Journal of
Classical
Analysis
Volume 6, Number 1 (2015), 29–37
doi:10.7153/jca-06-02

ORDER IDEAL LIMIT POINTS AND A GENERALIZED BOLZANO–THEOREM

BABLU BISWAS AND D. K. GANGULY

Abstract. In this paper we extend the concept of I-limit points and I-cluster points in a linearly ordered metric additive system and study the notion of OI-limit points and OI-cluster points and also establish a generalization of Bolzano-Weierstrass theorem connected to a bounded sequence.

1. INTRODUCTION

After the introduction of I-convergence (2000) by Kostyrko, Salat and Wilezynski [1] several authors [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], studied this concept from various aspects for the ideal I of subsets of natural numbers $\mathbb{N}$. The concepts of I-limit points and I-cluster points are introduced, those are generalization of statistical limit points [12], [13] associated to statistical convergence [14], [15], [16], [17], [18], [19]. Several authors established some results relating to the concepts of I-convergence on the real line. Our intention in this paper to extend the notion of I-limit points and I-cluster points in a linearly ordered additive system endowed with a suitable metric which is mentioned in the paper [20] in connection to the concept of order convergence [21], [22], [23].

Having the idea of OI-convergence [24] in the present paper we introduce the notion of OI-limit points and OI-cluster points analogous to I-limit points and I-cluster points and give some basic properties of these concepts. In the last part of this paper we also prove few results relating to the order ideally bounded sequence and it is also established that an order ideally bounded sequence has an order ideally convergent subsequence.

2. Definitions and notations

A partially ordered set or poset is a set P in which a binary relation $x \leqslant y$ is defined, which satisfies for all $x,y,z \in P$ the following conditions.

(i) $x \leqslant x$ for all $x \in P$,

(ii) if $x \leqslant y$ and $y \leqslant x$, then $x = y$,

(iii) if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$.

If $x \leqslant y$ and $x \neq y$, we write $x < y$. The relation $x \leqslant y$ is also written as $y \geqslant x$. Similarly, $x < y$ is also written as $y > x$.

Mathematics subject classification (2010): Primary 40A05; Secondary 46B42.

Keywords and phrases: Additive system, order convergence, natural density, ideal, I-convergence, OI-convergence.

Paper JCA-06-02

L is called an additive system, if every two elements $a,b \in L$ possess a least upper bound (l.u.b.) $a \vee b \in L$ and L is said to be a multiplicative system, if every two elements $a,b \in L$ possess a greatest lower bound (g.l.b.) $a \wedge b \in L$.

An element $\theta \in L$ is said to be the null element of L if $x \vee \theta = x$ and $x \wedge \theta = \theta$ for all $x \in L$.

If L is an additive system, we say that a sequence $\{a_i\} \in L$ is increasing (decreasing) if $a_i \leqslant a_j$ $(a_i \geqslant a_j)$ for $i < j$.

DEFINITION 2.1. [21] A sequence $\{x_n\}$ in an additive system L is said to be Order convergent (O-convergent) to $x \in L$, if there exists a sequence $\{y_n\}$ of elements of L with $y_n \downarrow \theta$ such that

$$|x_n - x| < y_n \text{, for each } n \in \mathbb{N},$$

where in L, $|x| = x^+ + x^-$ and $x^+ = x \vee \theta$, $x^- = (-x) \vee \theta$.

DEFINITION 2.2. [20] (i) Let L be an additive system and D be a real valued function defined on L. Then define a function γ on L by

$$\gamma(a,b) = 2D(a \vee b) - D(a) - D(b),\ \forall a,b \in L.$$

$D(a)$ is said to be monotone increasing (decreasing) when

$$D(a) \leqslant D(b)(D(a) \geqslant D(b)) \text{, for } a < b \text{ and } a,b \in L.$$

The function $D(a)$ is a norm if $\gamma(a,b)$ is a metric for L.

(ii) Let L be an additive system and $\gamma(a,b)$ be a real valued function defined for every $a,b \in L$; then define,

$$\Delta(a,b,c) = \tfrac{1}{2}\{\gamma(a,b) + \gamma(b,c) - \gamma(a,c)\} \text{, for } a,b,c \in L.$$

LEMMA 2.3. [20] (A) *If $D(a)$ is a real valued function defined on an additive system L, then for $a,b \in L$*

(i) $D(a) - D(b) = \gamma(a,b)$ *if* $a \geqslant b$.

(ii) *If $D(a)$ is monotone increasing, then* $|D(a) - D(b)| \leqslant \gamma(a,b)$.

(iii) $\gamma(a,b) = \gamma(b,a)$, $\gamma(a,a) = 0$.

(iv) $\Delta(a, a \vee b, b) = 0$.

(v) $D(a)$ *is monotone increasing if and only if* $\gamma(a,b) \geqslant 0$.

(vi) $D(a)$ *is properly monotone increasing if and only if* $\gamma(a,b) > 0$ *for* $a \neq b$.

(B) *If $D(a)$ is a real valued function defined on an additive system L and $\Delta(a,b,c) \geqslant 0$ for every $a,b,c \in L$, then the following statements are equivalent.*

(i) $\gamma(a \vee c, b \vee c) \leqslant \gamma(a,b)$ *for all* $a,b \in L$.

(ii) $\gamma(a \vee c, b \vee c) \leqslant \gamma(a,b)$ *for all* $b \leqslant a$.

(iii) $D(a \vee c) + D(b) \leqslant D(a) + D(c \vee b)$ *for* $b \leqslant a$.

(iv) $\gamma(a \vee c, b \vee d) \leqslant \gamma(a,b) + \gamma(c,d)$.

(C) *If $D(a)$ is monotone increasing, then $\Delta(a,b,c) \geqslant 0$ if and only if one of the equivalent statements of (B) holds.*

Note. If $D(a)$ is monotone increasing and $\Delta(a,b,c) \geqslant 0$ for $a,b,c \in L$, lemma 2.3 implies that γ is a metric on L.

DEFINITION 2.4. [25] If K is a subset of the set of positive integers $\mathbb{N}$, then the natural density of K, denoted by $\delta(K)$ is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n},$$

where $K_n = \{k \in K : k \leqslant n\}$ and $|K_n|$ is the number of elements of K_n.

DEFINITION 2.5. [14] A sequence $\{x_n\}$ of real numbers is said to be statistically convergent to some number ξ, if for any $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - \xi| \geqslant \varepsilon\}) = 0.$$

If $\{x_n\}$ is statistically convergent to ξ then we write $st - \lim_n x_n = \xi$.

We now recall the concept of ideal and filter of sets.

DEFINITION 2.6. [1] Let $X \neq \emptyset$. A family of sets $I \subseteq 2^X$ is said to be an ideal in X provided I satisfies the following conditions:

(a) $\emptyset \in I$,

(b) $A \cup B \in I$ if $A, B \in I$,

(c) If $A \in I$ and $B \subseteq A$ then $B \in I$.

DEFINITION 2.7. [1] Let X be a non-empty set. A non-empty family $F \subseteq 2^X$ is said to be a filter on X if the following conditions are satisfied:

(a) $\emptyset \notin F$,

(b) $A \cap B \in F$ if $A, B \in F$,

(c) If $A \in F$ and $A \subseteq B \subseteq X$ then $B \in F$.

An ideal I is said to be non-trivial if $I \neq \emptyset$ and $X \notin I$.

A non-trivial ideal I is said to be admissible in X if $\{x\} \in I$ for each $x \in X$.

LEMMA 2.8. [1] *I is said to be a non-trivial ideal in X if and only if the family of sets $F(I) = \{M \subseteq X : X - M \in I\}$ is a filter in X.*

It is called the filter associated with the ideal I.

DEFINITION 2.9. [1] Let I be a non-trivial ideal of subsets of $\mathbb{N}$, the set of natural numbers and (X, ρ) be a metric space. A sequence $x = \{x_n\}$ of elements of X is said to be I-convergent to $\xi \in X$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, \xi) \geqslant \varepsilon\} \in I$.

If $x = \{x_n\}$ is I-convergent to ξ then ξ is called the I-limit of the sequence x and we denote it by $I - \lim_{n\to\infty} x_n = \xi$.

DEFINITION 2.10. [1] Let I be a non-trivial ideal of subsets of $\mathbb{N}$ and (X, ρ) be a metric space. A sequence $x = \{x_n\}$ of elements of X is said to be I^*-convergent to $\xi \in X$ if there is a set $M \in F(I)$ with $M = \{m_1 < m_2 < m_3 < \ldots .\} \subseteq \mathbb{N}$ such that $\lim_{n\to\infty} \rho(x_{m_n}, \xi) = 0$.

DEFINITION 2.11. [1] Let I be a non-trivial ideal of subsets of $\mathbb{N}$ and $x=\{x_n\}$ be a sequence in a metric space (X,ρ).

(a) An element ξ is said to be an I-limit point of x provided there is a set $M=\{m_1<m_2<m_3<\ldots.\}\subseteq\mathbb{N}$ with $M\notin I$ such that $\lim_{n\to\infty}x_{m_n}=\xi$.

(b) An element ξ is said to be an I-cluster point of x if for each $\varepsilon>0$ we have $\{n\in\mathbb{N}:\rho(x_n,\xi)<\varepsilon\}\notin I$.

DEFINITION 2.12. [1] An admissible ideal I of subsets of $\mathbb{N}$ is said to have AP-property if for any sequence $\{A_1,A_2,A_3,\ldots\}$ of mutually disjoint sets of I, there is a sequence $\{B_1,B_2,B_3\ldots\}$ such that for each $i\in\mathbb{N}$ the symmetric difference $A_i\Delta B_i$ is finite and $\cup_{i=1}^{\infty}B_i\in I$.

DEFINITION 2.13. [24] Let I be a non-trivial ideal of subsets of $\mathbb{N}$ and (L,γ) be a metric additive system. A sequence $x=\{x_n\}$ of elements of L is said to be order ideal convergent to $\xi\in L$ if there is a sequence $y=\{y_n\}\in L$ with $y_n\downarrow\theta$ such that the set $A=\{n\in\mathbb{N}:\gamma(x_n,\xi)\geqslant D(y_n)\}\in I$, where D is a real valued monotone increasing function defined on L with $D(\theta)=0$ and $\Delta(a,b,c)\geqslant 0$ for all $a,b,c\in L$.

The number ξ is called the order ideal limit of the sequence $x=\{x_n\}$ and we write $OI-\lim x_n=\xi$.

Throughout the paper we consider D to be a monotone increasing real valued function with $D(\theta)=0$ and $\Delta(a,b,c)\geqslant 0$ for all $a,b,c\in L$.

LEMMA 2.14. [24] *If $x=\{x_n\}\in L$ be such that $\lim_{n\to\infty}x_n=\xi$ with respect to the metric γ then there is a sequence $\{\alpha_n\}\in L$ with $\alpha_n\downarrow\theta$ such that $\gamma(x_n,\xi)<D(\alpha_n)$, for all $n\in\mathbb{N}$.*

DEFINITION 2.15. [24] Let I be a non-trivial ideal of subsets of $\mathbb{N}$ and (L,γ) be a metric additive system. A sequence $x=\{x_n\}$ of elements of L is said to be order ideally bounded (i.e. OI-bounded) in L if there exists $B\in\mathbb{R}$ such that the set $\{n\in\mathbb{N}:D(x_n)\geqslant B\}\in I$.

3. Main results

In this section we first define OI-limit point and OI-cluster point analogous to the I-limit point and I-cluster point and investigate some properties relating to these points.

DEFINITION 3.1. Let I be a non-trivial ideal of subsets of $\mathbb{N}$ and (L,γ) be a metric additive system.

(a) An element $\xi\in L$ is said to be an OI-limit point of a sequence $x=\{x_n\}\in L$ provided there is a set $M=\{m_1<m_2<m_3<\ldots.\}\subseteq\mathbb{N}$ with $M\notin I$ and a sequence $\{y_n\}\in L$ can be choosen with $y_n\downarrow\theta$ such that $\gamma(x_{m_k},\xi)<D(y_{m_k})$, for all $k\in\mathbb{N}$.

(b) An element $\xi\in L$ is said to be an OI-cluster point of a sequence $x=\{x_n\}\in L$ provided there is a sequence $\{y_n\}\in L$ with $y_n\downarrow\theta$ such that $\{k\in\mathbb{N}:\gamma(x_k,\xi)<D(y_k)\}\notin I$.

NOTATION. We denote the set of all OI-limit points of a sequence x by $OIL(x)$ and $OIC(x)$ denotes the set of all OI-cluster points of x and $L(x)$ denotes the set of all ordinary limit points of the sequence x.

We give an example of a sequence x for which $L(x) \subseteq OSL(x)$ and $L(x) \subseteq OSC(x)$.

EXAMPLE 1. Consider $I = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$ and $L = \mathbb{R}$, the set of real numbers with D as the identity mapping, then γ be the usual metric on $\mathbb{R}$. Let

$$\begin{aligned} x_k &= 1; \text{ when } k = n^2, \; n = 1, 2, \ldots \\ &= 0; \text{ otherwise.} \end{aligned}$$

Clearly $L(x) = \{0, 1\}$. Also if $M = \{k \in \mathbb{N} : k \neq n^2\}$, $n = 1, 2, \ldots$, then $M^c \in I$, where $M^c = \mathbb{N} - M$ and $\gamma(x_n, 0) < \frac{1}{n}$ for all $n \in M$. This shows that $OIL(x) = OIC(x) = \{0\}$.

Now we determine some properties of the two sets $OIL(x)$ and $OIC(x)$.

THEOREM 3.2. *Let I be a non-trivial ideal of subsets of $\mathbb{N}$. For any sequence $x = \{x_n\}$ in L, $OIL(x) \subseteq OIC(x)$.*

Proof. Let $\lambda \in OIL(x)$. Then there exists a set $M = \{m_1 < m_2 < m_3 < \ldots.\} \subseteq \mathbb{N}$ with $M^c \in I$ and a sequence $\{y_n\} \in L$ can be choosen with $y_n \downarrow \theta$ such that $\gamma(x_{m_k}, \lambda) < D(y_{m_k})$, for all $k \in \mathbb{N}$.

$M^c \in I$ implies that $M \in F(I)$ and clearly $\{n \in \mathbb{N} : \gamma(x_n, \lambda) < D(y_n)\} \supseteq M$. Hence $\{n \in \mathbb{N} : \gamma(x_n, \lambda) < D(y_n)\} \in F(I)$ and thus $\lambda \in OCL(x)$.

Note. The converse of the above theorem is not true and it will be clear from the following example.

EXAMPLE 2. Consider $I = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$ then I is an admissible ideal and let $L = \mathbb{R}$, the set of real numbers with D as the identity mapping.

We consider the sequence $x = \{x_n\}$ as $\left\{0, 0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \ldots\right\}$.

Now for $\alpha \in [0, 1]$ let there exists a subsequence $\{x_{n_k}\}$ of x and a sequence $\{y_n\}$ of real numbers in $\mathbb{R}$ with $y_n \downarrow 0$ such that $\gamma(x_{n_k}, \alpha) < y_k$ for all $k \in \mathbb{N}$. Let $K = \{n_k : k \in \mathbb{N}\}$. Since $y_n \downarrow 0$ then for $\varepsilon > 0$ there exist $m \in \mathbb{N}$ such that $y_n < \varepsilon$ for all $n \geqslant m$.

Let $K_1 = \{n_k : k \in \mathbb{N}\} - \{1, 2, \ldots\ldots, m-1\}$.

Now $|K_1(n)| = |n_k \in K_1(n) : \gamma(x_{n_k}, \alpha) < \varepsilon| + |n_k \in K_1(n) : \gamma(x_{n_k}, \alpha) \geqslant \varepsilon|$.

$\Rightarrow \delta(K_1) \leqslant 2\varepsilon$.

Since ε is arbitrary, then $\delta(K_1) = 0$ and consequently $\delta(K) = 0$. This shows that $\alpha \notin OIL(x)$. i.e. $OIL(x) = \emptyset$.

Again let $\beta \in [0, 1]$. Then for any $\varepsilon > 0$,

$$\begin{aligned} \delta\left\{k \in \mathbb{N} : \gamma(x_k, \beta) < \frac{1}{k}\right\} &= \delta\{k \in \mathbb{N} : \gamma(x_k, \beta) < \varepsilon\} \\ &= \delta\{k \in \mathbb{N} : x_k \in (\beta - \varepsilon, \beta + \varepsilon)\} > \varepsilon > 0. \end{aligned}$$

So, for $y_k = \frac{1}{k}$, we have $\{k \in \mathbb{N} : \gamma(x_k, \beta) < D(y_k)\} \notin I$ and $OIC(x) = [0, 1]$ and thus $OSC(x) \neq OSL(x)$. $\square$

THEOREM 3.3. *If I is an admissible ideal of $\mathbb{N}$ then $OIC(x)$ is a closed set for any sequence $x=\{x_n\}$ in L.*

Proof. Let $x=\{x_n\}$ be a sequence in L and ξ be a limit point of $OIC(x)$. Consider a sequence $\{y_n\}$ in L with $y_n \downarrow \theta$.

Then for each $n \in \mathbb{N}$ there exists $\alpha_n \in OIC(x)$ such that $\alpha_n \in B(\xi, D(y_n))$, where $B(\xi, D(y_n))$ is the open ball with centre at ξ and radius $D(y_n)$.

Since $\alpha_n \in OIC(x)$ then there is a sequence $\{z_k^{(n)}\}_k$ with $z_k^{(n)} \downarrow \theta$ as $k \to \infty$ such that $A(n)=\{k \in \mathbb{N} : \gamma(x_k, \alpha_n) < D(z_k^{(n)})\} \notin I$ for $n=1,2,3\ldots$ and $A(n)$ is an infinite set.

Again $\alpha_n \in B(\xi, D(y_n))$ implies that $\gamma(\xi, \alpha_n) < D(y_n)$ and for $k \in A_n$,

$$\begin{aligned}\gamma(x_k,\xi) &\leqslant \gamma(x_k,\alpha_n)+\gamma(\alpha_n,\xi)\\ &< D(z_k^{(n)})+D(y_n);\ \text{ when } k \in A_n.\end{aligned}$$

Now $k \to \infty$ and $n \to \infty$ implies that $D(z_k^{(n)})+D(y_n) \to 0$. i.e. $\gamma(x_k,\xi) \to 0$ when $k,n \to \infty$. Using Lemma 2.14 we can choose a sequence $\{w_k\}$ in L with $w_k \downarrow \theta$ such that $\gamma(x_k,\xi) < D(w_k)$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : \gamma(x_k,\xi) < D(w_k)\} = \mathbb{N} \notin I$

So $\xi \in OIC(x)$ and hence $OIC(x)$ is closed. □

But it is the fact that the set $OIL(x)$ need not be closed and for this purpose we provide the following example.

EXAMPLE 3. Consider $I=\{A \subseteq \mathbb{N} : \delta(A)=0\}$ then I is an admissible ideal and let $L=\mathbb{R}$, the set of real numbers with D as the identity mapping. Then γ be the usual metric in $\mathbb{R}$. Let $x=\{x_n\}$ be a sequence in $\mathbb{R}$, such that, $x_n=\frac{1}{p}$ where $n=2^{p-1}(2q-1)$; p,q are positive integers.

Then for each p,

$$\begin{aligned}\delta(\{n\in\mathbb{N} : \gamma(x_n,\tfrac{1}{p})<\tfrac{1}{n}\}) &= \delta(\{n\in\mathbb{N} : |x_n-\tfrac{1}{p}|<\tfrac{1}{n}\})\\ &\geqslant \delta(\{n\in\mathbb{N} : x_n=\tfrac{1}{p}\})\\ &= 2^{-p}\\ &> 0.\end{aligned}$$

Choosing $y_n=\frac{1}{n}$ we have $\delta(\{n\in\mathbb{N} : \gamma(x_n,\frac{1}{p})<D(y_n)\})>0$. i.e. $\{n\in\mathbb{N} : \gamma(x_n,\frac{1}{p}) < D(y_n)\} \notin I$. Thus $\frac{1}{p} \in OIL(x)$ for each positive integer p.

Clearly 0 is a limit point of $OIL(x)$.

Let there is a subsequence $\{x_{n_k}\}$ of x and $\{y_k\}$ be a sequence of real number with $y_k \downarrow 0$ such that $\gamma(x_{n_k},0) < D(y_k)$ for all $k\in\mathbb{N}$ i.e. $x_{n_k} < y_k$ for all $k \in \mathbb{N}$.

Since $y_k \downarrow 0$ then $x_{n_k} \to 0$.

Let $K=\{n_k : k\in\mathbb{N}\}$. Then for each $p \in \mathbb{N}$,

$$\begin{aligned}|K_n| &= \left|\left\{k\in K_n : x_k \geqslant \frac{1}{p}\right\}\right| + \left|\left\{k\in K_n : x_k < \frac{1}{p}\right\}\right|\\ &\leqslant \left|\left\{k\in K_n : x_k \geqslant \frac{1}{p}\right\}\right| + \left|\left\{k\in \mathbb{N} : x_k < \frac{1}{p}\right\}\right| \leqslant \left|\left\{k\in K_n : x_k \geqslant \frac{1}{p}\right\}\right| + \frac{n}{2^p}.\end{aligned}$$

Thus $\delta(K) \leqslant \frac{1}{2^p}$. Since p is arbitrary then we have $\delta(K) = 0$. i.e., $K \in I$ and therefore $0 \notin OIL(x)$.

THEOREM 3.4. *Let I be an admissible ideal of subsets of $\mathbb{N}$ and $x = \{x_k\}$ and $y = \{y_k\}$ be two sequences in L such that $\{k \in \mathbb{N} : x_k \neq y_k\} \in I$. Then*
(i) $OIL(x) = OIL(y)$ and
(ii) $OIC(x) = OIC(y)$.

Proof. (i) Let $\xi \in OIL(x)$. Then there exists a set $M = \{m_1 < m_2 < m_3 < \ldots.\} \subseteq \mathbb{N}$ with $M \notin I$ and a sequence $\{\alpha_n\} \in L$ can be choosen with $\alpha_n \downarrow \theta$ such that $\gamma(x_{m_k}, \xi) < D(\alpha_{m_k})$, for all $k \in \mathbb{N}$.

Now $\{m_n \in \mathbb{N} : x_{m_n} \neq y_{m_n}\} \subseteq \{k \in \mathbb{N} : x_k \neq y_k\} \in I$. This implies $\{m_n \in \mathbb{N} : x_{m_n} = y_{m_n}\} \in F(I)$. If $\{m_n \in \mathbb{N} : x_{m_n} = y_{m_n}\} = \{p_n : n \in \mathbb{N}\}$ then, $\{p_k : k \in \mathbb{N}\} \notin I$ and $\gamma(x_{p_k}, \xi) < D(\alpha_{p_k})$, for all $k \in \mathbb{N}$. Consequently, $\gamma(y_{p_k}, \xi) < D(\alpha_{p_k})$, for all $k \in \mathbb{N}$. This shows that $\xi \in OIL(y)$ and hence $OIL(x) \subseteq OIL(y)$.

Similarly $OIL(y) \subseteq OIL(x)$ and hence $OIL(x) = OIL(y)$.

(ii) Let $\lambda \in OIC(x)$. Then there exists $\{\alpha_k\} \in L$ with $\alpha_k \downarrow \theta$ such that $A = \{k \in \mathbb{N} : \gamma(x_k, \lambda) < D(\alpha_k)\} \notin I$. Then $A \in F(I)$.

Let $B = \{k \in \mathbb{N} : x_k = y_k\} = \{k_n : n \in \mathbb{N}\}$. Clearly, $B \in F(I)$ and so $A \cap B \in F(I)$.

Consider $A \cap B = \{p_1, p_2, p_3 \ldots.\}$. Then

$$\{n \in \mathbb{N} : \gamma(x_{p_n}, \lambda) < D(\alpha_{p_n})\} \in F(I).$$

This implies that $\{n \in \mathbb{N} : \gamma(y_{p_n}, \lambda) < D(\alpha_{p_n})\} \in F(I)$.

So, $\lambda \in OIC(y)$ and $OIC(x) \subseteq OIC(y)$. By symmetry it can be shown that $OIC(y) \subseteq OIC(x)$ and hence $OIC(x) = OIC(y)$. $\square$

Several authors studied the concept of I-monotone sequence [26], [27] in real numbers. In this connection we establish few results related to OI-bounded sequence and prove the Bolzano-Weierstrass theorem on L.

THEOREM 3.5. *Let I be an admissible ideal of subsets of $\mathbb{N}$ and L be a linearly ordered additive system such that every subset of L has a supremum in L. If $\{x_n\}$ be an OI-bounded sequence in L such that the $M = \{n \in \mathbb{N} : x_{n+1} \geqslant x_n\} \notin I$ then $\{x_n\}$ is OI-convergent.*

Proof. Let $M = \{n_k : k \in \mathbb{N}\}$. Since $\{x_n\}$ is OI-bounded then there exists $B \in \mathbb{R}$ such that $\{n \in \mathbb{N} : D(x_n) \geqslant B\} \in I$ and so $\{n_k \in M : D(x_{k_n}) \geqslant B\} \subseteq \{n \in \mathbb{N} : D(x_n) \geqslant B\} \in I$.

Let $M_1 = \{p_k \in M : D(x_{p_k}) < B\}$. Then $M_1 \notin I$ and $x_{p_k} \leqslant x_{p_{k+1}}$ for all $k \in \mathbb{N}$.
Since D is monotone increasing, so $D(x_{p_1}) \leqslant D(x_{p_2}) \leqslant D(x_{p_3}) \leqslant \ldots < B$.
Here $sup_{k \in \mathbb{N}} x_{p_k} = \vee_{k \in \mathbb{N}} x_{p_k}$ exists.
Let $sup_{k \in \mathbb{N}} x_{p_k} = \alpha$. Since D is increasing then $sup_{k \in \mathbb{N}} D(x_{p_k}) = D(\alpha)$. Now,

$$\begin{aligned} \gamma(x_{p_k}, \alpha) &= 2D(x_{p_k} \vee \alpha) - D(x_{p_k}) - D(\alpha) \\ &= 2D(\alpha) - D(x_{p_k}) - D(\alpha); \textit{since } x_{p_k} \vee \alpha = \alpha \\ &= D(\alpha) - D(x_{p_k}). \end{aligned}$$

Again,

$$\begin{aligned}
x_{p_k} &\leqslant x_{p_{k+1}} \\
\Rightarrow D(x_{p_k}) &\leqslant D(x_{p_{k+1}}) \\
\Rightarrow D(\alpha) - D(x_{p_k}) &\geqslant D(\alpha) - D(x_{p_{k+1}}) \\
i.e.\ \ \gamma(x_{p_k}, \alpha) &\geqslant \gamma(x_{p_{k+1}}, \alpha).
\end{aligned}$$

Thus $\{\gamma(x_{p_k}, \alpha)\}$ is a monotone decreasing sequence of real numbers and $\gamma(x_{p_k}, \alpha) \to 0$ as $k \to \infty$.

Using Lemma 2.14 we have a sequence $\{z_n\}$ in L with $z_n \downarrow \theta$ such that $\gamma(x_{p_k}, \alpha) < D(z_{p_k})$ for all $k \in \mathbb{N}$.

Thus $\{p_k \in \mathbb{N} : \gamma(x_{p_k}, \alpha) < D(z_{p_k})\} = M_1 \in F(I)$ and $\{k \in \mathbb{N} : \gamma(x_k, \alpha) < D(z_k)\} \supseteq \{p_k \in \mathbb{N} : \gamma(x_{p_k}, \alpha) < D(z_{p_k})\}$ and hence $\{k \in \mathbb{N} : \gamma(x_k, \alpha) < D(z_k)\} \in F(I)$. Consequently, $\{k \in \mathbb{N} : \gamma(x_k, \alpha) \geqslant D(z_k)\} \in I$ and therefore $\{x_n\}$ is OI-convergent to α. □

COROLLORY 3.6. *Let I be an admissible ideal of subsets of $\mathbb{N}$ and L be a linearly ordered system such that every subset of L has a supremum in L. Then a monotone increasing OI-bounded sequence is OI-convergent.*

We state the following result in view of the Theorem 3.5

THEOREM 3.7. *Let I be an admissible ideal of subsets of $\mathbb{N}$ and L be a linearly ordered system such that every subset of L has an infimum in L. If $\{x_n\}$ is an OI-bounded sequence in L such that the $M = \{n \in \mathbb{N} : x_{n+1} \leqslant x_n\} \notin I$. Then $\{x_n\}$ is OI-convergent.*

COROLLORY 3.8. *Let Let I be an admissible ideal of subsets of $\mathbb{N}$ and L be a linearly ordered system such that every subset of L has an infimum in L. Then a monotone decreasing OI-bounded sequence is OI-convergent.*

THEOREM 3.9. *(Bolzano-Weierstrass Theorem) If I be an admissible ideal of subsets of $\mathbb{N}$ and L be a linearly ordered system such that every subset of L has an infimum as well as a supremum in L then an OI-bounded sequence has an OI-convergent subsequence.*

Proof. Let $\{x_n\}$ be an OI-bounded sequence in L. Since L is an ordered system then we can choose a monotone subsequence $\{x_{n_k}\}$ of $\{x_n\}$. $\{x_{n_k}\}$ is monotone and OI-bounded and hence it is OI-convergence. □

REFERENCES

[1] P. KOSTYRKO, T. SALAT, W. WILCZYNSKI, *I-convergence*, Real Analysis Exchange **26**, 2 (2000), 669–688.

[2] P. KOSTYRKO, M. MACAJ, T. SALAT, M. SLEZIAK, *I-convergence and External I-limit Points*, Math. Slovaca **55** (2005), 443–464.

[3] K. DEMS, *On I-Cauchy sequences*, Real Analysis Exchange **30** (2004), 123–128.

[4] F. GEZER, S. KARAKUS, *I and I^* convergent function sequences*, Math. Commun. **10** (2005), 71–80.

[5] B. C. TRIPATHY, *On I-convergent double sequence*, Soochow J. of Math. **31**, 4 (2005), 549–560.
[6] B. C. TRIPATHY, B. HAZARIKA, *I-convergent sequence spaces associated with multiplier sequence spaces*, Mathematical Inequalities and Applications **11**, 3 (2008), 543–548.
[7] B. C. TRIPATHY, S. MAHANTA, *On I-acceleration convergence of sequences*, Journal of the Franklin Institute **347** (2010), 591–598.
[8] B. C. TRIPATHY, M. SEN, S. NATH, *I-convergence in probabilistic n-normed space*, Soft Comput. **16** (2012), 1021–1027, DOI 10.1007/s00500-011-0799-8.
[9] B. C. TRIPATHY, A. J. DUTTA, *On I-acceleration convergence of sequences of fuzzy real numbers*, Math. Modell. Analysis **17**, 4 (2012), 549–557.
[10] B. C. TRIPATHY, B. SARMA, *On I-convergent double sequences of fuzzy real numbers*, Kyungpook Math. Journal **52**, 2 (2012), 189–200.
[11] T. SALAT, B. C. TRIPATHY, M. ZIMAN, *On I-convergence field*, Italian J. of Pure Appl. Math. **17** (2005), 45–54.
[12] J.A. FRIDY, *Statistical limit points*, Proc. Amer. Math. Soc. **118** (1993), 1187–1192.
[13] J. A. FRIDY, C. ORHAN, *Statistical limit superior and limit inferior*, Proc. Amer. Math. Soc. **125** (1997), 3625–3631.
[14] H. FAST, *Sur la convergence statistique*, Colloquium Matyhematicum **2** (1959), 241–244.
[15] I. J. SCHOENBERG, *The integrablity of certain functions and related summability methods*, The American mathematical Monthly **66**, 5 (1959), 361–375.
[16] H. STEINHAUS, *Sur la convergence ordinaire et la convergence asymptotique*, Colloquium Matyhematicum **2** (1951), 73–74.
[17] H.I.MILLER, *A measure theoritical subsequence characterization of statistical convergence*, Transactions of the American Mathematical Society **374**, 5 (1995), 1811–1819.
[18] J.A. FRIDY, *On statistical convergence*, Analysis **5**, 4 (1985), 301–313.
[19] T. SALAT, *On statistical convergence of real numbers*, Math. Slovaca. **30**, 2 (1980), 139–150.
[20] L. R. WILCOX AND M. F. SMILEY, *Metric Lattices*, Annals of Mathematics **40**, 2 (April 1939), 309–327.
[21] G. BIRKHOFF, *Lattice Theory*, Amer. Math. Soc. Colloq. Pubi. vol. XXV (First Edition), New York, 1940.
[22] O. FRINK, *Topology in lattices*, Trans. Amer. Math. Soc. **51** (1942), 569–582.
[23] CHARALAMBOS D. ALIPRANTIS, KIM C. BORDER, *Infinite Dimensional Analysis*, Springer Berlin Heidelberg (Third Edition), New York, 2006.
[24] D. K. GANGULY, B. BISWAS, *Order ideal convergence convergence in a metric space*, Communicated.
[25] I. NIVEN, H. S. ZUCKERMAN, *An Introduction to the Theory of Numbers*, John Wiley & Sons, New York, USA, 4-th edition, Chapter-11, 1980.
[26] B. C. TRIPATHY, B. HAZARIKA, *On I-monotonic and I-convergent sequences*, Kyungpook Math. Journal **51**, 2 (2011), 233–239.
[27] T. SALAT, B. C. TRIPATHY, M. ZIMAN, *On some properties of I-convergence*, Tatra Mt. Math. Publ. **28** (2004), 279–286.

(Received October 19, 2014)

Bablu Biswas
Department of Mathematics, P. N. Das College
Palta, India
e-mail: bablubiswas100@yahoo.com

D. K. Ganguly
Department of Pure Mathematics, University of Calcutta
Kolkata-700019, India
e-mail: gangulydk@yahoo.co.in