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LIPSCHITZ CONDITIONS FOR RANDOM PROCESSES FROM $L_p(\Omega)$ SPACES OF RANDOM VARIABLES

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Abstract. In this paper we study the Lipschitz continuity of random processes $X = (X(t), t \in \mathbb{T})$ from $L_p(\Omega)$ spaces, where $(\mathbb{T}, \rho)$ is some metric space, and provide estimates for the distribution of sample paths of such processes. Obtained results are used in Analysis in the study of the rate of approximation of functions by trigonometric polynomials.

1. Introduction

Let $(\mathbb{T}, \rho)$ be some metric space. We study conditions under which the sample paths of random processes $X = (X(t), t \in \mathbb{T})$ satisfy a Lipschitz condition. In particular, we consider a function f such that the following inequality holds:

$$\limsup_{\varepsilon \downarrow 0} \frac{\sup\limits_{0<\rho(t,s)\leqslant\varepsilon} |X(t)-X(s)|}{f(\varepsilon)} \leqslant 1.$$

This function is a modulus of continuity for the random process X belonging to the space $L_p(\Omega)$. The main interest for us is to estimate probabilities

$$\mathsf{P}\left\{ \sup_{0<\rho(t,s)\leqslant v} \frac{|X(t)-X(s)|}{f(\rho(t,s))} > x \right\}.$$

For Gaussian processes similar results were obtained by Dudley [5]. These results were generalized for some classes of processes belonging to the Orlitz spaces by Kozachenko [8, 9], see also Buldygin and Kozachenko [4], Zatula [13]. Lipschitz continuity of generalized sub-Gaussian processes was studied and estimates for the distribution of the norms of such processes were provided in [7].

There are applications of Lipschitz continuity of random processes to the study of the rate of approximation of functions by trigonometric polynomials. In particular, Kamenshchikova and Yanevich investigated an approximation of stochastic processes belonging to the spaces $L_p(\Omega)$ by trigonometric sums in the space $L_q[0, 2\pi]$ in [6].

Recently, various analytical properties of processes and fields that are not Gaussian, e.g. sub-Gaussian and Orlicz, were obtained. For example, the almost sure convergence of weighted sums of φ-subgaussian m-acceptable random variables was

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studied by Giuliano Antonini, Kozachenko and Volodin [2], an application to random Fourier series for φ-subgaussian random variables was investigated by Giuliano Antonini, Tien-Chung Hu, Kozachenko and Volodin [1], and necessary and sufficient conditions under which a symmetric measurable infinitely divisible process has sample paths in an Orlicz space L_ψ with a function ψ that satisfies Δ_2 condition were considered by Braverman and Samorodnitsky [3]. Krinik and Swift studied different properties of exponential Orlicz spaces and Fenchel-Orlicz spaces in the book [10]. Stochastic processes that take values in Orlicz spaces and properties of such processes were investigated by Rao and Ren [11]. Weber in [12] considered stochastic processes with value in exponential Orlicz spaces.

In all above-mentioned articles the authors consider spaces for which $\mathsf{E}|\xi|^p$ exists for each $1 \leqslant p < \infty$, but for $L_p(\Omega)$ spaces similar problems haven't been investigated, namely the task about the distribution of $\sup\limits_{0<\rho(t,s)\leqslant v} \frac{|X(t)-X(s)|}{f(\rho(t,s))}$.

The organization of the article is the following. Necessary technical results for the proof of the main theorem are introduced in Section 2. The main result (Theorem 2) is proved in Section 3. Lipschitz conditions for random processes belonging to the spaces $L_p(\Omega)$ are investigated in Section 4. Section 4 also contains an example of applying proven theorems for particular function $\sigma(h)$.

2. Definitions and technical results

Random variable ξ belongs to the space $L_p(\Omega)$, $1 \leqslant p < \infty$, if the condition

$$(\mathsf{E}|\xi|^p)^{1/p} \leqslant \infty$$

is satisfied.

It is well known that $L_p(\Omega)$, $1 \leqslant p < \infty$ is a space with the norm

$$\|\xi\|_{L_p} = (\mathsf{E}|\xi|^p)^{1/p}.$$

THEOREM 1. *Let ξ_1, ..., ξ_n be random variables belonging to the space $L_p(\Omega)$, $1 \leqslant p < \infty$. Denote $\eta = \max\limits_{1\leqslant k\leqslant n} |\xi_k|$, $a = \max\limits_{1\leqslant k\leqslant n} \|\xi_k\|_{L_p}$. Then $\forall x > 0$ the following inequality holds*

$$\mathsf{P}\left\{\eta > x\cdot a\cdot n^{2/p}\right\} \leqslant \frac{1}{nx^p}.$$

Proof. It follows from the Chebyshev inequality that for $p > 0$:

$$\mathsf{P}\{|\xi| > x\} \leqslant \frac{\|\xi\|_{L_p}^p}{x^p}. \tag{1}$$

Inequality (1) implies that

$$\begin{aligned}\mathsf{P}\left\{\eta > x\cdot a\cdot n^{2/p}\right\} &= \mathsf{E}\mathbb{1}\left\{\omega:\ \eta > x\cdot a\cdot n^{2/p}\right\}\\ &\leqslant \sum_{k=1}^{n} \mathsf{E}\mathbb{1}\{\eta = |\xi_k|\}\cdot \mathbb{1}\left\{\omega:\ |\xi_k| > x\cdot a\cdot n^{2/p}\right\}\\ &\leqslant n\cdot \max_{1\leqslant k\leqslant n} \mathsf{P}\left\{|\xi_k| > x\cdot a\cdot n^{2/p}\right\}\\ &\leqslant n\cdot \max_{1\leqslant k\leqslant n} \frac{\|\xi_k\|_{L_p}^p}{(x\cdot a\cdot n^{2/p})^p} \leqslant n\cdot \max_{1\leqslant k\leqslant n} \frac{a^p}{x^p\cdot a^p\cdot n^2} = \frac{1}{nx^p},\end{aligned}$$

which finishes the proof. □

Let $(\mathbb{T},\rho)$ be a metric space. The metric massiveness $N(u) := N_{(\mathbb{T},\rho)}(u)$ is the minimal number of closed balls (defined with respect to the metric ρ) of radius u that cover $\mathbb{T}$ [4]. Let us give some properties of the metric massiveness.

LEMMA 1. [4] *The following statements hold:*
1) For any $\varepsilon > 0$ we have $N_{(\mathbb{T},\rho)}(\varepsilon) \geqslant 1$. In this case if $\varepsilon \geqslant \operatorname{diam}\mathbb{T} = \sup_{t,s\in\mathbb{T}} \rho(t,s)$ then $N(\varepsilon) = 1$.
2) The function $N_{(\mathbb{T},\rho)}(\varepsilon)$ is right continuous and non-decreasing as ε decreases.
3) A space $\mathbb{T}$ contains a finite number of points if and only if $\sup_{\varepsilon>0} N_{(\mathbb{T},\rho)}(\varepsilon) < \infty$.

3. Theorem on moduli of continuity of random processes belonging to the spaces $L_p(\Omega)$

We now prove

THEOREM 2. *Let $(\mathbb{T},\rho)$ be a metric compact space. Consider a separable random process $X = (X(t), t\in\mathbb{T})$ belonging to the space $L_p(\Omega)$, $1\leqslant p<\infty$. Suppose that there is a monotonically increasing continuous function $\sigma = \{\sigma(h), h\geqslant 0\}$ such that $\sigma(h) > 0$ as $h > 0$, $\sigma(0) = 0$ and the following inequality holds*

$$\sup_{\rho(t,s)\leqslant h} \|X(t)-X(s)\|_{L_p} \leqslant \sigma(h). \tag{2}$$

Let $N(\varepsilon) = N_\rho(\mathbb{T},\varepsilon)$ be a metric massiveness of the space $(\mathbb{T},\rho)$. Consider $\varepsilon_0 = \sigma^{(-1)}\left(\sup_{t,s\in\mathbb{T}} \rho(t,s)\right)$, where $\sigma^{(-1)}(v)$ is the inverse function of the function $\sigma(v)$, and

$$g(\varepsilon) = \int_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{4/p} dt < \infty, \qquad \varepsilon > 0.$$

Then for $x>0$, $\varepsilon\in(0,\varepsilon_0)$ *the following inequality holds true*

$$\mathsf{P}\left\{\sup_{0<\rho(t,s)\leqslant\varepsilon}\frac{|X(t)-X(s)|}{(6+4\sqrt{2})B^{2/p}f(\rho(t,s))+(5+2\sqrt{6})B^{4/p}g(\rho(t,s))}>x\right\}$$
$$\leqslant\frac{2B^2+B}{(B^2-1)N(\varepsilon)\cdot x^p},$$

where $B>1$ *is some number,* $f(\varepsilon)=\int\limits_0^{\sigma(\varepsilon)}\left(N(\sigma^{(-1)}(t))\right)^{2/p}dt$, $\varepsilon>0$.

Proof of Theorem 2. Let $r\in(0,1)$, $\{v_k,\,k=0,1,2,\ldots\}$ be a sequence such that $v_0=\sup\limits_{t,s\in\mathbb{T}}\rho(t,s)$, $v_{k+1}=\min\{rv_k,\delta_k\}$ with

$$\delta_k=A\inf\{v:N(\sigma^{(-1)}(v))<BN(\sigma^{(-1)}(v_k))\},\tag{3}$$

where $B>1$ and A is a number such that $A>1$ and $Ar<1$. It follows from the second statement of Lemma 1 that the function $N(\sigma^{(-1)}(t))$ is nondecreasing as t decreases. Therefore, for every v_k there is a $v>v_k$ so that $N(\sigma^{(-1)}(v))<BN(\sigma^{(-1)}(v_k))$. Thus, the infimum exists. For the sequence $\{v_k,\,k=0,1,2,\ldots\}$ we have

$$v_{k+1}\leqslant rv_k,\quad k=0,1,2,\ldots\tag{4}$$

and, hence,

$$v_k\leqslant\frac{1}{1-r}(v_k-v_{k+1}).\tag{5}$$

The following inequality holds for $s<\inf\{v:N(\sigma^{(-1)}(v))<BN(\sigma^{(-1)}(v_k))\}$:

$$N(\sigma^{(-1)}(s))\geqslant BN(\sigma^{(-1)}(v_k)).$$

Thus, from (3), (4) and the last inequality we obtain that

$$N(\sigma^{(-1)}(v_{k+2}))\geqslant N(\sigma^{(-1)}(rv_{k+1}))\geqslant N(\sigma^{(-1)}(r\delta_k))\geqslant BN(\sigma^{(-1)}(v_k)).$$

Therefore

$$N(\sigma^{(-1)}(v_k))\geqslant BN(\sigma^{(-1)}(v_{k-2}))\geqslant B^2N(\sigma^{(-1)}(v_{k-4}))\geqslant\ldots\tag{6}$$

Let $\varepsilon_0=\sigma^{(-1)}(v_0)$, ..., $\varepsilon_k=\sigma^{(-1)}(v_k)$. The sequence $\{\varepsilon_k,\,k=0,1,2,\ldots\}$ is non-increasing and $\varepsilon_k\to0$ as $k\to\infty$. Let V_{ε_k}, $k=0,1,2,\ldots$ be a set of the centers of closed balls of the radius ε_k that form a minimal covering of the space $(\mathbb{T},\rho)$. The number of points in V_{ε_k} is equal to $N(\varepsilon_k)$. Denote $V_0=\bigcup\limits_{k=0}^{\infty}V_{\varepsilon_k}$. It follows from (2) using the Chebyshev inequality that the process X is continuous in probability. Therefore, the set V_0 is a set of separability of the process X. Let α_n be the mapping of the set V_0 into V_{ε_n}, where $\alpha_n(t)=t$ if $t\in V_{\varepsilon_n}$ and otherwise $\alpha_n(t)$ is a point in V_{ε_n} satisfying $\rho(t,\alpha_n(t))<\varepsilon_n$.

It follows from the Chebyshev inequality, (2) and (4) that

$$\mathsf{P}\left\{|X(t)-X(\alpha_n(t))|>r^{n/2}\right\}\leqslant\frac{\mathsf{E}(X(t)-X(\alpha_n(t)))^2}{r^n}=\frac{\|X(t)-X(\alpha_n(t))\|_{L_2}^2}{r^n}$$
$$\leqslant\frac{(\sigma(\rho(t,\alpha_n(t))))^2}{r^n}\leqslant\frac{(\sigma(\varepsilon_n))^2}{r^n}=\frac{v_n^2}{r^n}\leqslant\frac{r^{2n}v_0^2}{r^n}=v_0^2r^n.$$

Therefore

$$\sum_{n=1}^{\infty}\mathsf{P}\left\{|X(t)-X(\alpha_n(t))|>r^{n/2}\right\}<\infty.$$

Now it follows from the Borel-Cantelli lemma that $X(\alpha_n(t))\to X(t)$ with probability one as $n\to\infty$. Since the set V_0 is countable then $X(\alpha_n(t))\to X(t)$ as $n\to\infty$ with probability one $\forall t\in V_0$.

Let us prove two auxiliary results.

LEMMA 2. *Suppose that $\varepsilon\in(0,\varepsilon_0)$ and take an integer m such that the inequality $\varepsilon_{m+1}<\varepsilon\leqslant\varepsilon_m$ holds. Then*

$$\sup_{\substack{\rho(t,s)<\varepsilon\\ t,s\in\mathbb{T}}}|X(t)-X(s)|\leqslant 2\sum_{k=m+2}^{\infty}\max_{p\in V_{\varepsilon_k}}|X(p)-X(\alpha_{k-1}(p))|+$$
$$+\max_{\substack{v,w\in V_{\varepsilon_{m+1}}:\\ \|X(v)-X(w)\|_{L_p}\leqslant\sigma(\varepsilon)\frac{3-r}{1-r}}}|X(v)-X(w)|. \tag{7}$$

Proof of Lemma 2. Since V_0 is a set of separability of the process X, then with probability one

$$\sup_{\substack{\rho(t,s)<\varepsilon\\ t,s\in\mathbb{T}}}|X(t)-X(s)|=\sup_{\substack{\rho(t,s)<\varepsilon\\ t,s\in V_0}}|X(t)-X(s)|. \tag{8}$$

Let $t,s\in V_0$ and $\rho(t,s)<\varepsilon$. Let $k>m+1$. Denote $t_k=\alpha_k(t)$, $t_{k-1}=\alpha_{k-1}(t_k)$, ..., $t_m=\alpha_m(t_{m+1})$; $s_k=\alpha_k(s)$, $s_{k-1}=\alpha_{k-1}(s_k)$, ..., $s_m=\alpha_m(s_{m+1})$. Then for any t,s such that $\rho(t,s)<\varepsilon$ we obtain

$$X(t)-X(s)=(X(t)-X(t_k))+\sum_{l=m+2}^{k}(X(t_l)-X(t_{l-1}))-(X(s)-X(s_k))-$$
$$-\sum_{l=m+2}^{k}(X(s_l)-X(s_{l-1}))+(X(t_{m+1})-X(s_{m+1})). \tag{9}$$

It follows from (9) that

$$X(t_{m+1})-X(s_{m+1})=(X(t)-X(s))-(X(t)-X(t_k))+(X(s)-X(s_k))-$$
$$-\sum_{l=m+2}^{k}(X(t_l)-X(t_{l-1}))+\sum_{l=m+2}^{k}(X(s_l)-X(s_{l-1}))$$

and

$$\begin{aligned}
\|X(t_{m+1})-X(s_{m+1})\|_{L_p} &\leqslant \|X(t)-X(s)\|_{L_p}+\|X(t)-X(t_k)\|_{L_p}+\|X(s)-X(s_k)\|_{L_p}+\\
&\quad+\sum_{l=m+2}^{k}\|X(t_l)-X(t_{l-1})\|_{L_p}+\sum_{l=m+2}^{k}\|X(s_l)-X(s_{l-1})\|_{L_p}\\
&\leqslant \sigma(\rho(t,s))+\sigma(\rho(t,t_k))+\sigma(\rho(s,s_k))+\\
&\quad+\sum_{l=m+2}^{k}\sigma(\rho(t_l,t_{l-1}))+\sum_{l=m+2}^{k}\sigma(\rho(s_l,s_{l-1})) \qquad (10)\\
&\leqslant \sigma(\varepsilon)+2\sigma(\varepsilon_k)+2\sum_{l=m+2}^{k}\sigma(\varepsilon_{l-1})\leqslant \sigma(\varepsilon)+2\sum_{l=m+2}^{\infty}\sigma(\varepsilon_{l-1})\\
&=\sigma(\varepsilon)+2\sum_{l=m+2}^{\infty}\nu_{l-1}\leqslant \sigma(\varepsilon)+2\sum_{l=1}^{\infty}\nu_{m+l}\\
&\leqslant \sigma(\varepsilon)+2\sum_{l=1}^{\infty}\nu_{m+1}r^{l-1}=\sigma(\varepsilon)+\nu_{m+1}\frac{2}{1-r}\\
&\leqslant \sigma(\varepsilon)\left(1+\frac{2}{1-r}\right)=\sigma(\varepsilon)\frac{3-r}{1-r}.
\end{aligned}$$

It follows from (9) and (10) that $\forall t,s\in\mathbb{T}$ such that $\rho(t,s)<\varepsilon$ we have

$$\begin{aligned}
|X(t)-X(s)| &\leqslant \sum_{l=m+2}^{k}|X(t_l)-X(t_{l-1})|+\sum_{l=m+2}^{k}|X(s_l)-X(s_{l-1})|+|X(t)-X(t_k)|+\\
&\quad+|X(s)-X(s_k)|+|X(t_{m+1})-X(s_{m+1})|\\
&\leqslant 2\sum_{l=m+2}^{k}\max_{p\in V_{\varepsilon_l}}|X(p)-X(\alpha_{l-1}(p))|+|X(t)-X(t_k)|+|X(s)-X(s_k)|+\\
&\quad+\max_{\substack{v,w\in V_{\varepsilon_{m+1}}:\\ \|X(v)-X(w)\|_{L_p}\leqslant\sigma(\varepsilon)\frac{3-r}{1-r}}}|X(v)-X(w)|.
\end{aligned}$$

Now tending $k\to\infty$ we obtain that with probability one

$$|X(t)-X(s)|\leqslant 2\sum_{l=m+2}^{\infty}\max_{p\in V_{\varepsilon_l}}|X(p)-X(\alpha_{l-1}(p))|+\max_{\substack{v,w\in V_{\varepsilon_{m+1}}:\\ \|X(v)-X(w)\|_{L_p}\leqslant\sigma(\varepsilon)\frac{3-r}{1-r}}}|X(v)-X(w)|. \qquad (11)$$

The claim of the lemma follows from inequalities (8) and (11). □

LEMMA 3. *Suppose that* $\varepsilon\in(0,\varepsilon_0)$ *and choose the same integer* m *as in Lemma 2. Denote* $c_k=\sigma(\varepsilon_{k-1})\cdot(N(\varepsilon_k))^{2/p}$, $k=m+2,m+3,\dots$, *and let* $b_m(\varepsilon)=\frac{3-r}{1-r}\sigma(\varepsilon)\cdot(N(\varepsilon_{m+1}))^{4/p}$. *Then*

$$2\sum_{k=m+2}^{\infty}c_k+b_m(\varepsilon)\leqslant\frac{2(1+r)}{r(1-r)}B^{2/p}f(\varepsilon)+\frac{3-r}{r(1-r)}B^{4/p}g(\varepsilon), \qquad (12)$$

where $f(\varepsilon) = \int\limits_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{2/p} dt,\ g(\varepsilon) = \int\limits_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{4/p} dt.$

Proof of Lemma 3. Let us obtain a bound for the sum

$$\sum_{k=m+2}^{\infty} c_k = \sum_{k=m+2}^{\infty} \sigma(\varepsilon_{k-1}) \cdot (N(\varepsilon_k))^{2/p}.$$

Split the sum into two parts:

$$\sum_{k=m+2}^{\infty} c_k = \sum_{k=m+2}^{\infty} \nu_{k-1} \cdot (N(\varepsilon_k))^{2/p} = A_1 + A_2,$$

where

$$A_1 = \sum_{k \in D_1(m)} \nu_{k-1} \cdot (N(\varepsilon_k))^{2/p}, \qquad A_2 = \sum_{k \in D_2(m)} \nu_{k-1} \cdot (N(\varepsilon_k))^{2/p},$$

$$D_1(m) = \{k \geqslant m+2 : \nu_k = r\nu_{k-1}\}, \qquad D_2(m) = \{k \geqslant m+2 : \nu_k = \delta_{k-1}\}.$$

Inequalities (4) and (5) imply that

$$\begin{aligned} A_1 &= \frac{1}{r} \sum_{k \in D_1(m)} \nu_k \cdot \left(N(\sigma^{(-1)}(\nu_k))\right)^{2/p} \\ &\leqslant \frac{1}{r(1-r)} \sum_{k=m+2}^{\infty} (\nu_k - \nu_{k+1}) \cdot \left(N(\sigma^{(-1)}(\nu_k))\right)^{2/p} \\ &\leqslant \frac{1}{r(1-r)} \sum_{k=m+2}^{\infty} \int\limits_{\nu_{k+1}}^{\nu_k} \left(N(\sigma^{(-1)}(t))\right)^{2/p} dt \\ &= \frac{1}{r(1-r)} \int\limits_0^{\nu_{m+2}} \left(N(\sigma^{(-1)}(t))\right)^{2/p} dt. \end{aligned} \tag{13}$$

Since $N(\sigma^{(-1)}(\delta_k)) < BN(\sigma^{(-1)}(\nu_k))$, (4) and (5), we obtain

$$\begin{aligned} A_2 &= \sum_{k \in D_2(m)} \nu_{k-1} \cdot \left(N(\sigma^{(-1)}(\delta_{k-1}))\right)^{2/p} \\ &\leqslant \sum_{k \in D_2(m)} \nu_{k-1} \cdot \left(BN(\sigma^{(-1)}(\nu_{k-1}))\right)^{2/p} \\ &\leqslant \frac{1}{1-r} \sum_{k=m+2}^{\infty} (\nu_{k-1} - \nu_k) \cdot \left(BN(\sigma^{(-1)}(\nu_{k-1}))\right)^{2/p} \\ &\leqslant \frac{B^{2/p}}{1-r} \int\limits_0^{\nu_{m+1}} \left(N(\sigma^{(-1)}(t))\right)^{2/p} dt. \end{aligned} \tag{14}$$

Since $\nu_{m+2} < \nu_{m+1} < \sigma(\varepsilon)$ it follows from (13) and (14) that

$$2\sum_{k=m+2}^{\infty} c_k \leqslant \frac{2(1+r)}{r(1-r)} B^{2/p} \int_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{2/p} dt.$$

Let us estimate $b_m(\varepsilon)$. Since $\nu_{m+1} = \min\{r\nu_m, \delta_m\}$ let us consider two cases: $\nu_{m+1} = r\nu_m$ and $\nu_{m+1} = \delta_m$. If $\nu_{m+1} = r\nu_m$ then for $\varepsilon_{m+1} < \varepsilon \leqslant \varepsilon_m$ ($\nu_{m+1} < \sigma(\varepsilon) \leqslant \nu_m$):

$$\begin{aligned}
\sigma(\varepsilon)\cdot\left(N(\sigma^{(-1)}(\nu_{m+1}))\right)^{4/p} &= \sigma(\varepsilon)\cdot\left(N(\sigma^{(-1)}(r\nu_m))\right)^{4/p} \\
&\leqslant \sigma(\varepsilon)\cdot\left(N(\sigma^{(-1)}(r\sigma(\varepsilon)))\right)^{4/p} \\
&\leqslant \int_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(rv))\right)^{4/p} dv \\
&= \frac{1}{r}\int_0^{r\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{4/p} dt \\
&\leqslant \frac{1}{r}\int_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{4/p} dt.
\end{aligned}$$

If $\nu_{m+1} = \delta_m$ then it follows from (3)

$$\begin{aligned}
\sigma(\varepsilon)\cdot\left(N(\sigma^{(-1)}(\nu_{m+1}))\right)^{4/p} &= \sigma(\varepsilon)\cdot\left(N(\sigma^{(-1)}(\delta_m))\right)^{4/p} \\
&\leqslant \sigma(\varepsilon)\cdot\left(BN(\sigma^{(-1)}(\nu_m))\right)^{4/p} \\
&\leqslant \sigma(\varepsilon)\cdot\left(BN(\sigma^{(-1)}(\sigma(\varepsilon)))\right)^{4/p} \\
&\leqslant B^{4/p}\int_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{4/p} dt.
\end{aligned}$$

Therefore

$$b_m(\varepsilon) \leqslant \frac{3-r}{r(1-r)} B^{4/p} \int_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{4/p} dt.$$

Thus, we have the following estimation

$$2\sum_{k=m+2}^{\infty} c_k + b_m(\varepsilon) \leqslant \frac{2(1+r)}{r(1-r)} B^{2/p} \int_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{2/p} dt + \\ + \frac{3-r}{r(1-r)} B^{4/p} \int_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{4/p} dt,$$

which is the desired statement. □

Let us continue the proof of Theorem 2. Consider $0 < \varepsilon \leqslant \varepsilon_0$ and choose the same integer m as in Lemma 2.

Denote

$$\xi_k = \max_{t \in V_{\varepsilon_k}} |X(t) - X(\alpha_{k-1}(t))|,$$

$$\eta_m(\varepsilon) = \max_{\substack{v,w \in V_{\varepsilon_{m+1}}: \\ \|X(v)-X(w)\|_{L_p} \leqslant \sigma(\varepsilon)\frac{3-r}{1-r}}} |X(v) - X(w)|.$$

Lemma 2 implies that the inequality

$$\sup_{\substack{\rho(t,s)<y \\ t,s\in\mathbb{T}}} |X(t) - X(s)| \leqslant 2\sum_{k=m+2}^{\infty} \xi_k + \eta_m(y) \tag{15}$$

holds with probability one for any $\varepsilon_{m+1} < y \leqslant \varepsilon_m$. Let

$$G(y) = \frac{2(1+r)}{r(1-r)} B^{2/p} \int_0^{\sigma(y)} \left(N(\sigma^{(-1)}(t))\right)^{2/p} dt + \frac{3-r}{r(1-r)} B^{4/p} \int_0^{\sigma(y)} \left(N(\sigma^{(-1)}(t))\right)^{4/p} dt.$$

It follows from (15) and Lemma 3 that

$$\sup_{0<\rho(t,s)\leqslant\varepsilon} \frac{|X(t)-X(s)|}{G(\rho(t,s))} \leqslant \sup_{0<y\leqslant\varepsilon} \left[\sup_{0<\rho(t,s)\leqslant y} \frac{|X(t)-X(s)|}{G(y)}\right] \\ \leqslant \sup_{l\geqslant m+1} \sup_{\varepsilon_{l+1}<y\leqslant\varepsilon_l} \frac{2\sum_{p=l+1}^{\infty} \xi_p + \eta_l(y)}{2\sum_{p=l+1}^{\infty} c_p + b_l(y)}.$$

Therefore, for any $x > 0$ the following inequality holds:

$$\mathsf{P}\left\{\sup_{0<\rho(t,s)\leqslant\varepsilon} \frac{|X(t)-X(s)|}{\frac{1}{r(1-r)}(2(1+r)B^{2/p}f(\rho(t,s)) + (3-r)B^{4/p}g(\rho(t,s)))} > x\right\} \\ \leqslant \sum_{k=m+2}^{\infty} \mathsf{P}\left\{\frac{\xi_k}{c_k} > x\right\} + \sum_{l=m+1}^{\infty} \mathsf{P}\left\{\sup_{\varepsilon_{l+1}<v\leqslant\varepsilon_l} \frac{\eta_l(v)}{b_l(v)} > x\right\}, \tag{16}$$

where $f(\varepsilon)=\int\limits_0^{\sigma(\varepsilon)}\left(N(\sigma^{(-1)}(t))\right)^{2/p}dt,\ g(\varepsilon)=\int\limits_0^{\sigma(\varepsilon)}\left(N(\sigma^{(-1)}(t))\right)^{4/p}dt.$

Evaluate the probabilities in (16). It follows from Theorem 1 that $\forall x>0$:

$$\mathsf{P}\left\{\frac{\xi_k}{c_k}>x\right\}=\mathsf{P}\left\{\max_{t\in V_{\varepsilon_k}}|X(t)-X(\alpha_{k-1}(t))|>x\cdot\sigma(\varepsilon_{k-1})\cdot(N(\varepsilon_k))^{2/p}\right\}\leqslant\frac{1}{N(\varepsilon_k)\cdot x^p}; \tag{17}$$

$$\mathsf{P}\left\{\sup_{\varepsilon_{l+1}<v\leqslant\varepsilon_l}\frac{\eta_l(v)}{b_l(v)}>x\right\}=\mathsf{P}\left\{\sup_{\varepsilon_{l+1}<v\leqslant\varepsilon_l}\frac{\max\limits_{\substack{v,w\in V_{\varepsilon_{l+1}}:\\ \|X(v)-X(w)\|_{L_p}\leqslant\sigma(\varepsilon)\frac{3-r}{1-r}}}|X(v)-X(w)|}{\frac{3-r}{1-r}\sigma(\varepsilon)\cdot(N(\varepsilon_{l+1}))^{4/p}}>x\right\}$$

$$=\mathsf{P}\left\{\max_{\substack{v,w\in V_{\varepsilon_{l+1}}:\\ \|X(v)-X(w)\|_{L_p}\leqslant\sigma(\varepsilon)\frac{3-r}{1-r}}}|X(v)-X(w)|>x\cdot\frac{3-r}{1-r}\sigma(\varepsilon)\cdot(N(\varepsilon_{l+1}))^{4/p}\right\}$$
$$\leqslant\frac{1}{N^2(\varepsilon_{l+1})\cdot x^p}. \tag{18}$$

Finally, we obtain

$$\mathsf{P}\left\{\sup_{0<\rho(t,s)\leqslant\varepsilon}\frac{|X(t)-X(s)|}{\frac{1}{r(1-r)}(2(1+r)B^{2/p}f(\rho(t,s))+(3-r)B^{4/p}g(\rho(t,s)))}>x\right\}$$
$$\leqslant\sum_{k=m+2}^{\infty}\frac{1}{N(\varepsilon_k)}\cdot\frac{1}{x^p}+\sum_{l=m+1}^{\infty}\frac{1}{N^2(\varepsilon_{l+1})}\cdot\frac{1}{x^p}:=\frac{R(m)}{x^p},$$

where $f(\varepsilon)=\int\limits_0^{\sigma(\varepsilon)}\left(N(\sigma^{(-1)}(t))\right)^{2/p}dt,\ g(\varepsilon)=\int\limits_0^{\sigma(\varepsilon)}\left(N(\sigma^{(-1)}(t))\right)^{4/p}dt.$

Inequalities (6) imply that

$$\begin{aligned}R(m)&=\sum_{k=m+2}^{\infty}\frac{1}{N(\varepsilon_k)}+\sum_{l=m+1}^{\infty}\frac{1}{N^2(\varepsilon_{l+1})}\leqslant\frac{1}{N(\varepsilon_{m+2})}\sum_{p=0}^{\infty}\frac{1}{B^p}+\frac{1}{N^2(\varepsilon_{m+2})}\sum_{p=0}^{\infty}\frac{1}{B^{2p}}\\&=\frac{B}{(B-1)N(\varepsilon_{m+2})}+\frac{B^2}{(B^2-1)N^2(\varepsilon_{m+2})}\leqslant\frac{1}{N(\varepsilon)}\cdot\left(\frac{B}{B-1}+\frac{B^2}{B^2-1}\right)\\&=\frac{1}{N(\varepsilon)}\cdot\frac{2B^2+B}{B^2-1}.\end{aligned}$$

Since $\inf\limits_{0<r<1}\frac{2(1+r)}{r(1-r)}=6+4\sqrt{2}$ and $\inf\limits_{0<r<1}\frac{3-r}{r(1-r)}=5+2\sqrt{6}$ then for $x>0,\ \varepsilon\in(0,\varepsilon_0)$:

$$\mathsf{P}\left\{\sup_{0<\rho(t,s)\leqslant\varepsilon}\frac{|X(t)-X(s)|}{(6+4\sqrt{2})B^{2/p}f(\rho(t,s))+(5+2\sqrt{6})B^{4/p}g(\rho(t,s))}>x\right\}\leqslant\frac{2B^2+B}{(B^2-1)N(\varepsilon)\cdot x^p}.$$

The proof of Theorem 2 is completed. $\square$

Since $B^{4/p} > B^{2/p}$ for $B > 1$, $1 \leqslant p < \infty$, the following inequality holds for $x > 0$, $\varepsilon \in (0, \varepsilon_0)$:

$$\mathsf{P}\left\{ \sup_{0<\rho(t,s)\leqslant\varepsilon} \frac{|X(t)-X(s)|}{(6+4\sqrt{2})f(\rho(t,s))+(5+2\sqrt{6})g(\rho(t,s))} > B^{4/p}x \right\} \leqslant \frac{2B^2+B}{(B^2-1)N(\varepsilon)\cdot x^p}.$$

Denote $y = B^{4/p}x$. Minimizing the function $f(B) = \frac{(2B^2+B)B^4}{B^2-1}$, $B > 1$, we get approximately 18.977, which is $\approx f(B_0)$, where

$$B_0 = \frac{\sqrt{33}}{4}\cos\left(\frac{1}{3}\arctan\left(\frac{8\sqrt{41}}{37}\right)\right) - \frac{1}{8} \approx 1.24044.$$

Then the following corollary follows from Theorem 2.

COROLLARY 1. *Let the assumptions of Theorem 2 hold true. Then for $y > 0$, $\varepsilon \in (0, \varepsilon_0)$ the following inequality holds:*

$$\mathsf{P}\left\{ \sup_{0<\rho(t,s)\leqslant\varepsilon} \frac{|X(t)-X(s)|}{(6+4\sqrt{2})f(\rho(t,s))+(5+2\sqrt{6})g(\rho(t,s))} > y \right\} \leqslant \frac{C}{N(\varepsilon)\cdot y^p},$$

where $f(\varepsilon) = \int_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{2/p} dt$, $g(\varepsilon) = \int_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{4/p} dt$, $C \approx 18.977$.

4. Lipschitz conditions for random processes belonging to the spaces $L_p(\Omega)$

We prove

THEOREM 3. *Let the assumptions of Theorem 2 hold true. Then with probability one*

$$\limsup_{\varepsilon\downarrow 0} \frac{\Delta(X;\varepsilon)}{(6+4\sqrt{2})B^{2/p}f(\varepsilon)+(5+2\sqrt{6})B^{4/p}g(\varepsilon)} \leqslant 1,$$

where

$$\Delta(X;\varepsilon) = \sup_{\substack{t,s\in\mathbb{T} \\ 0<\rho(t,s)\leqslant\varepsilon}} |X(t)-X(s)|,$$

$$f(\varepsilon) = \int_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{2/p} dt < \infty,\ g(\varepsilon) = \int_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{4/p} dt,\ \varepsilon > 0.$$

Proof. It follows from (7) that with probability one

$$\sup_{\substack{\rho(t,s)<v \\ t,s\in\mathbb{T}}} |X(t)-X(s)| \leqslant 2\sum_{l=m+2}^{\infty} \xi_l + \eta_m(v). \tag{19}$$

It follows from (17) that for a sufficiently large l and for $x>0$: $\xi_l \leqslant xc_l$ with probability one. From (18) we have that for a sufficiently large m and for $x>0$: $\eta_m(v) \leqslant xb_m(v)$ with probability one. Therefore, for a sufficiently large l (or small enough ε) and for $x>0$ we have

$$\sup_{\substack{\rho(t,s)<v \\ t,s\in\mathbb{T}}} |X(t)-X(s)| \leqslant x\left(2\sum_{l=m+2}^{\infty} c_l + b_m(v)\right) \tag{20}$$

with probability one.

Now it follows from (12) that for a sufficiently small v

$$\sup_{\substack{\rho(t,s)\leqslant v \\ t,s\in\mathbb{T}}} |X(t)-X(s)| \leqslant (6+4\sqrt{2})B^{2/p}f(v)+(5+2\sqrt{6})B^{4/p}g(v)$$

with probability one. □

The following corollary follows from Theorem 3.

COROLLARY 2. *For a small enough v the following inequality holds*

$$\sup_{\rho(t,s)\leqslant v} |X(t)-X(s)| \leqslant (6+4\sqrt{2})B^{2/p}f(v)+(5+2\sqrt{6})B^{4/p}g(v)$$

with probability one.

EXAMPLE 1. *Let $\sigma(h)=dh^c$, $h,c,d>0$.*

The inverse function of the function $\sigma(h)$ is $\sigma^{(-1)}(h)=\sqrt[c]{\frac{h}{d}}$. Therefore, functions $f(\varepsilon)$ and $g(\varepsilon)$ take the following form:

$$f(\varepsilon)=\int_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{2/p} dt = \int_0^{d\varepsilon^c} \left(N\left(\sqrt[c]{\frac{t}{d}}\right)\right)^{2/p} dt;$$

$$g(\varepsilon)=\int_0^{\sigma(\varepsilon)} \left(N(\sigma^{(-1)}(t))\right)^{4/p} dt = \int_0^{d\varepsilon^c} \left(N\left(\sqrt[c]{\frac{t}{d}}\right)\right)^{4/p} dt.$$

In accordance with Theorem 2, for $\varepsilon\in(0,\varepsilon_0)$, $\varepsilon_0=\sigma^{(-1)}\left(\sup_{t,s\in\mathbb{T}}\rho(t,s)\right)$, $B>1$ and $\forall x>0$ the following inequality holds:

$$\mathsf{P}\left\{\sup_{0<\rho(t,s)\leqslant\varepsilon} \frac{|X(t)-X(s)|}{\gamma_B(\rho(t,s))} > x\right\} \leqslant \frac{2B^2+B}{(B^2-1)N(\varepsilon)\cdot x^p},$$

where

$$\gamma_B(\varepsilon)=(6+4\sqrt{2})B^{2/p}\int\limits_0^{d\varepsilon^c}\left(N\left(\sqrt[c]{\frac{t}{d}}\right)\right)^{2/p}dt+(5+2\sqrt{6})B^{4/p}\int\limits_0^{d\varepsilon^c}\left(N\left(\sqrt[c]{\frac{t}{d}}\right)\right)^{4/p}dt.$$

Moreover, according to Theorem 3, with probability one the following holds:

$$\limsup_{\varepsilon\downarrow 0}\frac{\sup\limits_{0<\rho(t,s)\leqslant\varepsilon}|X(t)-X(s)|}{\gamma_B(\varepsilon)}\leqslant 1.$$

Now let consider a space $\mathbb{T}=[0,T]$. Since the metric massiveness $N(u)$ denotes the minimal number of elements in an u–covering of the set $\mathbb{T}$ (in this case, the segment $[0,T]$), then $\frac{T}{2u}\leqslant N(u)\leqslant\frac{T}{2u}+1$. It implies that for the function $\sigma^{(-1)}(u)$:

$$N\left(\sqrt[c]{\frac{u}{d}}\right)=N\left(\sigma^{(-1)}(u)\right)\leqslant\frac{T}{2\sigma^{(-1)}(u)}+1=\frac{T}{2\sqrt[c]{\frac{u}{d}}}+1=\frac{T}{2}\sqrt[c]{\frac{d}{u}}+1.$$

Therefore we evaluate functions $f(\varepsilon)$ and $g(\varepsilon)$:

$$f(\varepsilon)=\int\limits_0^{d\varepsilon^c}\left(N\left(\sqrt[c]{\frac{t}{d}}\right)\right)^{2/p}dt\leqslant\int\limits_0^{d\varepsilon^c}\left(\frac{T}{2}\sqrt[c]{\frac{d}{t}}+1\right)^{2/p}dt;$$

$$g(\varepsilon)=\int\limits_0^{d\varepsilon^c}\left(N\left(\sqrt[c]{\frac{t}{d}}\right)\right)^{4/p}dt\leqslant\int\limits_0^{d\varepsilon^c}\left(\frac{T}{2}\sqrt[c]{\frac{d}{t}}+1\right)^{4/p}dt.$$

The last integral is finite if $c>\frac{4}{p}$. In accordance with Theorem 2, for $\varepsilon\in(0,\varepsilon_0)$, $B>1$, $c>\frac{4}{p}$ and $\forall x>0$ the following inequality holds

$$\mathsf{P}\left\{\sup_{0<\rho(t,s)\leqslant\varepsilon}\frac{|X(t)-X(s)|}{\gamma_{1,B}(\rho(t,s))}>x\right\}\leqslant\frac{2\varepsilon(2B^2+B)}{T(B^2-1)\cdot x^p},$$

where

$$\gamma_{1,B}(\varepsilon)=(6+4\sqrt{2})B^{2/p}\int\limits_0^{d\varepsilon^c}\left(\frac{T}{2}\sqrt[c]{\frac{d}{t}}+1\right)^{2/p}dt+(5+2\sqrt{6})B^{4/p}\int\limits_0^{d\varepsilon^c}\left(\frac{T}{2}\sqrt[c]{\frac{d}{t}}+1\right)^{4/p}dt.$$

Moreover, according to Theorem 3, the following holds

$$\limsup_{\varepsilon\downarrow 0}\frac{\sup\limits_{0<\rho(t,s)\leqslant\varepsilon}|X(t)-X(s)|}{\gamma_{1,B}(\varepsilon)}\leqslant 1$$

with probability one.

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