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SOME INEQUALITIES FOR THE VOLUME OF THE UNIT BALL

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Abstract. In the paper, the authors establish some new inequalities involving the volume of the unit ball in $\mathbb{R}^n$ and refine some results of Alzer.

1. Introduction

In the recent past, inequalities about the gamma function $\Gamma(x)$ have attracted attention of many experts. In particular, several authors established interesting monotonicity properties of the volume of the unit ball in $\mathbb{R}^n$,

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}, n = 1,2,\cdots. \tag{1.1}$$

Anderson, Vamanamurthy and Vuorinen [6] proved that $\Omega_n^{1/n}$ is strictly decreasing, and Klain and Rota [11] proved that the sequence $\frac{n\Omega_n}{\Omega_{n-1}}$ is increasing. A remarkable monotonicity theorem was published by Anderson and Qiu [5]: the sequence $\Omega_n^{1/(n\ln n)}$ decreases to $e^{-1/2}$. Guo and Qi [10] proved that for $n \geqslant 2$, the sequence $\Omega_n^{1/(n\ln n)}$ is logarithmically convex and the sequence

$$\frac{\Omega_n^{1/(n\ln n)}}{\Omega_{n+1}^{1/[(n+1)\ln(n+1)]}} \tag{1.2}$$

is decreasing.

Borgwardt [7] proved that for $n \geqslant 2$,

$$\sqrt{\frac{n}{2\pi}} \leqslant \frac{\Omega_{n-1}}{\Omega_n} \leqslant \sqrt{\frac{n+1}{2\pi}}. \tag{1.3}$$

Alzer [2] and Qiu [16] showed that for $n \geqslant 1$,

$$\sqrt{\frac{n+1/2}{2\pi}} < \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{n+\pi/2-1}{2\pi}}. \tag{1.4}$$

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This double inequality was recovered in the paper [9].

In [3], Alzer provided sharp upper and lower bounds for $\frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}}$: for all natural numbers $n \geqslant 2$,

$$\frac{\alpha*}{\sqrt{n}} < \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} < \frac{\beta*}{\sqrt{n}}, \tag{1.5}$$

where the constants $\alpha^* = \frac{3\sqrt{2}\pi}{6+4\pi} = 0.7178\cdots$ and $\beta^* = \sqrt{2\pi} = 2.5066\cdots$ are the best possible.

For more information on this topic, please refer to the papers [8, 9, 10, 13, 15, 17, 18] and closely related references therein.

The aim of this paper is to refine inequalities (1.3), (1.5), and the right hand side inequality of (1.4). In addition, we also give several new inequalities involving the volume of the unit ball in $\mathbb{R}^n$.

2. Lemmas

In order to prove the main results, we need the following lemmas.

LEMMA 2.1. ([17, p. 1178, Legendre]) *For every* $z \neq -1, -2, \cdots,$

$$2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \pi^{1/2}\Gamma(2z). \tag{2.1}$$

LEMMA 2.2. ([4, p. 383, Theorem 8]) *For every* n, *the functions*

$$F_n(x) = \ln\Gamma(x) - \left(x-\frac{1}{2}\right)\ln x + x - \frac{1}{2}\ln 2\pi - \sum_{j=1}^{2n}\frac{B_{2j}}{2j(2j-1)x^{2j-1}} \tag{2.2}$$

and

$$G_n(x) = -\ln\Gamma(x) + \left(x-\frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln 2\pi + \sum_{j=1}^{2n+1}\frac{B_{2j}}{2j(2j-1)x^{2j-1}} \tag{2.3}$$

are completely monotonic on $(0,\infty)$, *where the Bernoulli number* B_n *may be defined by*

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n\frac{x^n}{n!}, \quad |x| < 2\pi.$$

LEMMA 2.3. *The function* $h(x) = [g(x)]^2 - 2x$ *is strictly decreasing on* $[2,\infty)$, *where*

$$g(x) = \left(\frac{e}{x}\right)^x \frac{\Gamma(x+1)}{\sqrt{\pi}}. \tag{2.4}$$

Consequently, $h(\infty) < h(x) < h(2)$.

Proof. Using Lemma 2.2, we easily know $G_n > 0$ and $F_n' < 0$. Therefore, we have

$$\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x} < \exp\left[\sum_{j=1}^{2n+1} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}\right] \tag{2.5}$$

and

$$\psi(x) < \ln x - \frac{1}{2x} - \sum_{j=1}^{2n} \frac{B_{2j}}{2jx^{2j}}, \tag{2.6}$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function. Applying the above inequalities to $n = 0$ and $n = 1$ yields

$$\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x} < \exp\frac{1}{12x} \tag{2.7}$$

and

$$\psi(x) < \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4}, \tag{2.8}$$

from which we have

$$\ln g(x) < \frac{1}{2}\ln 2x + \frac{1}{12x} \tag{2.9}$$

and

$$\psi(x) - \ln x + \frac{1}{x} < \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4}. \tag{2.10}$$

Easy computation yields

$$h'(x) = 2g(x)g'(x) - 2 \tag{2.11}$$

and

$$g'(x) = g(x)\left[\psi(x) - \ln x + \frac{1}{x}\right]. \tag{2.12}$$

The requested inequality $h'(x) < 0$ can be equivalently written as

$$g'(x)g(x) < 1, \quad g^2(x)\left[\psi(x) - \ln x + \frac{1}{x}\right] < 1,$$

$$2\ln g(x) + \ln\left[\psi(x) - \ln x + \frac{1}{x}\right] < 0. \tag{2.13}$$

Taking into account inequalities (2.9) and (2.10), it suffices to show that

$$k(x) = \ln 2x + \frac{1}{6x} + \ln\left(\frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4}\right) < 0. \tag{2.14}$$

Because $k'(x) = \frac{P(x)}{6xQ(x)}$, where $P(x) = 10x^2 - 18x - 1$ and $Q(x) = 60x^4 - 10x^3 + x$, and all the coefficients of the polynomials $P(x+2)$ and $Q(x+2)$ are positive, we obtain $P(x) > 0$ and $Q(x) > 0$ for $x \in [2,\infty)$. As a result, $k'(x) > 0$ for $x \in [2,\infty)$. Therefore, $k(x)$ is strictly increasing on $x \in [2,\infty)$ with $k(\infty) = 0$, so $k(x) < 0$ on $x \in [2,\infty)$. The proof is complete. □

COROLLARY 2.1. *For $x \in [2,\infty)$, we have*

$$\sqrt{\pi}\left(\frac{x}{e}\right)^x\sqrt{2x+\alpha} < \Gamma(x+1) \leqslant \sqrt{\pi}\left(\frac{x}{e}\right)^x\sqrt{2x+\beta}, \tag{2.15}$$

where the constants $\alpha = \frac{1}{3}$ and $\beta = \frac{e^4}{4\pi} - 4$ are best possible.

REMARK 2.1. Corollary (2.1) is an improvement of [14, Theorem 1].

LEMMA 2.4. ([12, p. 390]) *Let $x_i \in (0,\infty)$ and $\sum_{i=1}^{n} x_i = nx$. Then*

$$\prod_{i=1}^{n}\Gamma(x_i) \geqslant [\Gamma(x)]^n. \tag{2.16}$$

3. Main results

In what follows, we always suppose $\beta = \frac{e^4}{4\pi} - 4$.

THEOREM 3.1. *For every integer $n > 3$, we have*

$$\frac{n+1/3}{\sqrt{\pi(2n+\beta)}} < \frac{\Omega_{n-1}}{\Omega_n} < \frac{n+\beta}{\sqrt{\pi(2n+1/3)}}. \tag{3.1}$$

Proof. It is easy to see that

$$\frac{\Omega_{n-1}}{\Omega_n} = \frac{\pi^{(n-1)/2}\Gamma(n/2+1)}{\pi^{n/2}\Gamma\big((n-1)/2+1\big)} = \frac{\Gamma\big((n+2)/2\big)}{\sqrt{\pi}\Gamma\big((n+1)/2\big)}. \tag{3.2}$$

Taking $z = \frac{n+1}{2}$ in Lemma 2.1, we obtain

$$2^n\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+2}{2}\right) = \pi^{1/2}n!. \tag{3.3}$$

Easy computation and simplification yield

$$\frac{\Omega_{n-1}}{\Omega_n} = \frac{2^n\left[\Gamma(n/2+1)\right]^2}{\pi n!}. \tag{3.4}$$

Using Corollary 2.1, we have

$$\sqrt{\pi}\left(\frac{n}{2e}\right)^{n/2}\sqrt{n+\frac{1}{3}} < \Gamma\left(\frac{n}{2}+1\right) < \sqrt{\pi}\left(\frac{n}{2e}\right)^{n/2}\sqrt{n+\beta} \tag{3.5}$$

and

$$\sqrt{\pi}\left(\frac{n}{e}\right)^n\sqrt{2n+\frac{1}{3}} < n! < \sqrt{\pi}\left(\frac{n}{e}\right)^n\sqrt{2n+\beta}. \tag{3.6}$$

Considering the left-hand side of (3.5) and the right-hand side of (3.6), we easily obtain

$$\frac{\Omega_{n-1}}{\Omega_n} > \frac{2^n \pi (n/2e)^n (n+1/3)}{\pi \sqrt{\pi} (n/e)^n \sqrt{2n+\beta}} = \frac{n+1/3}{\sqrt{\pi(2n+\beta)}}.$$

Similarly, we can also prove the right-hand side of (3.1) by the right-hand side of (3.5) and the left-hand side of (3.6). The proof is complete. □

THEOREM 3.2. *For every integer $n > 3$, we have*

$$\frac{\sqrt{\pi}(n+1)\sqrt{2n+1/3}}{(2\pi+n+1)(n+\beta)} < \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} < \frac{\sqrt{\pi}(n+1)\sqrt{2n+\beta}}{(2\pi+n+1)(n+1/3)}. \tag{3.7}$$

Proof. Using (3.3) and easy computation, we have

$$\frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} = \frac{\pi(n+1)n!}{2^n(2\pi+n+1)[\Gamma(n/2+1)]^2}. \tag{3.8}$$

Using (3.5) and (3.6), we have

$$\begin{aligned} \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} &< \frac{\pi(n+1)\left[\sqrt{\pi}(n/e)^n\sqrt{2n+\beta}\right]}{2^n(2\pi+n+1)\left[\sqrt{\pi}(n/2e)^{n/2}\sqrt{n+1/3}\right]^2} \\ &< \frac{\sqrt{\pi}(n+1)\sqrt{2n+\beta}}{(2\pi+n+1)(n+1/3)} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} &> \frac{\pi(n+1)\left[\sqrt{\pi}(n/e)^n\sqrt{2n+1/3}\right]}{2^n(2\pi+n+1)\left[\sqrt{\pi}(n/2e)^{n/2}\sqrt{n+\beta}\right]^2} \\ &> \frac{\sqrt{\pi}(n+1)\sqrt{2n+1/3}}{(2\pi+n+1)(n+\beta)}. \end{aligned} \tag{3.10}$$

The proof of Theorem 3.2 is complete. □

REMARK 3.1. It is clear that the double inequality (1.3) was thoroughly strengthened by (3.1) for $n > 3$. On the other hand, the right hand side of (3.1) is better than the right hand side of (1.4), whereas the left hand side of (3.1) and (1.4) are not included each other.

REMARK 3.2. Inequality (3.7) improves inequality (1.5).

THEOREM 3.3. *For every integer $n > 3$, we have*

$$\begin{aligned} \frac{(n+1)^2(n+3)(2n+1/3)^2}{4(n-1)(n+\beta)^4} &< \frac{\Omega_n^4}{\Omega_{n-3}\Omega_{n-1}\Omega_{n+1}\Omega_{n+3}} \\ &< \frac{(n+1)^2(n+3)(2n+\beta)^2}{4(n-1)(n+1/3)^4}. \end{aligned} \tag{3.11}$$

Proof. Setting $z = \frac{n+1}{2}, \frac{n+3}{2}, \frac{n-1}{2}, \frac{n+4}{2}$ in Lemma 2.1, we obtain (3.3) and

$$2^{n+2}\Gamma\Big(\frac{n+3}{2}\Big)\Gamma\Big(\frac{n+4}{2}\Big) = \pi^{1/2}(n+2)!, \tag{3.12}$$

$$2^{n-2}\Gamma\Big(\frac{n-1}{2}\Big)\Gamma\Big(\frac{n}{2}\Big) = \pi^{1/2}(n-2)!, \tag{3.13}$$

$$2^{n+3}\Gamma\Big(\frac{n+4}{2}\Big)\Gamma\Big(\frac{n+5}{2}\Big) = \pi^{1/2}(n+3)!. \tag{3.14}$$

Easy computation and simplification yield

$$\begin{aligned} &\frac{\Omega_n^4}{\Omega_{n-3}\Omega_{n-1}\Omega_{n+1}\Omega_{n+3}} \\ &= \frac{\Gamma\big((n-1)/2\big)\Gamma\big((n+1)/2\big)\Gamma\big((n+3)/2\big)\Gamma\big((n+5)/2\big)}{\big[\Gamma\big((n+2)/2\big)\big]^4} \\ &= \frac{\pi^2(n+1)^2(n+3)(n!)^4}{(n-1)2^{4n+2}\big[\Gamma\big((n+2)/2\big)\big]^8}, \end{aligned} \tag{3.15}$$

where we apply $\Gamma\left(\frac{n+4}{2}\right) = \frac{n+2}{2}\Gamma\left(\frac{n+2}{2}\right)$ and $\Gamma\left(\frac{n+2}{2}\right) = \frac{n}{2}\Gamma\left(\frac{n}{2}\right)$.

Using (3.5) and (3.6), we have

$$\begin{aligned} &\frac{\Omega_n^4}{\Omega_{n-3}\Omega_{n-1}\Omega_{n+1}\Omega_{n+3}} \\ &< \frac{\pi^2\left[\sqrt{\pi}\,(n/e)^n\sqrt{2n+\beta}\right]^4(n+1)^2(n+3)}{2^{4n+2}\left[\sqrt{\pi}\,(n/2e)^{n/2}\sqrt{n+1/3}\right]^8(n-1)} \\ &= \frac{(n+1)^2(n+3)(2n+\beta)^2}{4(n-1)(n+1/3)^4} \end{aligned} \tag{3.16}$$

and

$$\frac{\Omega_n^4}{\Omega_{n-3}\Omega_{n-1}\Omega_{n+1}\Omega_{n+3}} > \frac{(n+1)^2(n+3)(2n+1/3)^2}{4(n-1)(n+\beta)^4}. \tag{3.17}$$

The proof is complete. □

REMARK 3.3. Using Lemma 2.4, we can easily obtain

$$\begin{aligned} &\Gamma\Big(\frac{n-1}{2}\Big)\Gamma\Big(\frac{n+1}{2}\Big)\Gamma\Big(\frac{n+3}{2}\Big)\Gamma\Big(\frac{n+5}{2}\Big) \\ &\geqslant \Big[\Gamma\Big(\frac{(n-1)/2+(n+1)/2+(n+3)/2+(n+5)/2}{4}\Big)\Big]^4 \\ &= \Big[\Gamma\Big(\frac{n+2}{2}\Big)\Big]^4. \end{aligned}$$

Therefore, we have

$$\frac{\Omega_n^4}{\Omega_{n-3}\Omega_{n-1}\Omega_{n+1}\Omega_{n+3}} \geqslant \frac{\big[\Gamma\big((n+2)/2\big)\big]^4}{\big[\Gamma\big((n+2)/2\big)\big]^4} = 1.$$

By equalities

$$\frac{\Omega_n^3}{\Omega_{n-1}\Omega_{n+1}\Omega_{n+3}} = \frac{(n+1)^2(n+3)(n!)^3}{2^{3n+3}\left[\Gamma\left((n+2)/2\right)\right]^6} \tag{3.18}$$

and

$$(n+1)\Omega_{n+1} - \Omega_{n-1} = \left(2-\frac{1}{\pi}\right)\frac{\pi^{n/2}2^n\Gamma\left((n+2)/2\right)}{n!}, \tag{3.19}$$

completely similar to the proofs of Theorem 3.1 to Theorem 3.3, we can easily obtain the following results.

THEOREM 3.4. *For every integer $n > 3$, we have*

$$\begin{aligned}\frac{(n+1)^2(n+3)\left(\sqrt{2n+1/3}\right)^3}{8\left(\sqrt{\pi}\right)^3(n+\beta)^3} &< \frac{\Omega_n^3}{\Omega_{n-1}\Omega_{n+1}\Omega_{n+3}} \\ &< \frac{(n+1)^2(n+3)\left(\sqrt{2n+\beta}\right)^3}{8(\sqrt{\pi})^3(n+1/3)^3}.\end{aligned} \tag{3.20}$$

THEOREM 3.5. *For every integer $n > 3$, we have*

$$\begin{aligned}\left(2-\frac{1}{\pi}\right)\frac{\sqrt{n+1/3}}{\sqrt{2n+\beta}}\left(\frac{2\pi e}{n}\right)^{n/2} &< (n+1)\Omega_{n+1} - \Omega_{n-1} \\ &< \left(2-\frac{1}{\pi}\right)\frac{\sqrt{n+\beta}}{\sqrt{2n+1/3}}\left(\frac{2\pi e}{n}\right)^{n/2}.\end{aligned} \tag{3.21}$$

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