(z)+\frac{t}{\alpha}zG'(z),\quad z\in \mathrm{U},\quad t\geqslant 0, \tag{2.24}$$

we have

$$\left.\frac{\partial L(z,t))}{\partial z}\right|_{z=0}=\frac{\alpha-1+t}{\alpha}G'(0)=\frac{\alpha-1+t}{\alpha}\neq 0,\quad t\geqslant 0,$$

hence the function $L(z,t)=a_1(t)z+\ldots$ satisfies the conditions $a_1(t)\neq 0$ for all $t\geqslant 0$ and $\lim\limits_{t\to+\infty}|a_1(t)|=+\infty$.

From the definition (2.24) and the assumption (2.19), for all $t\geqslant 0$ we have

$$\frac{|L(z,t)|}{|a_1(t)|}=\frac{\left|\left(1-\frac{1}{\alpha}\right)G(z)+\frac{t}{\alpha}zG'(z)\right|}{\frac{\alpha-1+t}{\alpha}}\leqslant\frac{\left(1-\frac{1}{\alpha}\right)|G(z)|+\frac{t}{\alpha}|zG'(z)|}{\frac{\alpha-1+t}{\alpha}}. \tag{2.25}$$

Since G is convex and normalized, using the right-hand sides of the inequalities (2.17) in (2.25), we deduce that

$$\frac{|L(z,t)|}{|a_1(t)|} \leqslant \frac{r}{(1-r)^2}\,\frac{t+(\alpha-1)(1-r)}{\alpha-1+t} \leqslant \frac{r}{(1-r)^2}, \quad |z| \leqslant r, \ t \geqslant 0,$$

hence, the second assumption of Lemma 1.5 holds. Moreover,

$$\operatorname{Re}\left[z\frac{\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right] = \alpha - 1 + t\operatorname{Re}\left(1+\frac{zG''(z)}{G'(z)}\right) > 0, \quad z \in \mathrm{U}, \ t \geqslant 0,$$

and according to Lemma 1.5 the function $L(z,t)$ is a subordination chain. Therefore, according to Lemma 1.4 we conclude that the superordination condition (2.20) imply the superordination (2.21). Furthermore, since the differential equation (2.21) has the univalent solution G, it is the best subordinant of the given differential superordination, which complete the proof of the theorem. □

If we combine Theorem 2.1 and Theorem 2.2, then we obtain the following differential sandwich-type theorem:

THEOREM 2.3. *Let $\lambda,\mu,\eta,\nu \in \mathbb{R}$ and $p \in \mathbb{N}$, with $\mu < p+1$, $\lambda < \eta + p + 1$, $\nu < p$, and*

$$\alpha \equiv \frac{p(p+\eta-\lambda)}{p-\nu} > 1.$$

Let $g_k \in \mathscr{A}_p$ $(k=1,2)$, and set

$$\varphi_k(z) \equiv \frac{p-\nu}{p}\,\frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta} g_k(z)}{z^{p-1}} + \frac{\nu}{p}\,\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} g_k(z)}{z^{p-1}}.$$

Suppose that

$$\operatorname{Re}\left(1+\frac{z\varphi_k''(z)}{\varphi_k'(z)}\right) > -\rho, \quad z \in \mathrm{U}, \quad (k=1,2) \tag{2.26}$$

where ρ is given by (2.4). If $f \in \mathscr{A}_p$, suppose that the function

$$\frac{p-\nu}{p}\,\frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta} f(z)}{z^{p-1}} + \frac{\nu}{p}\,\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^{p-1}}$$

is univalent in U, *and $\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)/z^{p-1} \in \mathscr{H}[1,1] \cap \mathscr{Q}$. Then,*

$$\varphi_1(z) \prec \frac{p-\nu}{p}\,\frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta} f(z)}{z^{p-1}} + \frac{\nu}{p}\,\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^{p-1}} \prec \varphi_2(z) \tag{2.27}$$

implies

$$\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} g_1(z)}{z^{p-1}} \prec \frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^{p-1}} \prec \frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} g_2(z)}{z^{p-1}}.$$

Moreover, the functions $\mathscr{S}_{0,z}^{\lambda,\mu,\eta} g_1(z)/z^{p-1}$ and $\mathscr{S}_{0,z}^{\lambda,\mu,\eta} g_2(z)/z^{p-1}$ are the best subordinant the best dominant of (2.27), respectively.

The assumptions of Theorem 2.3 that the functions

$$\frac{p-\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p}+\frac{\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta}f(z)}{z^p}\quad\text{and}\quad\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)}{z^p}$$

need to be univalent in U, may be replaced by another simple conditions like it will be shown in the following result.

COROLLARY 2.1. *Let $\lambda,\mu,\eta,\nu\in\mathbb{R}$ and $p\in\mathbb{N}$, with $\mu<p+1$, $\lambda<\eta+p+1$, $\nu<p$, and*

$$\alpha\equiv\frac{p(p+\eta-\lambda)}{p-\nu}>1.$$

Let $f,g_k\in\mathscr{A}_p$ $(k=1,2)$, and let define the function ψ by

$$\psi(z)\equiv\frac{p-\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta}f(z)}{z^{p-1}}+\frac{\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)}{z^{p-1}}.$$

Suppose that the conditions (2.26) are satisfied, and

$$\operatorname{Re}\left(1+\frac{z\psi''(z)}{\psi'(z)}\right)>-\rho,\ \ z\in\mathrm{U},\tag{2.28}$$

where ρ is given by (2.4). Then,

$$\varphi_1(z)\prec\frac{p-\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta}f(z)}{z^{p-1}}+\frac{\nu}{p}\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)}{z^{p-1}}\prec\varphi_2(z)\tag{2.29}$$

implies

$$\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}g_1(z)}{z^{p-1}}\prec\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)}{z^{p-1}}\prec\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}g_2(z)}{z^{p-1}}.$$

Moreover, the functions $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}g_1(z)/z^{p-1}$ and $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}g_2(z)/z^{p-1}$ are the best subordinant the best dominant of (2.29), respectively.

Proof. In order to prove our result, we have to show that the condition (2.28) implies the univalence of the functions ψ and $F(z)=\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)/z^{p-1}$.

Since $0<\rho\leqslant 1/2$, the condition (2.28) means that ψ is a close-to-convex function in U (see [10]), hence ψ is univalent in U.

Furthermore, by using the same techniques as in the proof of Theorem 2.1, we can prove the convexity (univalence) of the function F, and so the details may be omitted. Therefore, by applying Theorem 2.3 we obtain the required result. □

THEOREM 2.4. *Let $\lambda,\mu,\eta,\beta\in\mathbb{R}$ and $p\in\mathbb{N}$, with $\mu<p+1$, $\lambda<\eta+p+1$, and $\beta+p>1$. Let $h_k\in\mathscr{A}_p$ $(k=1,2)$, and set*

$$\omega_k(z)\equiv\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}h_k(z)}{z^{p-1}}.\tag{2.30}$$

Suppose that

$$\operatorname{Re}\left(1+\frac{z\omega_k''(z)}{\omega_k'(z)}\right) > -\rho, \quad z \in \mathrm{U}, \quad (k=1,2) \tag{2.31}$$

where ρ *is given by*

$$\rho = \begin{cases} \dfrac{\beta+p-1}{2}, & \text{if } 1<\beta+p\leqslant 2,\\ \dfrac{1}{2(\beta+p-1)}, & \text{if } \beta+p>2. \end{cases} \tag{2.32}$$

If $f \in \mathscr{A}_p$, *suppose that the function* $\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)/z^{p-1}$ *is univalent in* U, *and* $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p} f(z)/z^{p-1} \in \mathscr{H}[1,1]\cap\mathscr{Q}$. *Then,*

$$\omega_1(z) \prec \frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{z^{p-1}} \prec \omega_2(z) \tag{2.33}$$

implies

$$\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p} h_1(z)}{z^{p-1}} \prec \frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p} f(z)}{z^{p-1}} \prec \frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p} h_2(z)}{z^{p-1}}.$$

Moreover, the functions $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p} h_1(z)/z^{p-1}$ *and* $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p} h_2(z)/z^{p-1}$ *are the best subordinant the best dominant of* (2.33), *respectively.*

Proof. Let define the functions F and h_k $(k=1,2)$ by

$$F(z) = \frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p} f(z)}{z^{p-1}} \quad \text{and} \quad H_k(z) = \frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p} h_k(z)}{z^{p-1}}. \tag{2.34}$$

From the definition of integral operator $\mathscr{F}_{\beta,p}$ given by (1.9), it is easy to check that

$$z\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p} f(z)\right)' = (\beta+p)\mathscr{S}_{0,z}^{\lambda,\mu,\eta} f(z) - \beta\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p} f(z). \tag{2.35}$$

From (2.35) combined with (2.30) and (2.34), we obtain that

$$(\beta+p)\omega_k(z) = (\beta+p-1)H_k(z) + zH_k'(z). \tag{2.36}$$

Setting

$$q_k(z) = 1+\frac{zH_k''(z)}{H_k'(z)} \qquad (k=1,2),$$

and differentiating both sides of (2.36), we get

$$1+\frac{z\omega_k''(z)}{\omega_k'(z)} = q_k(z) + \frac{zq_k'(z)}{q_k(z)+\beta+p-1}.$$

Now, the remaining part of the proof is similar to that of Theorem 2.3 (a combined proof of Theorem 2.1 and Theorem 2.2), and is therefore omitted. ☐

Using the same reasons and methods as in the proof of Corollary 2.1, from Theorem 2.4 we obtain the following result:

COROLLARY 2.2. *Let $\lambda,\mu,\eta,\beta\in\mathbb{R}$ and $p\in\mathbb{N}$, with $\mu<p+1$, $\lambda<\eta+p+1$, and $\beta+p>1$. Let $f,g_k\in\mathscr{A}_p$ $(k=1,2)$, and let define the function ψ by*

$$\psi(z)\equiv\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)}{z^{p-1}}.$$

Suppose that the conditions (2.31) are satisfied, and

$$\operatorname{Re}\left(1+\frac{z\psi''(z)}{\psi'(z)}\right)>-\rho,\ \ z\in\mathrm{U},$$

where ρ is given by (2.32). Then,

$$\omega_1(z)\prec\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)}{z^{p-1}}\prec\omega_2(z) \tag{2.37}$$

implies

$$\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}h_1(z)}{z^{p-1}}\prec\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}f(z)}{z^{p-1}}\prec\frac{\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}h_2(z)}{z^{p-1}}.$$

Moreover, the functions $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}h_1(z)/z^{p-1}$ and $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}h_2(z)/z^{p-1}$ are the best subordinant the best dominant of (2.37), respectively.

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Jugal K. Prajapat
Department of Mathematics
Central University of Rajasthan
Kishangarh, Distt.-Ajmer
Rajasthan, India
e-mail: jkp_0007@rediffmail.com

Teodor Bulboacă
Faculty of Mathematics and Computer Science
Babeş-Bolyai University
400084 Cluj-Napoca, Romania
e-mail: bulboaca@math.ubbcluj.ro