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MEANS AND NONREAL INTERSECTION POINTS OF TAYLOR POLYNOMIALS

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Abstract. Suppose that $f \in C^{r+1}(0,\infty)$, and let P_c denote the Taylor polynomial to f of order r at $x = c \in [a,b]$. In [2] it was shown that if r is an odd whole number and $f^{(r+1)}(x) \neq 0$ on $[a,b]$, then there is a unique x_0, $a < x_0 < b$, such that $P_a(x_0) = P_b(x_0)$. This defines a mean $M_f^r(a,b) \equiv x_0$. In this paper we discuss the *real parts* of the pairs of complex conjugate *nonreal* roots of $P_b - P_a$. We prove some results for r in general, but our most significant results are for the case $r = 3$. We prove in that case that if $f(z) = z^p$, where p is an *integer*, $p \notin \{0,1,2,3\}$, then $P_b - P_a$ has nonreal roots $x_1 \pm iy_1$, with $a < x_1 < b$ for any $0 < a < b$. This defines the countable family of means $M_{z^p}^3(a,b)$, where $p = n \in \mathbb{Z} - \{0,1,2,3\}$. We construct a cubic polynomial, g, whose real root gives the real part of the pair of complex conjugate nonreal roots of $P_b - P_a$. Instead of working directly with a formula for the roots of a cubic, we use the Intermediate Value Theorem to show that g has a root in (a,b).

1. Introduction

Suppose that $f \in C^{r+1}(0,\infty)$, and let P_c denote the Taylor polynomial to f of order r at $x = c > 0$. In ([2], Theorem 1.1) it was proved that if r is an odd whole number and $f^{(r+1)}(x) \neq 0$ on $[a,b]$, $0 < a < b$, then there is a unique real number x_0, $a < x_0 < b$, such that $P_a(x_0) = P_b(x_0)$. This, of course, defines a mean $M_f^r(a,b) \equiv x_0$. Further results and generalizations of the means $M_f^r(a,b)$ were proved in [3], where r is any positive integer, odd or even. The main purpose of this paper is to discuss the *real parts* of the pairs of complex conjugate *nonreal* roots of $P_b - P_a$. In Proposition 1 below we show, for any *odd* whole number, r, and under suitable assumptions on f, that $P_b - P_a$ has precisely one real zero x_0, $a < x_0 < b$. We also show that for any *even* whole number, r, $P_b - P_a$ has all nonreal zeros. The main question is then:

What can we say about the real parts of the pairs of complex conjugate nonreal roots of $P_b - P_a$? In particular, when do the real parts lie strictly between a and b? That is, what conditions on f imply that the real parts of the nonreal roots of $P_b - P_a$ define a mean? We cannot answer that question completely for r in general, but we are able to prove some results in § 2. If $f^{(r+1)}(x)$ is continuous and has no zeros in $[a,b]$, then the *averages* of the pairs of complex conjugate nonreal roots of $P_b - P_a$ lie strictly between a and b for any positive integer r (Proposition 2). This, of course, does not tell us what happens with the real parts of each *specific* nonreal root. However, for $r = 2$ one gets immediately that if $f'''(x) \neq 0$ on $[a,b]$, and if $x_1 \pm iy_1$ are the nonreal roots of $P_b - P_a$, then $a < x_1 < b$ for any $0 < a < b$ (Corollary 1). Also, if $f(z) = z^{r+1}$,

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then all of the nonreal roots of $P_b - P_a$ have real part given by the arithmetic mean $A(a,b) = \frac{a+b}{2}$ (Proposition 3). Most of the more detailed and complex proofs in this paper are in § 3, which involves the case $r = 3$. We prove (Theorem 1) that if $f(z) = z^p$, where p is an *integer*, $p \notin \{0,1,2,3\}$, then $P_b - P_a$ has nonreal roots $x_1 \pm iy_1$, with $a < x_1 < b$ for any $0 < a < b$. This defines the countable family of means $M^3_{z^p}(a,b)$, where $p \in \mathbb{Z} - \{0,1,2,3\}$. For example, if $p = -1$, one has $x_1 = \frac{ab(a+b)}{a^2+b^2}$. We have proven some partial results when p is *not* an integer, but Theorem 1 does *not* hold in general for $p \in \Re$, $p \notin \{0,1,2,3\}$. For example, let $p = \frac{3}{2}$, $a = 1$, and $b = 36$. Then $P_{36}(z) - P_1(z)$ has roots 6, $\frac{33}{43} \pm \frac{15}{43}\sqrt{291}i$, so that $x_0 < a$. In § 4 we give some alternate proofs and some partial results, which are perhaps interesting in their own right, and which also might lead to determining what conditions on f imply that the real parts of the pairs of complex conjugate nonreal roots of $P_b - P_a$ lie in (a,b). Finally we consider possible future research in § 5.

2. General r

A result very similar to Proposition 1(i) below was proved in [4]. Since the proof is short and we need some functions from that proof to prove Proposition 1(ii) as well as for some later material, we give the full proof here. Assume that $0 < a < b$ and that all functions f are real valued for the rest of the paper.

PROPOSITION 1. *Suppose that $f^{(r+1)}$ is continuous in some open interval containing $[a,b]$ and has no zeros in $[a,b]$. Let P_c denote the Taylor polynomial to f of order r at $x = c$.*

(i) If r is an odd positive integer, then $P_b - P_a$ has precisely one real zero x_0, $a < x_0 < b$.

(ii) If r is an even positive integer then $P_b - P_a$ has all nonreal zeros.

Proof. Let $E_c(x) = f(x) - P_c(x)$, $x \in [a,b]$. Note that

$$P_b(x) = P_a(x) \iff E_b(x) = E_a(x).$$

By the integral form of the remainder, we have

$$E_c(x) = \frac{1}{r!}\int_c^x f^{(r+1)}(t)(x-t)^r dt,$$

which implies that

$$E_a(x) = \frac{1}{r!}\int_a^x f^{(r+1)}(t)(x-t)^r dt$$

$$E_b(x) = -\frac{1}{r!}\int_x^b f^{(r+1)}(t)(x-t)^r dt.$$

Since $(E_a - E_b)(x) = \frac{1}{r!}\int_a^b f^{(r+1)}(t)(x-t)^r dt$ and $E_a(x) - E_b(x) = P_b(x) - P_a(x)$, $x \in [a,b]$, we have

$$(P_b - P_a)(x) = \frac{1}{r!}\int_a^b f^{(r+1)}(t)(x-t)^r dt. \tag{1}$$

Since (1) holds for $x \in [a,b]$ and $P_b - P_a$ is a polynomial, (1) holds for all $z \in \mathbb{C} =$ complex plane. We may assume, without loss of generality, that $f^{(r+1)}(x) > 0$ on $[a,b]$. Suppose first that r is odd. Using the formula $\frac{\partial}{\partial x}\int_a^x K(x,t)dt = \int_a^x \frac{\partial K}{\partial x}(x,t)dt + K(x,x)$ with $K(x,t) = f^{(r+1)}(t)(x-t)^{r-1}$, it follows that $E_a'(x) = \frac{1}{(r-1)!}\int_a^x f^{(r+1)}(t)(x-t)^{r-1}dt > 0$ for $a < x < b$, which implies that $E_a(x)$ is strictly increasing on (a,b).

$E_b'(x) = -\frac{1}{(r-1)!}\int_x^b f^{(r+1)}(t)(x-t)^{r-1}dt < 0$ for $a < x < b$, which implies that $E_b(x)$ is strictly decreasing on (a,b). Since $E_a(a) = E_b(b) = 0$, there is a unique x_0, $a < x_0 < b$, such that $E_b(x_0) - E_a(x_0) = 0$, which implies that $(P_b - P_a)(x_0) = 0$. Now $(P_b - P_a)'(x) = \frac{1}{(r-1)!}\int_a^b f^{(r+1)}(t)(x-t)^{r-1}dt > 0$ for $x \in \Re =$ real line. Hence $P_b - P_a$ has precisely one real zero. That proves (i).

Now suppose that r is even. Then $(P_b - P_a)(x) > 0$ for $x \in \Re$, which implies that $P_b - P_a$ has no real zeros. That proves (ii). □

(1) gives the following important equivalence: If P_c is the Taylor polynomial to f of order r at $x = c$, then

$$P_b(z) = P_a(z) \iff \int_a^b f^{(r+1)}(t)(z-t)^r dt = 0, z \in \mathbb{C}. \tag{2}$$

We now prove a general result which relates the *averages* of the real parts of the roots of $P_b - P_a$ to the center of mass of $[a,b]$ with density function $\left|f^{(r+1)}(t)\right|$.

PROPOSITION 2. *Suppose that $f^{(r+1)}$ is continuous in some open interval containing $[a,b]$ and has no zeros in $[a,b]$. Let P_c denote the Taylor polynomial to f of order r at $x = c$.*

(i) Suppose that r is odd and let $s = \frac{r-1}{2}$. By Proposition 1(ii), $P_b - P_a$ has precisely one real zero x_0, and $r-1$ nonreal zeros, $z_1, \bar{z}_1, \ldots, z_s, \bar{z}_s$. Let $x_k = \operatorname{Re} z_k$,

$k=1,\ldots,s$. Then

$$\frac{x_0+\sum_{k=1}^{s} 2\operatorname{Re} z_k}{r}=\frac{\int_a^b t f^{(r+1)}(t)dt}{\int_a^b f^{(r+1)}(t)dt}.$$

(ii) Suppose that r is even and let $s=\frac{r}{2}$. By Proposition 1(ii), P_b-P_a has all nonreal zeros, $z_1,\overline{z}_1,\ldots,z_s,\overline{z}_s$. Let $x_k=\operatorname{Re} z_k$, $k=1,\ldots,s$. Then

$$\frac{\sum_{k=1}^{s} 2\operatorname{Re} z_k}{r}=\frac{\int_a^b t f^{(r+1)}(t)dt}{\int_a^b f^{(r+1)}(t)dt}.$$

In either case, since P_b-P_a has real coefficients, Proposition 2 states that the average of the real parts of the roots of P_b-P_a is the center of mass of $[a,b]$, where the density function is $\left|f^{(r+1)}(t)\right|$.

Proof. We may assume, without loss of generality, that $f^{(r+1)}(x)>0$ on $[a,b]$. Define the monic polynomial $Q(z)=r!\frac{(P_b-P_a)(z)}{f^{(r)}(b)-f^{(r)}(a)}$. Since

$$(P_b-P_a)(z)=\sum_{k=0}^{r}\frac{f^{(k)}(b)(z-b)^k-f^{(k)}(a)(z-a)^k}{k!},$$

the coefficient of z^{r-1} in $Q(z)$ is

$$\frac{\left(rf^{(r-1)}(b)-rbf^{(r)}(b)\right)-\left(rf^{(r-1)}(a)-raf^{(r)}(a)\right)}{f^{(r)}(b)-f^{(r)}(a)}$$
$$=r\frac{\left(f^{(r-1)}(b)-bf^{(r)}(b)\right)-\left(f^{(r-1)}(a)-af^{(r)}(a)\right)}{f^{(r)}(b)-f^{(r)}(a)}.$$

Using Integration by Parts, it is easy to show that

$$\frac{\int_a^b t f^{(r+1)}(t)dt}{\int_a^b f^{(r+1)}(t)dt}=-\frac{\left(f^{(r-1)}(b)-bf^{(r)}(b)\right)-\left(f^{(r-1)}(a)-af^{(r)}(a)\right)}{f^{(r)}(b)-f^{(r)}(a)}.$$

Thus the coefficient of z^{r-1} in $Q(z)$ is $-r\frac{\int_a^b t f^{(r+1)}(t)dt}{\int_a^b f^{(r+1)}(t)dt}$.

Now suppose that r is odd. Since $P_b - P_a$ and Q have the same roots, $Q(z) = (z-x_0)(z-z_1)(z-\bar{z}_1)\cdots(z-z_s)(z-\bar{z}_s)$. Since the coefficient of z^{r-1} in $Q(z)$ is also given by

$$-x_0 - (z_1 + \bar{z}_1 + \cdots + z_s + \bar{z}_s) = -x_0 - 2\sum_{k=1}^{s} x_k,$$

we have $-x_0 - 2\sum_{k=1}^{s} x_k = -r\frac{\int_a^b t f^{(r+1)}(t)dt}{\int_a^b f^{(r+1)}(t)dt}$, which implies that

$$\frac{x_0 + \sum_{k=1}^{s} 2\mathrm{Re}\, z_k}{r} = \frac{\int_a^b t f^{(r+1)}(t)dt}{\int_a^b f^{(r+1)}(t)dt}.$$

Now suppose that r is even and write $Q(z) = (z-z_1)(z-\bar{z}_1)\cdots(z-z_s)(z-\bar{z}_s)$. Since the coefficient of z^{r-1} in $Q(z)$ is also given by

$$-(z_1 + \bar{z}_1 + \cdots + z_s + \bar{z}_s) = -2\sum_{k=1}^{s} x_k,$$

we have $-2\sum_{k=1}^{s} x_k = -r\frac{\int_a^b t f^{(r+1)}(t)dt}{\int_a^b f^{(r+1)}(t)dt}$, which implies that

$$\frac{\sum_{k=1}^{s} 2\mathrm{Re}\, z_k}{r} = \frac{\int_a^b t f^{(r+1)}(t)dt}{\int_a^b f^{(r+1)}(t)dt}. \quad \square$$

REMARK 1. Suppose that $f^{(r+1)}$ is continuous and has no zeros in $[a,b]$. If the average of the real parts of the nonreal roots of $P_b - P_a$ did not lie in (a,b), then it would not be possible for the real parts of all of the nonreal roots of $P_b - P_a$ to lie in (a,b). For each r, there are examples, such as $f(z) = z^{r+1}$ (see Proposition 3 below), where the real parts of all pairs of complex conjugate roots of $P_b - P_a$ do lie in (a,b) for all $0 < a < b$. Of course if $r = 3$, then there is only one pair of nonreal complex conjugate roots. As noted above, it is possible that the real parts of that complex conjugate pair do not lie in (a,b). For example, $f(z) = z^{3/2}$, $a = 1$, and $b = 36$. We also have examples for $r = 4$ and for $r = 5$ where only one pair of nonreal complex conjugate roots has real part lying in (a,b).

If $r = 2$, then $P_b - P_a$ has no real roots and only one pair of complex conjugate nonreal roots. Applying Proposition 2(ii) then yields the following corollary.

COROLLARY 1. *Suppose that f''' is continuous in some open interval containing $[a,b]$ and has no zeros in $[a,b]$. Let P_c denote the Taylor polynomial to f of order 2 at $x=c$. Let $x_1 \pm iy_1$ denote the nonreal roots of $P_b - P_a$ guaranteed by Proposition 1(ii). Then*

$$x_1 = \frac{\int_a^b t f'''(t)dt}{\int_a^b f'''(t)dt}$$

and thus $a < x_1 < b$.

The case $r=1$ of Proposition 2 above was given in [2], where

$$x_1 = \frac{\int_a^b t f''(t)dt}{\int_a^b f''(t)dt}$$

is just the x coordinate of the intersection point of tangent lines to a convex or concave function. This was actually the starting point that led to the generalization to intersection points of Taylor polynomials. Replacing f''' by f'' shows that this yields the same family of means as for the $r=2$ case.

PROPOSITION 3. *Let $f(z) = z^{r+1}$ and let P_c denote the Taylor polynomial to f of order r at $x=c$. Then all of the nonreal roots of $P_b - P_a$ have real part given by the arithmetic mean $A(a,b) = \frac{a+b}{2}$.*

Proof. By (2), it suffices to show that $\int_a^b f^{(r+1)}(t)(z-t)^r dt = 0 \iff \operatorname{Re} z = \frac{a+b}{2}$. Now $\int_a^b f^{(r+1)}(t)(z-t)^r dt = 0 \iff \int_a^b (z-t)^r dt = 0 \iff (z-b)^{r+1} - (z-a)^{r+1} = 0 \iff z-b = v(z-a)$, where $v = e^{2\pi ki/(r+1)}$ is an $(r+1)$st root of unity. Note that $v \neq 1$ since $a \neq b$. Thus $z = \frac{b-va}{1-v} = \frac{-v}{1-v}a + \frac{1}{1-v}b$.

$$\frac{1}{1-v} = \frac{1-\bar{v}}{(1-v)(1-\bar{v})} = \frac{1-\bar{v}}{1-2\operatorname{Re} v + |v|^2}$$
$$= \frac{1}{2(1-\operatorname{Re} v)}(1-\bar{v}),$$

therefore $\operatorname{Re}\left(\frac{1}{1-v}\right) = \frac{1}{2(1-\operatorname{Re} v)}(1-\operatorname{Re} v) = \frac{1}{2}$.

$1 = \frac{-v}{1-v} + \frac{1}{1-v}$ implies that $\operatorname{Re}\left(\frac{-v}{1-v}\right) = 1 - \operatorname{Re}\left(\frac{1}{1-v}\right) = \frac{1}{2}$. Hence $\operatorname{Re} z = \frac{1}{2}a + \frac{1}{2}b$. □

3. $r = 3$

We now state our main result for $r = 3$.

THEOREM 1. *Suppose that* $f(z) = z^p$, *where* p *is an integer,* $p \notin \{0,1,2,3\}$. *Let* P_c *denote the Taylor polynomial to* f *of order* 3 *at* $x = c$. *Then for any* $0 < a < b$, $P_b - P_a$ *has nonreal roots* $x_1 \pm iy_1$, *with* $a < x_1 < b$.

REMARK 2. Theorem 1 defines a countable family of means $M^3_{z^p}(a,b) = x_1$, where $p \in \mathbb{Z} - \{0,1,2,3\}$. By Proposition 3 with $r = 3$, amongst that family of means is the arithmetic mean.

REMARK 3. Finding z_1 such that $P_b(z_1) = P_a(z_1)$ of course involves solving a cubic polynomial equation. There are well-known formulas for the solutions of such equations, but the resulting expressions are complicated and it seems difficult to determine from such a formula that $a < x_1 < b$. For example, if $p = 5$ and $a = 1$, then one has

$$\begin{aligned} x_1 = &\frac{1}{3}\cdot\frac{b-1}{b+1}\cdot\frac{1}{20}\sqrt[3]{100\,(b^2+7b+1)\,(b-1)+150\sqrt{6}(b+1)\sqrt{q(b)}} \\ &-\frac{5}{2}\frac{b^2+4b+1}{\sqrt[3]{100\,(b^2+7b+1)\,(b-1)+150\sqrt{6}(b+1)\sqrt{q(b)}}}+2\left(b^2+b+1\right), \end{aligned}$$

where $q(b) = b^4 + 10b^3 + 28b^2 + 10b + 1$. Furthermore, we want to determine that for certain classes of functions, f, $a < \mathrm{Re}\,(z_1) < b$. Our proof of Theorem 1 also involves solving a certain cubic polynomial equation, $g(x) = 0$ (see (7) below). However, this time we are looking for a *real* solution, x_1, of $g(x) = 0$, with $a < x_1 < b$. That allows us to use the Intermediate Value Theorem to show that there is such a solution. That is, we show that for certain classes of functions, f, $g(a)g(b) < 0$. That avoids actually working with a formula for the solution of a cubic polynomial equation.

If P_c is the Taylor polynomial to f of order 3 at $x = c$, then (2) becomes

$$P_b(z) = P_a(z) \iff \int_a^b f''''(t)(z-t)^3 dt = 0. \tag{3}$$

For the rest of this section we prove some lemmas and propositions which are used to prove Theorem 1. Important for our proofs are the following integrals. Let

$$A = \int_a^b f''''(t)dt, \quad B = \int_a^b t f''''(t)dt, \quad C = \int_a^b t^2 f''''(t)dt, \quad D = \int_a^b t^3 f''''(t)dt. \tag{4}$$

We suppress the dependence of A, B, C, and D on a, b, and on f in our notation. We now prove a lemma which gives an equivalent condition for $P_b(z_1) = P_a(z_1)$ to hold when $r = 3$.

LEMMA 1. *Let P_c denote the Taylor polynomial to f of order 3 at $x=c$, and let $z_1=x_1+iy_1$ with $y_1\neq 0$. Then $P_b(z_1)=P_a(z_1)$ if and only if the following system of equations holds.*

$$\begin{aligned} Ax_1^3-3Bx_1^2+3Cx_1-D+3(B-Ax_1)y_1^2&=0 \\ 3Ax_1^2-6Bx_1+3C-Ay_1^2&=0. \end{aligned} \tag{5}$$

Proof. Using the formulas

$$\begin{aligned} \operatorname{Re}\left((z_1-t)^3\right) &= (\operatorname{Re}(z_1)-t)^3-3(\operatorname{Re}(z_1)-t)\operatorname{Im}^2(z_1) \\ \operatorname{Im}\left((z_1-t)^3\right) &= 3(\operatorname{Re}(z_1)-t)^2\operatorname{Im}(z_1)-(\operatorname{Im}(z_1))^3, \end{aligned}$$

we have

$$\operatorname{Re}\left(\int_a^b f''''(t)(z_1-t)^3dt\right)=\int_a^b f''''(t)\left[(x_1-t)^3-3(x_1-t)y_1^2\right]dt$$

and

$$\operatorname{Im}\left(\int_a^b f''''(t)(z_1-t)^3dt\right)=\int_a^b f''''(t)\left[3(x_1-t)^2y_1-y_1^3\right]dt.$$

$P_b(z_1)=P_a(z_1) \iff \int_a^b f''''(t)(z_1-t)^3dt=0$ by (3). If $y_1\neq 0$, then $\int_a^b f''''(t)(z_1-t)^3dt=0$ is equivalent to the following two equations:

$$\begin{aligned} \int_a^b f''''(t)\left[(x_1-t)^3-3(x_1-t)y_1^2\right]dt&=0 \\ \int_a^b f''''(t)\left[3(x_1-t)^2-y_1^2\right]dt&=0. \end{aligned} \tag{6}$$

Simplifying (6) shows that (x_1,y_1) satisfies (6) if and only if (x_1,y_1) satisfies (5). □

We now define the following very important cubic polynomial, g, which depends upon the given function, f, as well as on a and b:

$$g(x)=8A^2x^3-24ABx^2+6(AC+3B^2)x+AD-9BC, \tag{7}$$

where A,B,C, and D are given by (4).

LEMMA 2. *Let $A\neq 0$, $B,C\in\Re$ and not necessarily given by (4). If $B^2-AC<0$, then g is increasing on $\Re$.*

Proof.

$$g'(x) = 24A^2x^2 - 48ABx + 6AC + 18B^2$$
$$= 24A^2\left(x^2 - \frac{2B}{A}x + \frac{AC+3B^2}{4A^2}\right) = 24A^2\left(\left(x - \frac{B}{A}\right)^2 + \frac{AC-B^2}{4A^2}\right) > 0. \quad \square$$

REMARK 4. Let $A \neq 0$, $B, C \in \Re$. If $B^2 - AC < 0$ and $x_1 \in \Re$, then

$$3x_1^2 - \frac{6B}{A}x_1 + \frac{3C}{A} = 3\left(\left(x_1 - \frac{B}{A}\right)^2 + \frac{AC-B^2}{A^2}\right) > 0.$$

Critical for our proof of Theorem 1 below is the following proposition.

PROPOSITION 4. *Suppose that f'''' is continuous in some open interval containing $[a,b]$ and has no zeros in $[a,b]$ and let P_c denote the Taylor polynomial to f of order 3 at $x = c$. Then the polynomial g given by (7) has a unique real zero. In addition, if $x_1 \in \Re, y_1 = \sqrt{3x_1^2 - \frac{6B}{A}x_1 + \frac{3C}{A}}$, and $z_1 = x_1 + iy_1$, then $g(x_1) = 0 \iff P_b(z_1) = P_a(z_1)$.*

Proof. If $f'''' > 0$ on $[a,b]$, then $\left(\int_a^b t f''''(t)dt\right)^2 = \left(\int_a^b \sqrt{f''''(t)}\left(t\sqrt{f''''(t)}\right)dt\right)^2 < \left(\int_a^b f''''(t)dt\right)\left(\int_a^b t^2 f''''(t)dt\right)$ by the Cauchy-Bunyakowsky inequality ([6]). Note that the strict inequality follows since $\sqrt{f''''(t)}$ and $t\sqrt{f''''(t)}$ cannot be proportional to one another. Since $B^2 - AC$ does not depend on the sign of f'''', where A, B, C are given by (4), we have $B^2 - AC < 0$ when $f'''' \neq 0$ on $[a,b]$. Thus g has a unique real zero by Lemma 2 and the fact that g is a cubic polynomial. Note that by Remark 4, y_1 is real and positive. By Lemma 1, $P_b(z_1) = P_a(z_1) \iff$ (5) holds. Since $A \neq 0$, solving the second equation in (5) for y_1^2 and substituting into the first equation in (5) to obtain $y_1^2 = 3x_1^2 - \frac{6B}{A}x_1 + \frac{3C}{A}$ shows that (5) holds if and only if

$$Ax_1^3 - 3Bx_1^2 + 3Cx_1 - D + 3(B - Ax_1)\left(3x_1^2 - \frac{6B}{A}x_1 + \frac{3C}{A}\right) = 0 \iff$$
$$(A - 9A)x_1^3 + (-3B + 18B + 9B)x_1^2 + \left(3C - 9C - 18\frac{B^2}{A}\right)x_1 - D + \frac{9BC}{A} = 0 \iff$$
$$-8Ax_1^3 + 24Bx_1^2 - 6\left(C + 3\frac{B^2}{A}\right)x_1 + \frac{9BC}{A} - D = 0 \iff$$
$$-8A^2x_1^3 + 24ABx_1^2 - 6(AC + 3B^2)x_1 + 9BC - AD = 0 \iff g(x_1) = 0. \quad \square$$

We now focus on the case where $f(z) = z^p$, $p \in \Re - \{0,1,2,3\}$. For the purpose of proving Theorem 1, it will suffice(as shown in the proof below) to just consider the case when $a=1$, which we assume from now on. For $f(z)=z^p$, (4) then yields

$$A = p(p-1)(p-2)\left(b^{p-3}-1\right), \qquad B = p(p-1)(p-3)\left(b^{p-2}-1\right), \tag{8}$$
$$C = p(p-2)(p-3)\left(b^{p-1}-1\right), \qquad D = (p-1)(p-2)(p-3)\left(b^{p}-1\right).$$

Let $g(x)$ be given by (7), where A,B,C, and D are given by (8). For $p \in \Re$, $p \neq 0,1$, it is more convenient to define the following functions of b:

$$\begin{aligned} V(b) &= \frac{g(1)}{p(p-1)} \\ W(b) &= \frac{g(b)}{p(p-1)}. \end{aligned} \tag{9}$$

It is important to note that g is a cubic polynomial in x where the coefficients involve b. V and W are functions of the variable b.

Using (8) and substituting for A,B,C, and D in (7), yields

$$\begin{aligned} g(1) &= 8A^2 - 24AB + 6(AC+3B^2) + AD - 9BC \\ &= 8\left(p(p-1)(p-2)\right)^2\left(b^{p-3}-1\right)^2 \\ &\quad -24\left(p(p-1)\right)^2(p-2)(p-3)\left(b^{p-3}-1\right)\left(b^{p-2}-1\right) \\ &\quad +6\left(p(p-2)\right)^2(p-1)(p-3)\left(b^{p-3}-1\right)\left(b^{p-1}-1\right) \\ &\quad +18\left(p(p-1)(p-3)\right)^2\left(b^{p-2}-1\right)^2 \\ &\quad +\left((p-1)(p-2)\right)^2 p(p-3)\left(b^{p-3}-1\right)\left(b^{p}-1\right) \\ &\quad -9\left(p(p-3)\right)^2(p-1)(p-2)\left(b^{p-2}-1\right)\left(b^{p-1}-1\right), \end{aligned}$$

which implies, after some simplification, that

$$\begin{aligned} V(b) = {} & 12(p+1) - 2(p-2)(p-3)\left(4p^2-12p-1\right)b^{2p-3} \\ & +6p(p-3)\left(4p^2-16p+13\right)b^{2p-4} - 24p(p-1)(p-2)(p-3)b^{2p-5} \\ & +8p(p-1)(p-2)^2b^{2p-6} - (p-1)(p-2)^2(p-3)b^{p} \\ & +3p(p-2)(p-3)(p-5)b^{p-1} - 3p(p-3)\left(p^2-9p+2\right)b^{p-2} \\ & +(p-2)(p+1)\left(p^2-13p+6\right)b^{p-3}. \end{aligned} \tag{10}$$

$$\begin{aligned} g(b) &= 8A^2b^3 - 24ABb^2 + 6(AC+3B^2)b + AD - 9BC \\ &= 8\left(p(p-1)(p-2)\right)^2\left(b^{p-3}-1\right)^2 b^3 \\ &\quad -24\left(p(p-1)\right)^2(p-2)(p-3)\left(b^{p-3}-1\right)\left(b^{p-2}-1\right)b^2 \\ &\quad +6\left(p(p-2)\right)^2(p-1)(p-3)\left(b^{p-3}-1\right)\left(b^{p-1}-1\right)b \\ &\quad +18\left(p(p-1)(p-3)\right)^2\left(b^{p-2}-1\right)^2 b \\ &\quad +\left((p-1)(p-2)\right)^2 p(p-3)\left(b^{p-3}-1\right)\left(b^{p}-1\right) \end{aligned}$$

$$-9\left(p(p-3)\right)^2(p-1)(p-2)\left(b^{p-2}-1\right)\left(b^{p-1}-1\right),$$

which implies, after some simplification, that

$$\begin{aligned}W(b) = {} & 12(p+1)b^{2p-3}+(p-2)(p+1)\left(p^2-13p+6\right)b^p \\ & -3p(p-3)\left(p^2-9p+2\right)b^{p-1}+3p(p-2)(p-3)(p-5)b^{p-2} \\ & -(p-1)(p-2)^2(p-3)b^{p-3}+8p(p-1)(p-2)^2b^3 \\ & -24p(p-1)(p-2)(p-3)b^2+6p(p-3)\left(4p^2-16p+13\right)b \\ & -2(p-2)(p-3)\left(4p^2-12p-1\right).\end{aligned} \tag{11}$$

Much of the work in proving Theorem 1 is embodied in the following two propositions.

PROPOSITION 5. *Suppose that* $p=n\in\mathbb{N}$, $n\notin\{1,2,3\}$. *Then* $V(b)=Q(b)(b-1)^5$, *where* Q *is a polynomial with negative nonzero coefficients.*

Proof. While the cases $n=4$ thru 8 could be absorbed into the proof below, we find it more convenient to treat those cases separately. $n=4$ gives $V(b)=-60(b-1)^5$, $n=5$ gives $V(b)=-36(13b^2+10b+2)(b-1)^5$, $n=6$ gives $V(b)=-12(142b^4+161b^3+105b^2+35b+7)(b-1)^5$, $n=7$ gives $V(b)=-24(185b^6+246b^5+220b^4+140b^3+60b^2+20b+4)(b-1)^5$, and $n=8$ gives $V(b)=-36(265b^8+385b^7+395b^6+327b^5+210b^4+105b^3+45b^2+15b+3)(b-1)^5$, so that Proposition 5 holds in those cases. So assume now that $n\geqslant 9$. We list the derivatives of V evaluated at $b=0$ (simplified somewhat) and which are required for our proof.

$$\begin{aligned}V(0) &= 12n+12, \qquad V^{(i)}(0)=0, \quad i=1,\ldots,n-4 \\ V^{(n-3)}(0) &= (n-2)!\,(n+1)\left(n^2-13n+6\right) \\ V^{(n-2)}(0) &= -3(n-2)!\,(n-3)n\left(n^2-9n+2\right) \\ V^{(n-1)}(0) &= 3n!\,(n-2)(n-3)(n-5) \\ V^{(n)}(0) &= -n!\,(n-1)(n-2)^2(n-3) \\ V^{(i)}(0) &= 0, i=n+1,\ldots,2n-7.\end{aligned} \tag{12}$$

Note first that $Q(0)=-V(0)<0$, so we only need to show that $Q^{(r)}(0)\leqslant 0$ for $r\geqslant 1$. $Q(b)=(b-1)^{-5}V(b)$ yields

$$Q^{(r)}(b)=\frac{d^r}{dy^r}\left((b-1)^{-5}V(b)\right)=\sum_{j=0}^{r} j!\binom{r}{j}\binom{-5}{j}(b-1)^{-5-j}V^{(r-j)}(b),$$

which implies that $Q^{(r)}(0)=\sum_{j=0}^{r} j!\binom{r}{j}\binom{-5}{j}(-1)^{j+1}V^{(r-j)}(0)$. Using the identity $\binom{-5}{j}=(-1)^j\binom{j+4}{j}$ yields $Q^{(r)}(0)=-\sum_{j=0}^{r} j!\binom{r}{j}\binom{j+4}{j}V^{(r-j)}(0)$ or

$$Q^{(r)}(0)=-V^{(r)}(0)-\sum_{j=1}^{r}\left(\prod_{i=0}^{j-1}(r-i)\right)\binom{j+4}{j}V^{(r-j)}(0). \tag{13}$$

Case 1: $1 \leqslant r \leqslant n-4$. By (12), in (13) $V^{(r)}(0)=0$ and $V^{(r-j)}(0)=0$ for $1 \leqslant j \leqslant r-1$, so we are left with $j=r$, which yields $Q^{(r)}(0)=-(r!)\binom{r+4}{r}V(0)=-12(r!)\binom{r+4}{r}(n+1)<0$

Case 2: $r=n-k$, $k=0,1,2,3$.

If $r=n-3$, then by (12) the only nonzero derivatives which appear in (13) are $-V^{(r)}(0)$ or when $j=r$, which gives

$$\begin{aligned} Q^{(n-3)}(0) &= -V^{(n-3)}(0)-(n-3)!\binom{n+1}{n-3}V(0) \\ &= -(n-2)!\,(n+1)\left(n^2-13n+6\right)-12(n-3)!\binom{n+1}{n-3}(n+1), \end{aligned}$$

which implies that

$$\begin{aligned} -\frac{Q^{(n-3)}(0)}{(n+1)\,(n-3)!} &= (n-2)\left(n^2-13n+6\right)+\frac{1}{2}(n+1)n(n-1)(n-2) \\ &= \frac{1}{2}\,(n-2)\,(n-4)\left(n^2+6n-3\right)>0. \end{aligned}$$

Since $n^2+6n-3>0$, $n>1$, $Q^{(n-3)}(0)<0$.

If $r=n-2$, then by (12) the only nonzero derivatives which appear in (13) are $-V^{(r)}(0)$ or when $j=1$ or $j=r$ in (13), which gives

$$\begin{aligned} Q^{(n-2)}(0) &= -V^{(n-2)}(0)-5(n-2)V^{(n-3)}(0)-(n-2)!\binom{n+2}{n-2}V(0) \\ &= 3(n-2)!\,(n-3)\,n\left(n^2-9n+2\right)-5(n-2)(n-2)!\,(n+1)\left(n^2-13n+6\right) \\ &\quad -12(n-2)!\binom{n+2}{n-2}(n+1), \end{aligned}$$

which implies that

$$\begin{aligned} \frac{Q^{(n-2)}(0)}{(n-2)!} &= 3\,(n-3)\,n\left(n^2-9n+2\right)-5(n-2)\,(n+1)\left(n^2-13n+6\right) \\ &\quad -\frac{1}{2}(n+2)(n+1)^2n(n-1) \\ &= -\frac{1}{2}\,(n-5)\,(n+2)\,p(n), \end{aligned}$$

where $p(x)=x^3+10x^2-27x+12$. $p'(x)=3x^2+20x-27>0$ for $x>2$, which implies that p is increasing on $(2,\infty)$. Since $p(2)>0$, $p(x)>0$ for $x>2$. Thus $Q^{(n-2)}(0)<0$.

If $r = n-1$, then by (12) the only nonzero derivatives which appear in (13) are $-V^{(r)}(0)$ or when $j=1$, $j=2$, or $j=r$, which gives

$$\begin{aligned}
Q^{(n-1)}(0) &= -V^{(n-1)}(0) - 5(n-1)V^{(n-2)}(0) - 15(n-1)(n-2)V^{(n-3)}(0) \\
&\quad -(n-1)!\binom{n+3}{n-1}V(0) \\
&= -3n!\,(n-2)\,(n-3)\,(n-5) + 15(n-1)(n-2)!\,(n-3)\,n\left(n^2-9n+2\right) \\
&\quad -15(n-1)(n-2)(n-2)!\,(n+1)\left(n^2-13n+6\right) \\
&\quad -12(n-1)!\binom{n+3}{n-1}(n+1),
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{Q^{(n-1)}(0)}{(n-1)!} &= -3n\,(n-2)\,(n-3)\,(n-5) + 15\,(n-3)\,n\left(n^2-9n+2\right) \\
&\quad -15(n-2)\,(n+1)\left(n^2-13n+6\right) - \frac{1}{2}(n+3)(n+2)(n+1)^2 n \\
&= -\frac{1}{2}\,(n-1)\,(n-6)\,(n+3)\left(n^2+17n-20\right).
\end{aligned}$$

Since $n^2+17n-20>0$, $n>2$, $Q^{(n-1)}(0)<0$.

If $r=n$, then by (12) the only nonzero derivatives which appear in (13) are $-V^{(r)}(0)$ or when $j=1$, $j=2$, $j=3$, or $j=r$, which gives

$$\begin{aligned}
Q^{(n)}(0) &= -V^{(n)}(0) - 5nV^{(n-1)}(0) - 15n(n-1)V^{(n-2)}(0) \\
&\quad - 35n(n-1)(n-2)V^{(n-3)}(0) - n!\binom{n+4}{n}V(0) \\
&= n!\,(n-1)\,(n-2)^2\,(n-3) - 15n\,(n!)\,(n-2)\,(n-3)\,(n-5) \\
&\quad + 45n(n-1)(n-2)!\,(n-3)\,n\left(n^2-9n+2\right) \\
&\quad - 35n(n-1)(n-2)(n-2)!\,(n+1)\left(n^2-13n+6\right) - 12n!\binom{n+4}{n}(n+1),
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{Q^{(n)}(0)}{n!} &= (n-1)\,(n-2)^2\,(n-3) - 15n\,(n-2)\,(n-3)\,(n-5) \\
&\quad + 45\,(n-3)\,n\left(n^2-9n+2\right) - 35(n-2)\,(n+1)\left(n^2-13n+6\right) \\
&\quad - \frac{1}{2}(n+4)(n+3)(n+2)(n+1)^2 \\
&= -\frac{1}{2}\,(n+4)\,(n-7)\,p(n),
\end{aligned}$$

where $p(x) = x^3+22x^2-45x+30$. $p'(x) = 3x^2+44x-45>0$ for $x>1$, which implies that p is increasing on $(1,\infty)$. Since $p(1)>0$, $p(x)>0$ for $x>1$. Thus $Q^{(n)}(0)<0$.

Note that we only need to go up to $r=2n-8$ since $\deg Q=2n-8$. Since $n>8$, $2n-8>n$.

Case 3: $r=n+k$, $k=1,\ldots,n-8$. Note that, by (12), in (13) the only other nonzero derivatives(not including the 0th derivative) which appear are when $r-j=n-l$, $l=0,1,2,3\Rightarrow j=r-n+l=k+l$ for $l=0,1,2,3$. So let $j=k,k+1,k+2,k+3$ and also let $j=r$ in (13) to obtain

$$
\begin{aligned}
Q^{(n+k)}(0) = &-\left(\prod_{i=0}^{k-1}(n+k-i)\right)\binom{k+4}{k}V^{(n)}(0)-\left(\prod_{i=0}^{k}(n+k-i)\right)\binom{k+5}{k+1}V^{(n-1)}(0)\\
&-\left(\prod_{i=0}^{k+1}(n+k-i)\right)\binom{k+6}{k+2}V^{(n-2)}(0)-\left(\prod_{i=0}^{k+2}(n+k-i)\right)\binom{k+7}{k+3}V^{(n-3)}(0)\\
&-(n+k)!\binom{n+k+4}{n+k}V(0)\\
=&\left(\prod_{i=0}^{k-1}(n+k-i)\right)n!\,(n-1)\,(n-2)^2\,(n-3)\,\frac{1}{24}\left(\prod_{i=1}^{4}(k+i)\right)\\
&-3\left(\prod_{i=0}^{k}(n+k-i)\right)n!\,(n-2)\,(n-3)\,(n-5)\,\frac{1}{24}\left(\prod_{i=2}^{5}(k+i)\right)\\
&+3\left(\prod_{i=0}^{k+1}(n+k-i)\right)(n-2)!\,(n-3)\,n\left(n^2-9n+2\right)\frac{1}{24}\left(\prod_{i=3}^{6}(k+i)\right)\\
&-\left(\prod_{i=0}^{k+2}(n+k-i)\right)(n-2)!\,(n+1)\left(n^2-13n+6\right)\frac{1}{24}\left(\prod_{i=4}^{7}(k+i)\right)\\
&-\frac{1}{2}(n+k)!\left(\prod_{i=1}^{4}(n+k+i)\right)(n+1),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\frac{24Q^{(n+k)}(0)}{\left(\prod_{i=0}^{k-1}(n+k-i)\right)} = &\; n!\,(n-1)\,(n-2)^2\,(n-3)\left(\prod_{i=1}^{4}(k+i)\right)\\
&-3n(n-1)!n\,(n-2)\,(n-3)\,(n-5)\left(\prod_{i=2}^{5}(k+i)\right)\\
&+3n(n-1)(n-2)!\,(n-3)\,n\left(n^2-9n+2\right)\left(\prod_{i=3}^{6}(k+i)\right)\\
&-n(n-1)(n-2)(n-3)!(n-2)\,(n+1)\left(n^2-13n+6\right)\left(\prod_{i=4}^{7}(k+i)\right)\\
&-12(n+1)!\left(\prod_{i=1}^{4}(n+k+i)\right),
\end{aligned}
$$

therefore

$$\begin{aligned}\frac{24Q^{(n+k)}(0)}{(n-3)!\left(\prod_{i=0}^{k-1}(n+k-i)\right)} &= n(n-1)^2(n-2)^3(n-3)\left(\prod_{i=1}^{4}(k+i)\right)\\ &\quad - 3n^2(n-1)(n-2)^2(n-3)(n-5)\left(\prod_{i=2}^{5}(k+i)\right)\\ &\quad + 3(n-2)(n-3)n^2(n-1)\left(n^2-9n+2\right)\left(\prod_{i=3}^{6}(k+i)\right)\\ &\quad - (n-2)^2(n+1)n(n-1)\left(n^2-13n+6\right)\left(\prod_{i=4}^{7}(k+i)\right)\\ &\quad - 12(n+1)n(n-1)(n-2)\left(\prod_{i=1}^{4}(n+k+i)\right),\end{aligned}$$

thus

$$\begin{aligned}\frac{24Q^{(n+k)}(0)}{(n!)\left(\prod_{i=0}^{k-1}(n+k-i)\right)} &= (n-1)(n-2)^2(n-3)\left(\prod_{i=1}^{4}(k+i)\right)\\ &\quad - 3n(n-2)(n-3)(n-5)\left(\prod_{i=2}^{5}(k+i)\right)\\ &\quad + 3(n-3)n\left(n^2-9n+2\right)\left(\prod_{i=3}^{6}(k+i)\right)\\ &\quad - (n-2)(n+1)\left(n^2-13n+6\right)\left(\prod_{i=4}^{7}(k+i)\right)\\ &\quad - 12(n+1)\left(\prod_{i=1}^{4}(n+k+i)\right)\\ &= -12(n-7-k)(n+k+4)\left(n^3+2(3k+11)n^2\right.\\ &\quad \left. + \left(k^2-7k-45\right)n+(k+5)(k+6)\right)\\ &= 12(n-7-k)(n+k+4)p(n),\end{aligned}$$

where

$$p(x) = x^3 + 2(3k+11)x^2 + \left(k^2-7k-45\right)x + (k+5)(k+6).$$

Since $k^2-7k-45>0$ for $k \geqslant 12$, p has all positive coefficients for $k \geqslant 12$, and hence no roots in $\mathbb{N}$. We now show that p has only one real root when $0 \leqslant k \leqslant 11$. $\lim_{x\to-\infty} p(x) = -\infty$ and $p(0)>0$, implies that p has a negative real root. In general, the polynomial $y = x^3 + A_1x^2 + A_2x + A_3$ has all real roots if and only if its discriminant,

$D = 18A_1A_2A_3 + A_1^2A_2^2 - 27A_3^2 - 4A_2^3 - 4A_1^3A_3$, is non-negative. The discriminant of p, after simplifying, is the polynomial in k given by $D(k) = 32k^6 - 912k^5 - 22943k^4 - 175366k^3 - 606963k^2 - 947132k - 492060$. D has one positive real root by Descarte rule of signs since there is one sign change in D. Since $D(46) < 0$ and $D(47) > 0$, $D(k) = 0$ for $46 < k < 47$. Since $D(0) < 0$ and D cannot vanish in $[0,11]$, $D(k) < 0$ for $0 \leqslant k \leqslant 11$. Hence p has only one real root when $0 \leqslant k \leqslant 11$. Since that real root is negative and $p(0) > 0$, $p(n) > 0$ for $n \in \mathbb{N}$ and $0 \leqslant k \leqslant 11$. That proves that $\dfrac{24Q^{(n+k)}(0)}{(n!)\left(\prod_{i=0}^{k-1}(n+k-i)\right)} = -12\,(n-7-k)\,(n+k+4)\,p(n) < 0$ since $k < n-7$, which in turn implies that $Q^{(n+k)}(0) < 0$. □

PROPOSITION 6. *Suppose that p is a negative integer. Then $V\left(\frac{1}{b}\right) = S(b)(b-1)^5$, where S is a polynomial with positive nonzero coefficients.*

Proof. Let $p = -n, n \in \mathbb{N}$ and let $K(b) = V\left(\frac{1}{b}\right), b > 0$. Then by (10),

$$\begin{aligned} K(b) = {} & -12(n-1) + 8n(n+1)(n+2)^2b^{2n+6} - 24n(n+1)(n+2)(n+3)b^{2n+5} \\ & +6n(n+3)\left(4n^2+16n+13\right)b^{2n+4} - 2(n+2)(n+3)\left(4n^2+12n-1\right)b^{2n+3} \\ & +(n+2)(n-1)\left(n^2+13n+6\right)b^{n+3} - 3(n+3)n\left(n^2+9n+2\right)b^{n+2} \\ & +3n(n+2)(n+3)(n+5)b^{n+1} - (n+1)(n+2)^2(n+3)b^n, \end{aligned}$$

and $S(b) = \dfrac{K(b)}{(b-1)^5}$. As in the proof of Proposition 5, with $V(b)$ replaced by $K(b)$, we have $S^{(r)}(0) = -K^{(r)}(0) - \sum_{j=0}^{r} j!\binom{r}{j}\binom{j+4}{j}K^{(r-j)}(0)$ or

$$S^{(r)}(0) = -K^{(r)}(0) - \sum_{j=1}^{r}\left(\prod_{i=0}^{j-1}(r-i)\right)\binom{j+4}{j}K^{(r-j)}(0). \tag{14}$$

It is more convenient to do $n = 1$ separately. In that case,

$$V\left(\frac{1}{b}\right) = 72b\left(2b^2+2b+1\right)(b-1)^5$$

and Proposition 6 holds. So assume now that $n \geqslant 2$.

We list the derivatives of K evaluated at $b = 0$ (simplified somewhat) and which are required for our proof.

$$\begin{aligned} K(0) &= -12(n-1); \quad K^{(i)}(0) = 0, \quad i = 1, \ldots, n-1 \\ K^{(n)}(0) &= -(n+3)!(n+2); \quad K^{(n+1)}(0) = 3(n+3)!n(n+5) \end{aligned}$$

$$\begin{aligned} K^{(n+2)}(0) &= -3(n+3)!n\left(n^2+9n+2\right) \\ K^{(n+3)}(0) &= (n+3)!(n+2)(n-1)\left(n^2+13n+6\right) \\ K^{(i)}(0) &= 0, \quad i = n+4, \dots, 2n+2. \end{aligned} \tag{15}$$

Note first that $S(0) = -K(0) > 0$, so we only need to show that $S^{(r)}(0) \geqslant 0$ for $r \geqslant 1$.

Case 1: $1 \leqslant r \leqslant n-1$

By (15), in (14), $K^{(r)}(0) = 0$ and $K^{(r-j)}(0) = 0$ for $1 \leqslant j \leqslant r-1$, so we are left with $S^{(r)}(0) = -(r!)\binom{r+4}{r}K(0) = 12(r!)\binom{r+4}{r}(n-1) > 0$

Case 2: $r = n+k$, $k = 0,1,2$

If $r = n$, then by (15) the only nonzero derivatives which appear in (14) are $-K^{(n)}(0)$ or when $j = r$, which gives

$$\begin{aligned} S^{(n)}(0) &= -K^{(n)}(0) - (n!)\binom{n+4}{n}K(0) \\ &= (n+3)!(n+2) + (n!)\binom{n+4}{n}12(n-1) > 0. \end{aligned}$$

If $r = n+1$, then by (15) the only nonzero derivatives which appear in (14) are $-K^{(n+1)}(0)$, or when $j = 1$, or $j = r$, which gives

$$\begin{aligned} S^{(n+1)}(0) &= -K^{(n+1)}(0) - 5(n+1)K^{(n)}(0) - (n+1)!\binom{n+5}{n+1}K(0) \\ &= -3(n+3)!n\,(n+5) + 5(n+3)!(n+2)(n+1) + 12(n-1)(n+1)!\binom{n+5}{n+1}, \end{aligned}$$

and thus

$$\begin{aligned} \frac{S^{(n+1)}(0)}{(n+1)!} &= -3n(n+2)(n+3)(n+5) + 5(n+1)(n+2)^2(n+3) \\ &\quad + \frac{1}{2}(n-1)(n+5)(n+4)(n+3)(n+2) \\ &= \frac{1}{2}(n+11)(n+3)(n+2)(n+1)n > 0. \end{aligned}$$

If $r = n+2$, then by (15) the only nonzero derivatives which appear in (14) are $-K^{(n+2)}(0)$, or when $j = 1$, $j = 2$, or $j = r$, which gives

$$\begin{aligned} S^{(n+2)}(0) &= -K^{(n+2)}(0) - 5(n+2)K^{(n+1)}(0) - 15(n+2)(n+1)K^{(n)}(0) \\ &\quad - (n+2)!\binom{n+6}{n+2}K(0) \\ &= 3(n+3)!n\left(n^2+9n+2\right) - 15(n+2)(n+3)!n\,(n+5) \\ &\quad + 15(n+3)!(n+2)^2(n+1) + 12(n-1)(n+2)!\binom{n+6}{n+2}, \end{aligned}$$

which implies that

$$\begin{aligned}\frac{S^{(n+2)}(0)}{3(n+3)!} &= n\left(n^2+9n+2\right) - 5n(n+2)(n+5) + 5(n+1)(n+2)^2 \\ &\quad + \frac{1}{6}(n-1)(n+6)(n+5)(n+4) \\ &= \frac{1}{6}np(n),\end{aligned}$$

where $p(x) = x^3 + 20x^2 + 53x - 2$. p has one positive real root by Descarte rule of signs. Since $p(0) < 0$ and $p(1) > 0$, that root lies in $(0,1)$. Hence $p(n) > 0$ for $n \geqslant 1$, which implies that $S^{(n+2)}(0) > 0$.

If $r = n+3$, then by (15) the only nonzero derivatives which appear in (14) are $-K^{(n+3)}(0)$, or when $j = 1$, $j = 2$, $j = 3$, or $j = r$, which gives

$$\begin{aligned}S^{(n+3)}(0) &= -K^{(n+3)}(0) - 5(n+3)K^{(n+2)}(0) - 15(n+3)(n+2)K^{(n+1)}(0) \\ &\quad - 35(n+3)(n+2)(n+1)K^{(n)}(0) - (n+3)!\binom{n+7}{n+3}K(0) \\ &= -(n+3)!(n+2)(n-1)\left(n^2+13n+6\right) + 15(n+3)(n+3)!n\left(n^2+9n+2\right) \\ &\quad - 45(n+3)(n+2)(n+3)!n(n+5) \\ &\quad + 35(n+3)(n+2)(n+1)(n+3)!(n+2) + 12\binom{n+7}{n+3}(n-1),\end{aligned}$$

which implies that

$$\begin{aligned}\frac{S^{(n+3)}(0)}{(n+3)!} &= -(n+2)(n-1)\left(n^2+13n+6\right) + 15(n+3)n\left(n^2+9n+2\right) \\ &\quad - 45(n+3)(n+2)n(n+5) + 35(n+3)(n+2)(n+1)(n+2) \\ &\quad + \frac{1}{2}(n+7)(n+6)(n+5)(n+4)(n-1) \\ &= \frac{1}{2}(n-1)(n+4)p(n),\end{aligned}$$

where $p(x) = x^3 + 26x^2 + 75x - 6$. p has one positive real root by Descarte rule of signs. Since $p(0) < 0$ and $p(1) > 0$, that root lies in $(0,1)$. Hence $p(n) > 0$ for $n \geqslant 1$, which implies that $S^{(n+3)}(0) > 0$.

Note that we only need to go up to $r = 2n+1$ since $\deg S = 2n+1$. So consider

Case 3: $r = n+k$, $k = 4, \ldots, n+1$

Note that, by (15), in (14) the only nonzero derivatives which appear in (14) (not including the 0th derivative) are when $r - j = n + l$, $l = 0,1,2,3$, or when $r - j = 0$. That gives $j = r - n - l = k - l$ for $l = 0,1,2,3$ or $j = r$. So let $j = k, k-1, k-2, k-3$

and also let $j = r = n+k$ in (14) to obtain

$$
\begin{aligned}
S^{(n+k)}(0) = &-\left(\prod_{i=0}^{k-1}(n+k-i)\right)\binom{k+4}{k}K^{(n)}(0) \\
&-\left(\prod_{i=0}^{k-2}(n+k-i)\right)\binom{k+3}{k-1}K^{(n+1)}(0)-\left(\prod_{i=0}^{k-3}(n+k-i)\right)\binom{k+2}{k-2}K^{(n+2)}(0) \\
&-\left(\prod_{i=0}^{k-4}(n+k-i)\right)\binom{k+1}{k-3}K^{(n+3)}(0)-(n+k)!\binom{n+k+4}{n+k}K(0) \\
= &\left(\prod_{i=0}^{k-1}(n+k-i)\right)\binom{k+4}{k}(n+3)!(n+2) \\
&-3\left(\prod_{i=0}^{k-2}(n+k-i)\right)\binom{k+3}{k-1}(n+3)!n(n+5) \\
&+3\left(\prod_{i=0}^{k-3}(n+k-i)\right)\binom{k+2}{k-2}(n+3)!n\left(n^2+9n+2\right) \\
&-\left(\prod_{i=0}^{k-4}(n+k-i)\right)\binom{k+1}{k-3}(n+3)!(n+2)(n-1)\left(n^2+13n+6\right) \\
&+12(n-1)(n+k)!\binom{n+k+4}{n+k},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\frac{24S^{(n+k)}(0)}{\left(\prod_{i=0}^{k-4}(n+k-i)\right)(n+3)!} = &\left(\prod_{i=k-3}^{k-1}(n+k-i)\right)\left(\prod_{i=1}^{4}(k+i)\right)(n+2) \\
&-3\left(\prod_{i=k-3}^{k-2}(n+k-i)\right)\left(\prod_{i=1}^{4}(k-1+i)\right)n(n+5) \\
&+3(n+3)\left(\prod_{i=1}^{4}(k-2+i)\right)n\left(n^2+9n+2\right) \\
&-\left(\prod_{i=1}^{4}(k-3+i)\right)(n+2)(n-1)\left(n^2+13n+6\right) \\
&+12(n-1)\left(\prod_{i=1}^{4}(n+k+i)\right) \\
= &\,12(n+2-k)(n+k+1)\left(n^3+(6k+8)n^2\right. \\
&\left.+\left(k^2+17k+15\right)n+k-k^2\right).
\end{aligned}
$$

Let $q(x) = x^3+(6k+8)x^2+\left(k^2+17k+15\right)x+k-k^2$. Since $k \geqslant 3$, q has one sign change, so one positive real root by Descarte rule of signs. Since $q(0) = k-k^2 < 0$ and

$q(1) = 24 + 24k > 0$, that positive real root lies between 0 and 1. Thus $q(n) > 0$ for $n \geqslant 1$ and that proves that $S^{(n+k)}(0) > 0$ for $k \leqslant n+1$. $\square$

Proof (of Theorem 1). Let $z_1 = x_1 + iy_1$. Then we want to prove that $\int_a^b f''''(t)(z_1 - t)^3 dt = 0$ with $a < \operatorname{Re} z_1 < b$. Theorem 1 will then follow from (3). Given $0 < a < b$, suppose that $\int_1^{b/a} f''''(t)(z_1 - t)^3 dt = 0$ where $1 < z_1 < \frac{b}{a}$. Then $\int_1^{b/a} t^s (z_1 - t)^3 dt = 0$, where $s = p(p-1)(p-2)(p-3)$. Letting $u = at$, we have $0 = \int_1^{b/a} t^s (z_1 - t)^3 dt = \frac{1}{a^{s+1}} \int_a^b u^s \left(z_1 - \frac{u}{a}\right)^3 dt = \frac{1}{a^{s+4}} \int_a^b u^s (az_1 - u)^3 du$ with $a < az_1 < b$. That shows that it suffices to prove Theorem 1 when $a = 1$, which we assume for the rest of the proof.

Suppose first that $p = n$, a positive integer, $n \notin \{1,2,3\}$. By Proposition 5, $V(b) = Q(b)(b-1)^5$, where Q is a polynomial with negative nonzero coefficients. Thus $V(b) > 0$ for $0 < b < 1$ and $V(b) < 0$ for $b > 1$. Now suppose that $p = -n$, $n \in \mathbb{N}$. By Proposition 6, $V\left(\frac{1}{b}\right) = S(b)(b-1)^5$, where S is a polynomial with positive nonzero coefficients. It follows again that $V(b) > 0$ for $0 < b < 1$ and $V(b) < 0$ for $b > 1$. Assuming now that p is an integer, $p \notin \{0,1,2,3\}$, by (10) and (11),

$$\begin{aligned}
b^{2p-3} V\left(\frac{1}{b}\right) &= 12(p+1)b^{2p-3} - 2(p-2)(p-3)\left(4p^2 - 12p - 1\right) \\
&\quad - 2(p-2)(p-3)\left(4p^2 - 12p - 1\right) \\
&\quad + 6p(p-3)\left(4p^2 - 16p + 13\right)b - 24p(p-1)(p-2)(p-3)b^2 \\
&\quad + 8p(p-1)(p-2)^2 b^3 - (p-1)(p-2)^2(p-3)b^{p-3} \\
&\quad + 3p(p-2)(p-3)(p-5)b^{p-2} \\
&\quad - 3(p-3)p\left(p^2 - 9p + 2\right)b^{p-1} + (p-2)(p+1)\left(p^2 - 13p + 6\right)b^p \\
&= W(b).
\end{aligned}$$

That is,

$$W(b) = b^{2p-3} V\left(\frac{1}{b}\right), \quad b > 0. \tag{16}$$

Using (16), it then follows immediately that $W(b) < 0$ for $0 < b < 1$ and $W(b) > 0$ for $b > 1$. Let $g(x)$ be given by (7), where A, B, C, and D are given by (8). Since $p(p-1) > 0$, $V(b) < 0$ and $W(b) > 0$ for $b > 1$ implies, using (9), that $g(1) < 0$ and $g(b) > 0$ for $b > 1$. By the Intermediate Value Theorem, $g(x_1) = 0$ for some $a < x_1 < b$. $f(z) = z^p$ clearly satisfies the hypotheses of Proposition 4, which then implies that $P_b(z_1) = P_a(z_1)$, where $y_1 \neq 0$ is given by Proposition 4 and $z_1 = x_1 + iy_1$. $\square$

REMARK 5. Theorem 1 probably holds in the more general case when $p \in \Re$, $p > 3$ or $p < 0$. If $p = \frac{n}{m}$ is rational, then a proof similar to the proofs of Proposition 5 or Proposition 6 might work to obtain a similar factorization of $V\left(b^{1/m}\right)$. After trying some of the details, it looks somewhat tedious, and a different approach might lead to proving Theorem 1 for a much larger class of functions, such as $f(z)=e^{z}$.

4. Alternate proofs and partial results

In this section we give the proof for $f(z)=z^{p}$ and $r=3$ which are not covered by Theorem 1. The results are partial because we either prove that $a<x_{1}$ for certain real values of p or that $x_{1}<b$ for certain real values of p, but not both. We also give an alternate proof of Theorem 1 when $p\in\mathbb{N}$, $p\geqslant 13$ which is somewhat different than the proof of Theorem 1 given above. First we need the following lemma.

LEMMA 3. *(i) Let* $k(x)=3\frac{r-1}{r}\frac{x^{r}-1}{x^{r-1}-1}-x$, *where* $r\geqslant\frac{3}{2}$. *Then* $k(x)>2$ *for* $x>1$

(ii) Let $l(x)=3\frac{r-1}{r}\frac{x^{r}-1}{x^{r-1}-1}-2x-1$, *where* $r<0$. *Then* $l(x)<0$ *for* $x>1$

Proof. Consider the family of means

$$E_{r,s}(x,y)=\begin{cases}\left(\frac{s}{r}\frac{x^{r}-y^{r}}{x^{s}-y^{s}}\right)^{1/(r-s)} & \text{if } r,s\neq 0,\ r\neq s,\ x\neq y\\ \left(\frac{1}{r}\frac{x^{r}-y^{r}}{\log x-\log y}\right)^{1/r} & \text{if } r\neq 0,\ s=0,\ x\neq y\\ e^{-1/r}\left(\frac{x^{x^{r}}}{y^{y^{r}}}\right)^{1/(x^{r}-y^{r})} & \text{if } s=r\neq 0,\ x\neq y\\ \sqrt{xy} & \text{if } r=s=0,\ x\neq y\\ x & \text{if } x=y\end{cases},$$

known as the Stolarsky means. It is well known ([5]) that, for fixed x and y, $E_{r,s}(x,y)$ is increasing in the parameters r and s.

To prove (i), $k(x)=3E_{r-1,r}(x,1)-x>3E_{1/2,3/2}(x,1)-x=\sqrt{x}+1>2$

To prove (ii), $l(x)=3E_{r-1,r}(x,1)-2x-1<3E_{-1,0}(x,1)-2x-1=\frac{x\ln x}{x-1}-2x-1$. From $\ln x<x-1$ we have $\frac{\ln x}{x-1}<1<2+\frac{1}{x}$ for $x>1$, which implies that $\frac{x\ln x}{x-1}<2x+1$.

We now use Lemma 3 to prove part of the conclusion of Theorem 1 for $p\in\Re$, $p\geqslant\frac{7}{2}$ or for $p\in\Re$, $p<2$, $p\neq 0,1$. $\square$

THEOREM 2. *Suppose that* $f(z)=z^{p}$, $p\in\Re$, *and let* P_{c} *denote the Taylor polynomial to* f *of order* 3 *at* $x=c$.

(i) If $p \geqslant \frac{7}{2}$*, then for any* $0 < a < b$*,* $P_b - P_a$ *has nonreal roots* $x_1 \pm iy_1$*, with* $a < x_1$*.*

(ii) If $p < 2,\ p \neq 0,1$*, then for any* $0 < a < b$*,* $P_b - P_a$ *has nonreal roots* $x_1 \pm iy_1$*, with* $x_1 < b$*.*

Proof. As in the proof of Theorem 1, we may assume that $a = 1$, so that $b > 1$. Let

$$Q(z) = \frac{6}{f'''(b) - f'''(1)}\left(P_b(z) - P_1(z)\right) = z^3 + a_1 z^2 + a_2 z + a_3,$$

where $a_1 = \dfrac{3\left(f''(b) - bf'''(b) - f''(1) + f'''(1)\right)}{f'''(b) - f'''(1)} = -3\dfrac{p-3}{p-2}\dfrac{b^{p-2}-1}{b^{p-3}-1}$. Note that Q and $P_b - P_1$ have the same roots. Write $Q(z) = (z - x_0)(z - z_1)(z - \overline{z}_1)$, where x_0 is the real root of $Q(z) = 0$ with $1 < x_0 < b$ guaranteed by Proposition 1(i) with $r = 3$. Then $x_0 + 2\operatorname{Re} z_1 = -a_1 = 3\dfrac{p-3}{p-2}\dfrac{b^{p-2}-1}{b^{p-3}-1}$, which implies that

$$\operatorname{Re} z_1 = \frac{1}{2}\left(3\frac{s-1}{s}\frac{b^s - 1}{b^{s-1}-1} - x_0\right),$$

where $s = p - 2$.

To prove (i), since $r \geqslant \frac{3}{2}$ and $x_0 < b$, $\operatorname{Re} z_1 \geqslant \frac{1}{2}\left(3\frac{s-1}{s}\frac{b^s-1}{b^{s-1}-1} - b\right) > 1$ by Lemma 3(i). Thus we have shown that $P_b(z_1) = P_1(z_1)$ with $\operatorname{Re} z_1 > 1$. To prove (ii), since $r < 0$ and $x_0 > 1$, $\operatorname{Re} z_1 \leqslant \frac{1}{2}\left(3\frac{s-1}{s}\frac{b^s-1}{b^{s-1}-1} - 1\right) < b$ by Lemma 3(ii). Thus we have shown that $P_b(z_1) = P_1(z_1)$ with $\operatorname{Re} z_1 < b$. □

We shall now give an alternate proof of Theorem 1 when $p \in N$, $p \geqslant 13$ (the cases $p \in N$, $p = 4, \ldots, 12$ can be checked directly). The method used here is somewhat different from the proof of Theorem 1 and could possibly lead to a proof for $p > 3$ in general. First we need the following lemmas.

LEMMA 4. *For any* $n \in \mathbb{N}, n \geqslant 4$*, and* $j \leqslant n - 4$

$$\sum_{k=j}^{n-4}(8k^3 + 60k^2 + 130k + 75)\binom{n}{k+4}(-1)^{k-j}\binom{k}{j} =$$
$$-\frac{1}{2}(n + j + 1)\left(n^2 - (10j + 13)n + j^2 + 5j + 6\right).$$

Proof. One can first derive formulas for $\sum_{k=j}^{n-4} k^i \binom{n}{k+4}(-1)^{k-j}\binom{k}{j}$ for $i = 0,1,2,3$. We leave the details to the reader. □

LEMMA 5. *For any* $n \in \mathbb{N}$, $n \geqslant 13$, *define the polynomial of degree* n,

$$M(x) = \sum_{k=0}^{n-4} (8k^3 + 60k^2 + 130k + 75) \binom{n}{k+4} x^{k+4}.$$

Then M *has exactly one root in the interval* $(-1,0)$.

Proof. $M(x) = x^4 N(x)$, where $N(x) = \sum_{k=0}^{n-4} (8k^3 + 60k^2 + 130k + 75) \binom{n}{k+4} x^k$, which avoids the zero of order 4 at $x = 0$. Note that the number of roots of M and N in $(-1,0)$ are identical. For $j \leqslant n-4$,

$$\begin{aligned} N^{(j)}(x) &= \sum_{k=0}^{n-4} (8k^3 + 60k^2 + 130k + 75) \binom{n}{k+4} (j!) \binom{k}{j} x^{k-j} \\ &= (j!) \sum_{k=j}^{n-4} (8k^3 + 60k^2 + 130k + 75) \binom{n}{k+4} \binom{k}{j} x^{k-j}. \end{aligned}$$

Hence $N^{(j)}(0) = (j!)(8j^3 + 60j^2 + 130j + 75) \binom{n}{j+4} > 0$, which implies that the sequence $\left\{ N^{(j)}(0) \right\}_{j=0}^{n-4}$ has 0 sign changes.

$N^{(j)}(-1) = (j!) \sum_{k=j}^{n-4} (8k^3 + 60k^2 + 130k + 75) \binom{n}{k+4} (-1)^{k-j} \binom{k}{j}$. We now show that the sequence $\left\{ N^{(j)}(-1) \right\}_{j=0}^{n-4}$ has 1 sign change. Let

$$w(j) = n^2 - 13n - 10jn + j^2 + 5j + 6 = j^2 - 5(2n-1)j + n^2 - 13n + 6.$$

Then by Lemma 4, $N^{(j)}(-1) = -\frac{1}{2}(j!)(n+j+1)w(j)$, which implies that the number of sign changes in $\left\{ N^{(j)}(-1) \right\}_{j=0}^{n-4}$ equals the number of sign changes in $\{w(j)\}_{j=0}^{n-4}$.

$w'(j) = 2j - 10n + 5 \leqslant 2(n-4) - 10n + 5 = -8n - 3 < 0$, which implies that w is decreasing for $1 \leqslant j \leqslant n-4$.

$w(1) = n^2 - 23n + 12$, which is $\begin{cases} < 0 \text{ if } 1 \leqslant n \leqslant 22 \\ > 0 \text{ if } 23 \leqslant n \end{cases}$, and $w(n-4) = -8n^2 + 24n + 2$, which is $\begin{cases} > 0 \text{ if } 1 \leqslant n \leqslant 3 \\ < 0 \text{ if } 4 \leqslant n \end{cases}$.

Case 1: $13 \leqslant n \leqslant 22$. Then $w(1) < 0$ & $w(n-4) < 0$, which implies that $w(j) < 0$ for $1 \leqslant j \leqslant n-4$. Then $N^{(j)}(-1) > 0$.

Case 2: $n \geqslant 23$. Then $w(1) > 0$ & $w(n-4) < 0$. Since w is decreasing for $1 \leqslant j \leqslant n-4$, there is a $j = j_0$ such that $w(j) > 0$ for $1 \leqslant j \leqslant j_0$ & $w(j) < 0$ for $j_0 + 1 \leqslant j \leqslant n-4$.

Then $N^{(j)}(-1)<0$ for $1\leqslant j\leqslant j_0$ & $N^{(j)}(-1)>0$ for $j_0+1\leqslant j\leqslant n-4$. Since $j=0$ gives

$$N(-1)=\sum_{k=0}^{n-4}(8k^3+60k^2+130k+75)\binom{n}{k+4}(-1)^k=-\frac{1}{2}(n+1)\left(n^2-13n+6\right)<0$$

for $n\geqslant 13$, there is one sign change in $N(-1)$, $N'(-1)$, $N''(-1),...,N^{(n-4)}(-1)$. By the Fourier Budan Theorem, N has precisely one real root in the interval $(-1,0)$. □

Alternate Proof of Theorem 1. when $p\in N$, $p\geqslant 13$: The critical step in the proof of Theorem 1 was showing that $V(b)>0$ for $0<b<1$ and $V(b)<0$ for $b>1$. The rest of the proof is exactly the same as in the proof of Theorem 1. First, to show that $V(b)<0$ for $b>1$, let $L(b)=\dfrac{V'(b)}{b^{p-4}(p-2)(p-3)}$, where V is given in (10). It follows after some computation and simplification that

$$\begin{aligned}L^{(k)}(1)=&-2(2p-3)\left(4p^2-12p-1\right)\left(\prod_{j=0}^{k-1}(p-j)\right)\\&+12p\left(4p^2-16p+13\right)\left(\prod_{j=1}^{k}(p-j)\right)-24p(p-1)(2p-5)\left(\prod_{j=2}^{k+1}(p-j)\right)\\&+16p(p-1)(p-2)\left(\prod_{j=3}^{k+2}(p-j)\right).\end{aligned}$$

Some more simplification yields $L^{(k)}(1)=-2(8k^3-36k^2+34k+3)\left(\prod_{j=0}^{k-1}(p-j)\right)$, which holds for any $p\in\Re$, $p\neq 2,3$. Assume now that $p=n\in N-\{0,1,2,3\}$. Then one can write $L(b)=-2\sum_{k=4}^{n}(8k^3-36k^2+34k+3)\binom{n}{k}(b-1)^k$, $b\in\Re$ since $L^{(k)}(1)=0$ if $k>n$. Making a change of variable in the summation yields,

$$L(b)=-2\sum_{k=0}^{n-4}C_k\binom{n}{k+4}(b-1)^{k+4}, \tag{17}$$

where $C_k=8k^3+60k^2+130k+75$. Note that for real values of p in general the series $\sum_{k=0}^{\infty}C_k\binom{p}{k}(b-1)^{k+4}$ does not converge if $|b-1|>1$, which is one of the difficulties present in using this approach for such values of p. Now

$$\begin{aligned}\frac{V'(b)}{(n-2)(n-3)}&=b^{n-4}L(b)=L(b)\left(\sum_{j=0}^{n-4}\binom{n-4}{j}(b-1)^j\right)\\&=\left(-2\sum_{k=0}^{n-4}C_k\binom{n}{k+4}(b-1)^{k+4}\right)\left(\sum_{j=0}^{n-4}\binom{n-4}{j}(b-1)^j\right)\end{aligned}$$

$$= -2\sum_{k=0}^{n-4}\sum_{j=0}^{n-4} C_k \binom{n}{k+4}\binom{n-4}{j}(b-1)^{j+k+4},$$

and thus

$$V'(b) = -2(n-2)(n-3)\sum_{k=0}^{n-4}\sum_{j=0}^{n-4} C_k \binom{n}{k+4}\binom{n-4}{j}(b-1)^{j+k+4},$$

which implies that

$$V(b) = -2(n-2)(n-3)\sum_{k=0}^{n-4}\sum_{j=0}^{n-4} C_k \binom{n}{k+4}\binom{n-4}{j}\frac{(b-1)^{j+k+5}}{j+k+5}, \tag{18}$$

which is a polynomial of degree $2n-3$ in b. Since $C_k > 0$ for $n \geqslant 4$, it follows immediately from (18) that $V(b) < 0$ for $b > 1$.

Now we show that $V(b) > 0$ for $0 < b < 1$. Since $V^{(k)}(1) = 0$, $k = 0,\dots,4$, and $V^{(5)}(1) < 0$, V is decreasing on some open interval containing $b = 1$. Since $V(0) = 12(n+1) > 0$, V must have an even number of roots in $(0,1)$, multiplicities included. If V has two or more roots in $(0,1)$, it then follows that V' also must have two or more roots in $(0,1)$. One of those roots follows from Rolle's Theorem, and the other root follows from the fact that V must have a local maximum at $t \in (0,1)$, where t is the largest root in $(0,1)$. Since $V'(b) = (n-2)(n-3)b^{n-4}L(b)$, if V has two or more roots in $(0,1)$, then L must have two or more roots in $(0,1)$. By (17), $L(b) = -2M(b-1)$, where M is the polynomial from Lemma 5. This contradicts Lemma 5, which implies that L has exactly one root in the interval $(0,1)$. Since V must have an even number of roots in $(0,1)$ and V cannot have two or more roots in $(0,1)$, V does not vanish in $(0,1)$. Since $V(0) > 0$, one has $V(b) > 0$ for $0 < b < 1$. $\square$

5. Future research

5.1. $r = 3$

It would be nice to prove Theorem 1 for a much larger class of functions than just certain powers of z. An approach along these lines might be similar to the alternate proof of Theorem 1 given above. Equivalent to (7), we have

$$g(x) = 9\left(\int_a^b f''''(t)(x-t)^2\,dt\right)\left(\int_a^b f''''(t)(x-t)\,dt\right) - \left(\int_a^b f''''(t)(x-t)^3\,dt\right)\left(\int_a^b f''''(t)dt\right).$$

Now let $h(b) = g(a)$ (this is almost identical to $V(b)$ used extensively above). We were then able to derive the following formula:

$$h^{(k)}(a) = \sum_{j=4}^{k-1} \frac{\prod_{l=0}^{j-2}(k-l)}{j(j-1)(j-4)!} \left(8j^2 - (8j+1)r + 2j - 1\right) f^{(j)}(a) f^{(k-j+3)}(a).$$

It is not too hard to show that

$$8j^2 - (8j+1)r + 2j - 1 < 0$$

for $j \geqslant 4$. One can then try to use the series expansion $h(b) = \sum_{k=0}^{\infty} h^{(r)}(a)(b-a)^k$ to determine when $h(b) > 0$ for $b > a$. Of course one has to worry about convergence of this series. If $\sum_{k=0}^{\infty} h^{(r)}(a)(b-a)^k$ does converge for all $b > a$ and if $f^{(l)}(a) f^{(m)}(a) > 0$ for all l, m, then one does obtain $h(b) < 0$ for $b > a$. This should work for $f(z) = e^z$, say. A similar (but more complicated) formula could then be derived for $s^{(k)}(a)$, where $s(b) = g(b)$ (this is almost identical to $W(b)$ used above). One would then try to show that $s(b) > 0$ for $b > a$.

Of course, as noted earlier, Theorem 1 does *not* hold for all values of p, $p \neq 0, 1, 2, 3$. It probably fails for $0 < p < 3$, $p \neq 0, 1, 2$.

It is also interesting to ask which means arise amongst the class of means given by Theorem 1. We know that the arithmetic mean arises as the real part of any of the nonreal roots of $f(z) = z^4$ (and as the real part of any of the nonreal roots of $f(z) = z^{r+1}$ for r in general by Proposition 3). However, it is not clear, even for $r = 3$, whether the geometric or harmonic means also arise in this fashion. We believe that the geometric and harmonic means do not appear, but have no proof of that fact. This contrasts with the means which arise amongst the class of means $M_f^r(a,b)$, where r is odd and $M_f^r(a,b)$ is the unique real root of $P_b - P_a$ in (a,b). In ([2], Theorem 1.1) it was proved that if $f(x) = x^{r/2}$, then $M_f^r(a,b)$ equals the geometric mean $G(a,b) = \sqrt{ab}$, and if $f(x) = x^{-1}$, then $M_f^r(a,b)$ equals the harmonic mean $H(a,b) = \dfrac{2ab}{a+b}$.

5.2. $r > 3$

First we make the following conjectures.

CONJECTURE 1. *Suppose that $f \in C^{r+1}(0,\infty)$ with $f^{(r+1)}(x) \neq 0$ on $[a,b]$, and let P_c denote the Taylor polynomial to f of order r at $x = c$. Then at least one pair of complex conjugate roots of $P_b - P_a$ has real part lying between a and b.*

CONJECTURE 2. *Suppose that $f(z) = z^n$, $n \in \mathbb{N}$, $n \geqslant r+1$, and let P_c denote the Taylor polynomial to f of order r at $x = c$. Then every pair of complex conjugate roots of $P_b - P_a$ has real part lying between a and b.*

The only part of Conjecture 1 that we have proven so far for general r is when $f(x) = x^{r+1}$. We have proven Conjecture 2 with $r = 3$ in Theorem 1 with $p \in \mathbb{N} - \{0, 1, 2, 3\}$.

Conjecture 2 does not hold in general for $f(z) = z^{-n}, n \in \mathbb{N}$. The function $f(z) = \frac{1}{z}$ seems to be a good source of examples for various values of r. This is perhaps not surprising since it was shown in [2] that the odd order Taylor polynomials to $f(z) = \frac{1}{z}$ always intersect at a point whose x coordinate is the harmonic mean $H(a,b) = \frac{2ab}{a+b}$. For the focus of this paper, if $r = 4$, then the two nonreal complex conjugate pairs of roots of $P_b - P_a$ have real parts $x_1 = \frac{1}{2}\left(5+\sqrt{5}\right)\frac{ab(a+b)}{2b^2+(1+\sqrt{5})ab+2a^2}$ and $x_2 = \frac{1}{2}\left(5-\sqrt{5}\right)\frac{ab(a+b)}{2b^2-(\sqrt{5}-1)ab+2a^2}$. Since $x_1 - a = \frac{1}{2}\left(1+\sqrt{5}\right)\frac{a\left(b-a+a\sqrt{5}\right)(b-a)}{2b^2+ab+ab\sqrt{5}+2a^2} > 0$ and $x_1 - b = -\frac{1}{2}\frac{b\left(4b+a+a\sqrt{5}\right)(b-a)}{2b^2+ab+ab\sqrt{5}+2a^2} < 0$, we have $a < x_1 < b$ and thus $m(a,b) = \frac{1}{2}\left(5+\sqrt{5}\right)\frac{ab(a+b)}{2b^2+(1+\sqrt{5})ab+2a^2}$ is a mean. However, if $a = 1$ and $b = 4$, then $x_2 = 5\frac{5-\sqrt{5}}{19-2\sqrt{5}} < 1$, and thus x_2 does *not* lie in (a,b). For $r = 5$, the two nonreal complex conjugate pairs of roots of $P_b - P_a$ have real parts $x_1 = \frac{(a+b)ab}{2(a^2-ab+b^2)}$ and $x_2 = \frac{3(a+b)ab}{2(a^2+ab+b^2)}$. In a similar fashion, one can show that x_1 does *not* lie in (a,b), while $a < x_2 < b$. Jumping to $r = 7$, one can easily show that the reals parts of two of the nonreal complex conjugate pairs of roots of $P_b - P_a$ have real parts lying in (a,b), while the third does *not* lie in (a,b).

5.3. Nonreal nodes

One can try to extend some of the results in this paper to the case where P_c is the Taylor polynomial to f of order r at $z = c$, and where c can be nonreal. For example, consider $f(z) = z^4$, $r = 3$, $a = 2+4i$, and $b = 4+2i$. A simple computation shows that $P_{2+4i}(z) - P_{4+2i}(z) = (8-8i)(-z+2+2i)(-z+3+3i)(-z+4+4i)$, so that the roots of $P_{2+4i} - P_{4+2i}$ are $z_1 = 2+2i$, $z_2 = 3+3i$, and $z_3 = 4+4i$. Note that $a \leqslant \operatorname{Re} z_j \leqslant b$ and $a \leqslant \operatorname{Im} z_j \leqslant b$ for $j = 1,2,3$, but there is not a strict inequality in each case. Also, $3+3i$ is the arithmetic mean of $a = 2+4i$ and $b = 4+2i$, something we saw for the case of real a and b.

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