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TIGHT FRAMELET PACKETS ON LOCAL FIELDS OF POSITIVE CHARACTERISTIC

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Abstract. An important tool for the construction of tight wavelet frames on local fields of positive characteristic with the help of unitary extension principles was presented by Shah and Debnath [Tight wavelet frames on local fields, *Analysis*, **33** (2013), 293–307]. In this paper, we continue the study based on the extension principles and give an explicit construction of a class of tight framelet packets on local fields of positive characteristic.

1. Introduction

A field K equipped with a topology is called a *local field* if both the additive K^+ and multiplicative groups K^* of K are locally compact Abelian groups. The local fields are essentially of two types: zero and positive characteristic (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). Examples of local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$ where as local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and multiresolution analysis theory are quite different. In recent years, local fields have attracted the attention of several mathematicians, and have found innumerable applications not only to number theory but also to representation theory, division algebras, quadratic forms and algebraic geometry. As a result, local fields are now consolidated as part of the standard repertoire of contemporary mathematics. For more about local fields and their applications, we refer to the monographs [14, 25].

In recent years, there has been a considerable interest in the problem of constructing wavelet bases on various spaces other than $\mathbb{R}$, such as abstract Hilbert spaces [24], locally compact Abelian groups [7], Cantor dyadic groups [10], p-adic fields [9], zero-dimensional groups [12] and Vilenkin groups [13]. Recently, R. L. Benedetto and J. J. Benedetto [2] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets. The concept of multiresolution analysis on a local field K of positive characteristic was introduced by Jiang *et al.* [8]. They pointed out a method for constructing orthogonal wavelets on local field K with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(K)$. Subsequently, the tight wavelet

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frames on local fields of positive characteristic were constructed by Shah and Debnath [22] using extension principles. Recently, Shah and Abdullah [18] have introduced the notion of non-uniform multiresolution analysis on local field K of positive characteristic and obtained the necessary and sufficient condition for a function φ to generate a non-uniform multiresolution analysis on local fields. More results in this direction can also be found in [6, 19, 20] and in the references therein.

The traditional wavelet frames provide poor frequency localization in many applications as they are not suitable for signals whose domain frequency channels are focused only on the middle frequency region. Therefore, in order to make more kinds of signals suited for analyzing by wavelet frames, it is necessary to extend the concept of wavelet frames to a library of wavelet frames, called *framelet packets* or *wavelet frame packets*. The original idea of framelet packets was introduced by Coifman *et al.* [5] to provide more efficient decomposition of signals containing both transient and stationary components. Well known Daubechies orthogonal wavelets are a special of wavelet packets. Chui and Li [4] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be employed to the spline wavelets and so on. Shen [23] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. The construction of wavelet packets and wavelet frame packets on local fields of positive characteristic were recently reported by Behera and Jahan in [1]. They proved lemmas on the so-called splitting trick and several theorems concerning the Fourier transform of the wavelet packets and the construction of wavelet packets to show that their translates form an orthonormal basis of $L^2(K)$. Other notable generalizations are the wavelet packets and framelet packets on a positive half-line $\mathbb{R}^+$ [16, 17, 21], the vector-valued wavelet packets [3] and the tight framelet packets on $\mathbb{R}^d$ [11]. Inspired by the above described work on local fields [1, 8, 22], in this article we derive explicit formulations to construct a class of tight framelet packets in $L^2(K)$ using the unitary extension principle of Ron and Shen [15].

This paper is structured as follows. In Section 2, we discuss some preliminary facts about local fields of positive characteristic and review some major concepts concerning tight wavelet frames on local fields. In Section 3, we prove a crucial lemma called the *splitting lemma* which play a key role in the construction of tight framelet packets on local field K. By virtue of this lemma, we construct a class of tight framelet packets in $L^2(K)$ by decomposing the wavelet spaces $W_{j,\ell},\ j\in\mathbb{Z},\ 1\leqslant\ell\leqslant L$ using the framelet symbols $h_\ell,\ 1\leqslant\ell\leqslant L$. In Section 4, we derive another approach to construct tight framelet packets in $L^2(K)$ by directly decomposing the MRA space V_J for a fixed level $J>0$ to the level 0 with any combined mask $\mathbf{h}=[h_0,h_1,\dots,h_L]$ satisfying the unitary extension principle condition $\mathscr{M}(\xi)\mathscr{M}^*(\xi)=I_q$, where $\mathscr{M}(\xi)=\left\{h_\ell\big(\mathfrak{p}\xi+\mathfrak{p}u(k)\big)\right\}_{\ell,k=0}^{q-1}$.

2. Preliminaries on local fields

Let K be a field and a topological space. Then K is called a *local field* if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence by a local field, we mean a field K which is locally compact, non-discrete and totally disconnected. The p-adic fields are examples of local fields. More details are referred to [14, 25]. In the rest of this paper, we use the symbols $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let K be a fixed local field. Then there is an integer $q = \mathfrak{p}^r$, where $\mathfrak{p}$ is a fixed prime element of K and r is a positive integer, and a norm $|.|$ on K such that for all $x \in K$ we have $|x| \geqslant 0$ and for each $x \in K \setminus \{0\}$ we get $|x| = q^k$ for some integer k. This norm is non-Archimedean, that is $|x+y| \leqslant \max\{|x|,|y|\}$ for all $x,y \in K$ and $|x+y| = \max\{|x|,|y|\}$ whenever $|x| \neq |y|$. Let dx be the Haar measure on the locally compact, topological group $(K,+)$. This measure is normalized so that $\int_{\mathfrak{D}} dx = 1$, where $\mathfrak{D} = \{x \in K : |x| \leqslant 1\}$ is the *ring of integers* in K. Define $\mathfrak{B} = \{x \in K : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$ and hence as result $\mathfrak{B}$ is both principal and prime. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Then, it is easy to verify that $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x', x' \in \mathfrak{D}^*$. Moreover, each $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \left\{x \in K : |x| < q^{-k}\right\}$ is a compact subgroup of K^+ and usually known as the *fractional ideals* of K^+ (see [14]). Let $\mathscr{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathscr{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(yx), x \in K$. Suppose that χ_u is any character on K^+, then clearly the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [25], the set $\left\{\chi_{u(n)} : n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

The Fourier transform $\hat{f}$ of a function $f \in L^1(K) \cap L^2(K)$ is defined by

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}dx. \tag{2.1}$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)dx.$$

Furthermore, the properties of Fourier transform on local field are much similar to those of on the real line. In particular Fourier transform is unitary on $L^2(K)$.

Let us now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. Since $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(q)$, we choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that $\operatorname{span}\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ such that $0 \leqslant n < q$, we have

$$n = a_0 + a_1 p + \ldots + a_{c-1} p^{c-1}, \quad 0 \leqslant a_k < p, \; k = 0, 1, \ldots, c-1.$$

Define

$$u(n) = (a_0 + a_1 \zeta_1 + \ldots + a_{c-1} \zeta_{c-1}) \mathfrak{p}^{-1}. \tag{2.2}$$

For $n \geqslant 0$ and $0 \leqslant b_k < q$, $k = 0, 1, 2, \ldots, s$, we write

$$n = b_0 + b_1 q + b_2 q^2 + \ldots + b_s q^s,$$

such that

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \ldots + u(b_s)\mathfrak{p}^{-s}. \tag{2.3}$$

If $r, k \geqslant 0$ and $0 \leqslant s < q^k$, then it follows that

$$u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s).$$

Further, it is easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \geqslant 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \ldots, c-1 \text{ or } j \neq 1. \end{cases} \tag{2.4}$$

Let $\mathscr{Z} = \{u(n) : n \in \mathbb{N}_0\}$, where $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of (distinct) coset representation of $\mathfrak{D}$ in K^+. Then

$$l^2(\mathscr{Z}) = \Big\{ z : \mathscr{Z} \to \mathbb{C} : \sum_{n \in \mathbb{N}_0} \big| z\big(u(n)\big) \big|^2 < \infty \Big\}$$

is a Hilbert space with an inner product

$$\langle z, w \rangle = \sum_{n \in \mathbb{N}_0} z\big(u(n)\big) \overline{w\big(u(n)\big)}.$$

Moreover, the Fourier transform on $l^2(\mathscr{Z})$ is a map $\wedge : l^2(\mathscr{Z}) \to L^2(\mathfrak{D})$ defined by

$$\hat{z}(\xi) = \sum_{n \in \mathbb{N}_0} z\big(u(n)\big) \chi_{u(n)}(\xi), \quad z\big(u(n)\big) \in \mathscr{Z}$$

and its inverse is

$$z\big(u(n)\big) = \langle f, \chi_{u(n)} \rangle = \int_{\mathfrak{D}} f(x) \overline{\chi_{u(n)}(x)}\, dx, \quad f \in L^2(\mathfrak{D}).$$

For all $z, w \in \mathscr{Z}$, we have Parseval's relation:

$$\langle z, w \rangle = \sum_{n \in \mathbb{N}_0} z\big(u(n)\big) \overline{w\big(u(n)\big)} = \int_{\mathfrak{D}} \hat{z}(\xi) \overline{\hat{w}(\xi)}\, d\xi = \langle \hat{z}, \hat{w} \rangle,$$

and Plancherel's relation:

$$||z||^2 = \sum_{n\in\mathbb{N}_0} \left|z\big(u(n)\big)\right|^2 = \int_{\mathfrak{D}} \left|\hat{z}(\xi)\right|^2 d\xi = ||\hat{z}||^2.$$

For given $\Psi := \{\psi_1, \ldots, \psi_L\} \subset L^2(K)$, define the wavelet system

$$X(\Psi) := \left\{\psi^{\ell}_{j,k} : 1 \leqslant \ell \leqslant L, j \in \mathbb{Z}, k \in \mathbb{N}_0\right\} \tag{2.5}$$

where $\psi^{\ell}_{j,k} = q^{j/2}\psi^{\ell}\big(\mathfrak{p}^{-j}\cdot - u(k)\big)$. The wavelet system $X(\Psi)$ is called a *framelet system*, if there exist positive numbers $0 < A \leqslant B < \infty$ such that for all $f \in L^2(K)$

$$A\|f\|_2^2 \leqslant \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \left|\langle f, \psi^{\ell}_{j,k}\rangle\right|^2 \leqslant B\|f\|_2^2. \tag{2.6}$$

The largest A and the smallest B for which (2.6) holds are called *wavelet frame bounds*. A wavelet frame is a *tight wavelet frame* if A and B are chosen such that $A = B$ and then generators $\psi_1, \psi_2, \ldots, \psi_L$ are often referred as *tight framelets*. If only the right-hand inequality in (2.6) holds, then $X(\Psi)$ is called a Bessel sequence.

The construction of framelet systems often starts with the construction of MRA, which is built on refinable functions. A function $\varphi \in L^2(K)$ is called *refinable* if it satisfies a refinement equation:

$$\varphi(x) = \sqrt{q}\sum_{k\in\mathbb{N}_0} a_k\, \varphi\big(\mathfrak{p}^{-1}x - u(k)\big), \tag{2.7}$$

for some $\{a_k : k \in \mathbb{N}_0\} \in l^2(\mathbb{N}_0)$. The Fourier transform of (2.7) yields

$$\hat{\varphi}(\xi) = h_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi), \tag{2.8}$$

where

$$h_0(\xi) = \frac{1}{\sqrt{q}}\sum_{k\in\mathbb{N}_0} a_k\, \overline{\chi_k(\xi)},$$

is an integral periodic function in $L^2(\mathfrak{D})$ and is often called the *refinement symbol* of φ.

For a compactly supported refinable function $\varphi \in L^2(K)$ with $\hat{\varphi}(0) \neq 0$, let V_0 be the closed shift invariant space generated by $\{\varphi(\cdot - u(k) : k \in \mathbb{N}_0\}$ and $V_j = \{\varphi\big(\mathfrak{p}^{-j} - u(k)\big) : k \in \mathbb{N}_0\}$, $j \in \mathbb{Z}$. It is known that when φ is compactly supported, then $\{V_j : j \in \mathbb{Z}\}$ forms a multiresolution analysis for $L^2(K)$ (see [6]). Recall that a multiresolution analysis is a family of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$ that satisfies (i) $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$; (ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$ and (iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$. We further assume that

$$\lim_{\xi\to 0} \hat{\varphi}(\xi) = 1 \quad \text{for } a.e.\ \ \xi \in \mathfrak{D}, \tag{2.9}$$

and

$$\sum_{k\in\mathbb{N}_0} \left|\hat{\varphi}\big(\xi + u(k)\big)\right|^2 = 1 \quad \text{for } a.e.\ \ \xi \in \mathfrak{D}. \tag{2.10}$$

Given an MRA generated by the refinable function φ, one can construct a set of basic tight framelets $\Psi := \{\psi_1, \dots, \psi_L\} \subset V_1$ satisfying

$$\hat{\psi}^\ell(\xi) = h_\ell(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi), \tag{2.11}$$

where

$$h_\ell(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} a_k^\ell \overline{\chi_k(\xi)}, \quad \ell = 1, \dots, L \tag{2.12}$$

are the integral periodic functions in $L^2(\mathfrak{D})$ and are called the *framelet symbols* or *wavelet masks* (see [8]).

With $h_\ell(\xi), \ell = 0, 1, \dots, L$ as framelet symbols, we formulate the $q \times (L+1)$ matrix $\mathscr{M}(\xi)$ as:

$$\mathscr{M}(\xi) = \begin{pmatrix} h_0\big(\mathfrak{p}\xi + \mathfrak{p}u(0)\big) & h_0\big(\mathfrak{p}\xi + \mathfrak{p}u(1)\big) & \dots & h_0\big(\mathfrak{p}\xi + \mathfrak{p}u(q-1)\big) \\ h_1\big(\mathfrak{p}\xi + \mathfrak{p}u(0)\big) & h_1\big(\mathfrak{p}\xi + \mathfrak{p}u(1)\big) & \dots & h_1\big(\mathfrak{p}\xi + \mathfrak{p}u(q-1)\big) \\ \vdots & \vdots & \ddots & \vdots \\ h_L\big(\mathfrak{p}\xi + \mathfrak{p}u(0)\big) & h_L\big(\mathfrak{p}\xi + \mathfrak{p}u(1)\big) & \dots & h_L\big(\mathfrak{p}\xi + \mathfrak{p}u(q-1)\big) \end{pmatrix}. \tag{2.13}$$

The so-called *unitary extension principle* (UEP) provides a sufficient condition on $\Psi = \{\psi_1, \dots, \psi_L\}$ such that the resulting wavelet system $X(\Psi)$ forms a tight frame of $L^2(K)$. In this connection, Shah and Debnath [22] gave an explicit construction scheme for the construction of tight framelets on local fields of positive characteristic using unitary extension principles in the following way.

THEOREM 2.1. *Suppose that the refinable function φ and the framelet symbols $h_0, h_1, \dots, h_L$ satisfy (2.8)-(2.10). Define $\psi_1, \dots, \psi_L$ by (2.11). Let $\mathscr{M}(\xi)$ be the matrix as defined in (2.13) such that*

$$\mathscr{M}(\xi)\mathscr{M}^*(\xi) = I_q, \quad \textit{for a.e. } \xi \in \sigma(V_0) \tag{2.14}$$

where $\sigma(V_0) := \big\{\xi \in \mathfrak{D} : \sum_{k \in \mathbb{Z}} |\hat{\varphi}\big(\xi + u(k)\big)|^2 \neq 0\big\}$, then the wavelet system $X(\Psi)$ given by (2.5) constitutes a normalized tight wavelet frame for $L^2(K)$.

Moreover, if the framelet symbols $h_\ell, \ell = 0, 1, \dots, L$, satisfy the UEP condition (2.14). Then, for any $\xi \in K$, we have

$$\sum_{k=0}^{q-1} \big|h_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)\big|^2 \leqslant 1, \tag{2.15}$$

and

$$\sum_{\ell=0}^{L} h_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(r)\big)\overline{h_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(s)\big)} = \delta_{r,s}, \quad 0 \leqslant r, s \leqslant q-1. \tag{2.16}$$

For each $j \in \mathbb{Z}$, we define

$$V_j = \overline{span}\big\{\varphi_{j,k} : k \in \mathbb{N}_0\big\},$$

and

$$W_{j,\ell}=\overline{span}\{\psi^{\ell}_{j,k}:k\in\mathbb{N}_0\},\quad \ell=0,1,\ldots,L.$$

Therefore, in view of tight frame decomposition, we have

$$V_j=V_{j-1}+\sum_{\ell=1}^{L}W_{j-1,\ell}. \tag{2.17}$$

It is immediate from the above decomposition that these $L+1$ *spaces are in general not orthogonal. Therefore, by the repeated applications of* (2.17), *we can further split the* V_j *spaces as:*

$$V_j=V_{j-1}+\sum_{\ell=1}^{L}W_{j-1,\ell}=V_{j-2}+\sum_{r=j-2}^{j-1}\sum_{\ell=1}^{L}W_{r,\ell}=\ldots=V_{j_0}+\sum_{r=j_0}^{j-1}\sum_{\ell=1}^{L}W_{r,\ell}=\sum_{r=-\infty}^{j-1}\sum_{\ell=1}^{L}W_{r,\ell}.$$

3. Tight framelet packets on local fields via wavelet spaces $W_{j,\ell}$

We start this section with a *splitting lemma* which plays a crucial role in the construction of tight framelet packets on local field K of positive characteristic. We split the wavelet spaces $W_{j,\ell}$ by framelet symbols $h_\ell,\ell=0,1,\ldots,L$ and then by selecting and recursively decomposition, we will obtain various tight framelet packets of $L^2(K)$.

LEMMA 3.1. (Splitting Lemma) *Let* $g\in L^2(K)$ *and* $\{g_{j,k}:k\in\mathbb{N}_0\}$ *be a Bessel's sequence in* $L^2(K)$ *i.e.,*

$$\sum_{k\in\mathbb{N}_0}\left|\hat{g}\big(\xi+u(k)\big)\right|^2\leqslant B,\quad \xi\in K \tag{3.1}$$

for any fixed $j\in\mathbb{Z}$. *Let* $h_\ell, 0\leqslant\ell\leqslant L$ *be the framelet masks associated with the refinable function* φ *and the tight framelets* ψ_ℓ, $1\leqslant\ell\leqslant L$ *satisfying the UEP condition (2.14). Suppose*

$$g^{\ell}(x)=q\sum_{k\in\mathbb{N}_0}h_\ell\big(u(n)\big)\,g\big(\mathfrak{p}^{-1}x-u(n)\big), \tag{3.2}$$

$$G_\ell=\overline{span}\{g^{\ell}_{j-1,k}:k\in\mathbb{N}_0\}, \tag{3.3}$$

and $G=\overline{span}\{g_{j,k}:k\in\mathbb{N}_0\}$, *for* $0\leqslant\ell\leqslant L$. *Then*

(i) *For* $\ell=0,1,\ldots,L$, *each set* $\left\{g^{\ell}_{j-1,k}:k\in\mathbb{N}_0\right\}$ *forms a Bessel's sequence with* $||g^{\ell}||_2^2\leqslant B$ *and* $||g||_2^2\leqslant B$.

(ii) *For any sequence* $z\in l^2(\mathscr{Z})$, *there exists* $L+1$ *sequences* $\{z_\ell\}_{\ell=0}^{L}$ *defined by*

$$z_\ell\big(u(k)\big)=\sqrt{q}\sum_{k\in\mathbb{N}_0}\overline{h_\ell}\big(u(n)-\mathfrak{p}^{-1}u(k)\big)\,z\big(u(n)\big),\quad k\in\mathbb{N}_0 \tag{3.4}$$

such that

$$||z||^2_{l^2(\mathscr{Z})} = \sum_{\ell=0}^{L} ||z_\ell||^2, \tag{3.5}$$

and

$$\sum_{k\in\mathbb{N}_0} z\big(u(k)\big)\, g_{j,k} = \sum_{\ell=0}^{L}\sum_{k\in\mathbb{N}_0} z_\ell\big(u(k)\big) g^\ell_{j-1,k}. \tag{3.6}$$

(iii) *In particular for any* $f \in L^2(K)$, *let* $z\big(u(k)\big) = \langle f, g_{j,k}\rangle$, $k \in \mathbb{N}_0$, *then* $z \in l^2(\mathscr{Z})$ *and (3.4)–(3.6) gives*

$$z_\ell\big(u(k)\big) = \Big\langle f, g^\ell_{j-1,k}\Big\rangle, \quad k \in \mathbb{N}_0, \;\; \ell = 0,1,\ldots,L, \tag{3.7}$$

$$\sum_{k\in\mathbb{N}_0} \big|\langle f, g_{j,k}\rangle\big|^2 = \sum_{\ell=0}^{L}\sum_{k\in\mathbb{N}_0} \Big|\langle f, g^\ell_{j-1,k}\rangle\Big|^2, \tag{3.8}$$

and

$$\sum_{k\in\mathbb{N}_0} \langle f, g_{j,k}\rangle g_{j,k} = \sum_{\ell=0}^{L}\sum_{k\in\mathbb{N}_0} \Big\langle f, g^\ell_{j-1,k}\Big\rangle g^\ell_{j-1,k}, \tag{3.9}$$

respectively.

(iv) G *has the decomposition*

$$G = G_0 + G_1 + \ldots + G_L.$$

Proof. (i) By Plancherel's formula, we have

$$\begin{aligned} ||g||_2^2 &= ||\hat{g}||_2^2 \\ &= \int_K \big|\hat{g}(\xi)\chi_k(\xi)\big|^2 d\xi \\ &= \int_{\mathfrak{D}} \sum_{k\in\mathbb{N}_0} \big|\hat{g}\big(\xi + u(k)\big)\big|^2 \big|\chi_k(\xi)\big|^2 d\xi. \end{aligned}$$

Using equation (3.1) and the fact that the set $\big\{\chi_{u(n)} : n \in \mathbb{N}_0\big\}$ is a complete orthonormal system on $\mathfrak{D}$, we obtain $||g||_2^2 \leqslant B$.

On taking Fourier transform of equation (3.2), we obtain

$$\hat{g}_\ell(\xi) = h_\ell(\mathfrak{p}\xi)\,\hat{g}(\mathfrak{p}\xi). \tag{3.10}$$

Using (2.15) and (3.1), we have

$$\begin{aligned} \sum_{k\in\mathbb{N}_0} \big|\hat{g}_\ell\big(\xi + u(k)\big)\big|^2 &= \sum_{k\in\mathbb{N}_0} \big|h_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)\big|^2 \big|\hat{g}\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)\big|^2 \\ &= \sum_{s=0}^{q-1}\sum_{k\in\mathbb{N}_0} \big|h_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+s)\big)\big|^2 \big|\hat{g}\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+s)\big)\big|^2 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{s=0}^{q-1} \left|h_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(s)\big)\right|^2 \sum_{k\in\mathbb{N}_0} \left|\hat{g}\big(\mathfrak{p}\xi + \mathfrak{p}u(qk+s)\big)\right|^2 \\
&= \sum_{s=0}^{q-1} \left|h_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(s)\big)\right|^2 \sum_{k\in\mathbb{N}_0} \left|\hat{g}\big(\mathfrak{p}\xi + \mathfrak{p}u(s)\big) + u(k)\right|^2 \\
&\leqslant B \sum_{s=0}^{q-1} \left|h_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(s)\big)\right|^2 \\
&\leqslant B, \quad \text{for } \ell = 0,1,\ldots,L.
\end{aligned}
$$

(ii) For each $0 \leqslant \ell \leqslant L$, the Fourier transform of (3.4) gives

$$
\hat{z}_\ell(\xi) = q^{-1/2} \sum_{r=0}^{q-1} h_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(r)\big)\hat{z}\big(\mathfrak{p}\xi + \mathfrak{p}u(r)\big). \tag{3.11}
$$

By summing equation (3.11) over $\ell = 0$ to L and using (2.16), we obtain

$$
\begin{aligned}
\sum_{\ell=0}^{L} \left|\hat{z}_\ell(\xi)\right|^2 &= q^{-1} \sum_{\ell=0}^{L} \sum_{r,s=0}^{q-1} \overline{h_\ell}\big(\mathfrak{p}\xi + \mathfrak{p}u(r)\big)\hat{z}\big(\mathfrak{p}\xi + \mathfrak{p}u(s)\big)h_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(s)\big)\overline{\hat{z}}\big(\mathfrak{p}\xi + \mathfrak{p}u(r)\big) \\
&= q^{-1} \sum_{r,s=0}^{q-1} \hat{z}\big(\mathfrak{p}\xi + \mathfrak{p}u(s)\big)\overline{\hat{z}}\big(\mathfrak{p}\xi + \mathfrak{p}u(r)\big) \sum_{\ell=0}^{L} h_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(s)\big)\overline{h_\ell}\big(\mathfrak{p}\xi + \mathfrak{p}u(r)\big) \\
&= q^{-1} \sum_{r,s=0}^{q-1} \hat{z}\big(\mathfrak{p}\xi + \mathfrak{p}u(s)\big)\overline{\hat{z}}\big(\mathfrak{p}\xi + \mathfrak{p}u(r)\big)\, \delta_{r,s} \\
&= q^{-1} \sum_{r=0}^{q-1} \left|\hat{z}\big(\mathfrak{p}\xi + \mathfrak{p}u(r)\big)\right|^2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sum_{\ell=0}^{L} \left|\left|\hat{z}_\ell\right|\right|^2_{\ell^2(\mathscr{Z})} &= \sum_{\ell=0}^{L} \sum_{k\in\mathbb{N}_0} \left|\hat{z}_\ell\big(u(k)\big)\right|^2 = \sum_{\ell=0}^{L} \int_{\mathfrak{D}} \left|\hat{z}_\ell\big(u(k)\big)\right|^2 d\xi \\
&= \int_{\mathfrak{D}} \sum_{\ell=0}^{L} \left|\hat{z}_\ell\big(u(k)\big)\right|^2 d\xi = q^{-1} \int_{\mathfrak{D}} \sum_{r=0}^{q-1} \left|\hat{z}\big(\mathfrak{p}\xi + \mathfrak{p}u(r)\big)\right|^2 d\xi \\
&= \int_{\mathfrak{D}} \left|\hat{z}(\xi)\right|^2 d\xi = \int_{\mathfrak{D}} \left|\sum_{n\in\mathbb{N}_0} z\big(u(n)\big)\chi_{u(n)}(\xi)\right|^2 d\xi \\
&= \sum_{n\in\mathbb{N}_0} \left|z\big(u(n)\big)\right|^2 = \left|\left|\hat{z}\right|\right|^2_{\ell^2(\mathscr{Z})}.
\end{aligned}
$$

Equation (3.6) can be recast in the frequency domain as:

$$
q^{-j/2}\hat{z}(\mathfrak{p}^j\xi)\,\hat{g}(\mathfrak{p}^j\xi) = q^{\frac{1-j}{2}} \sum_{\ell=0}^{L} \hat{z}_\ell(\mathfrak{p}^{j-1}\xi)\,\hat{g}_\ell(\mathfrak{p}^{j-1}\xi). \tag{3.12}
$$

Thus, in order to show that (3.6) holds, it suffices to verify the equality (3.12).

$$\begin{aligned}
R.H.S. &= q^{\frac{1-j}{2}} \sum_{\ell=0}^{L} \hat{z}_\ell(\mathfrak{p}^{j-1}\xi)\,\hat{g}_\ell(\mathfrak{p}^{j-1}\xi) \\
&= q^{\frac{1-j}{2}} \sum_{\ell=0}^{L} \hat{z}_\ell(\mathfrak{p}^{j-1}\xi)\,h_\ell(\mathfrak{p}^{j}\xi)\hat{g}_\ell(\mathfrak{p}^{j}\xi) \quad \text{(By splitting lemma)} \\
&= q^{\frac{1-j}{2}} \hat{g}(\mathfrak{p}^{j}\xi) \sum_{\ell=0}^{L} \left[q^{-1} \sum_{r=0}^{q-1} \hat{z}\big(\mathfrak{p}\xi + \mathfrak{p}u(r)\big)\overline{h_\ell}\big(\mathfrak{p}\xi + \mathfrak{p}u(r)\big) \right] h_\ell(\mathfrak{p}^{j}\xi) \\
&= q^{-j/2} \hat{g}(\mathfrak{p}^{j}\xi) \sum_{r=0}^{q-1} \hat{z}\big(\mathfrak{p}\xi + \mathfrak{p}u(r)\big) \sum_{\ell=0}^{L} \left[\overline{h_\ell}\big(\mathfrak{p}\xi + \mathfrak{p}u(r)\big)h_\ell(\mathfrak{p}^{j}\xi)\right] \\
&= q^{-j/2} \hat{g}(\mathfrak{p}^{j}\xi) \sum_{r=0}^{q-1} \hat{z}\big(\mathfrak{p}\xi + \mathfrak{p}u(r)\big)\,\delta_{r,0} \\
&= q^{-j/2} \hat{g}(\mathfrak{p}^{j}\xi)\hat{z}\big(\mathfrak{p}^{j}\xi\big) = L.H.S.
\end{aligned}$$

(iii). For the proof of the part (iii) of the splitting lemma, it is sufficient to verify equation (3.7) only. The equations (3.8) and (3.9) are direct consequences of equations (3.5) and (3.6) which have been proved. Moreover, from equation (3.4), we have

$$\begin{aligned}
z_\ell\big(u(k)\big) &= q^{1/2} \sum_{n\in\mathbb{N}_0} \overline{h_\ell}\big(u(n) - \mathfrak{p}^{-1}u(k)\big)z\big(u(n)\big) \\
&= q^{1/2} \sum_{n\in\mathbb{N}_0} \overline{h_\ell}\big(u(n) - \mathfrak{p}^{-1}u(k)\big)\langle f, g_{j,n}\rangle \\
&= \langle f, q^{1/2} \sum_{n\in\mathbb{N}_0} \overline{h_\ell}\big(u(n) - \mathfrak{p}^{-1}u(k)\big) g_{j,n}\rangle \\
&= \langle f, g^{\ell}_{j-1,k}\rangle, \quad \ell = 0,1,\ldots,L.
\end{aligned}$$

(iv). This is immediate from equations (3.2) and (3.3). $\square$

In the following sub-section we construct tight framelet packets for $L^2(K)$ via multiresolution analysis generated by the framelet symbols. To do this, let $\{\psi^\ell, h_\ell\}_{\ell=0}^{L}$ satisfy the conditions of the unitary extension principle and $\omega_0 = \varphi$. Define the functions $\omega_n(x)$, $n = 0,1,2,\ldots,$ associated with the refinable function φ recursively by

$$\hat{\omega}_n(\xi) = \hat{\omega}_{(L+1)r+\ell}(\xi) = h_\ell(\mathfrak{p}\xi)\,\omega_r(\mathfrak{p}\xi), \quad \ell = 0,1,\ldots,L,\ r \in \mathbb{N}_0. \tag{3.13}$$

Note that for $r = 0$ and $\ell = 0,1,\ldots,L$, we have

$$\hat{\omega}_\ell(\xi) = h_\ell(\mathfrak{p}\xi)\,\omega_0(\mathfrak{p}\xi) = h_\ell(\mathfrak{p}\xi)\varphi(\mathfrak{p}\xi), \tag{3.14}$$

which shows that $\omega_\ell(\cdot) = \psi^\ell(\cdot)$, $\ell = 0,1,\ldots,L.$

For $n \in \mathbb{N}_0$, define a family of subspaces of $L^2(K)$ by

$$U_n = \overline{\text{span}}\{\omega_{n,0,k} : k \in \mathbb{N}_0\}. \tag{3.15}$$

Clearly $U_0 = V_0$ and $U_\ell = W_{0,\ell}$, for $\ell = 1,\ldots,L$. Since $X(\Psi)$ is a tight wavelet frame constructed via UEP in an MRA generated by φ. Therefore, we have

$$\sum_{n\in\mathbb{N}_0} \left|\hat{\omega}_0\big(\xi + u(k)\big)\right|^2 \leqslant 1, \quad \xi \in K.$$

By invoking Lemma 3.1, for $n = 1,2,\ldots$, we obtain

$$\sum_{k\in\mathbb{N}_0} \left|\hat{\omega}_n\big(\xi + u(k)\big)\right|^2 \leqslant 1, \quad U_n^1 = \sum_{t=(L+1)n}^{(L+1)(n+1)-1} U_t,$$

and for any $f \in L^2(K)$,

$$\sum_{k\in\mathbb{N}_0} \left|\langle f, \omega_{n,1,k}\rangle\right|^2 = \sum_{t=(L+1)n}^{(L+1)(n+1)-1} \sum_{k\in\mathbb{N}_0} \left|\langle f, \omega_{t,0,k}\rangle\right|^2.$$

A repeated application of the Splitting Lemma 3.1 for $j = 1,2,\ldots,$ yields

$$U_n^j = \sum_{t=(L+1)^j n}^{(L+1)^j(n+1)-1} U_t \tag{3.16}$$

and for any $f \in L^2(K)$

$$\sum_{k\in\mathbb{N}_0} \left|\langle f, \omega_{n,j,k}\rangle\right|^2 = \sum_{t=(L+1)^j n}^{(L+1)^j(n+1)-1} \sum_{k\in\mathbb{N}_0} \left|\langle f, \omega_{t,0,k}\rangle\right|^2. \tag{3.17}$$

Substituting $n = 0$ in (3.16) and (3.17), we get

$$V_j = \sum_{t=0}^{(L+1)^j-1} U_t \tag{3.18}$$

and

$$\sum_{k\in\mathbb{N}_0} \left|\langle f, \varphi_{j,k}\rangle\right|^2 = \sum_{t=0}^{(L+1)^j-1} \sum_{k\in\mathbb{N}_0} \left|\langle f, \omega_{t,0,k}\rangle\right|^2 \tag{3.19}$$

for any $f \in L^2(K)$, respectively. Moreover, for $n = \ell, \ell = 1,\ldots,L$, (3.16) and (3.17) yields

$$W_{j,\ell} = W_{0,\ell}^j = U_\ell^j = \sum_{t=(L+1)^j\ell}^{(L+1)^j(\ell+1)-1} U_t, \tag{3.20}$$

and for any $f \in L^2(K)$

$$\sum_{k\in\mathbb{N}_0} \left|\langle f, \psi_{\ell,j,k}\rangle\right|^2 = \sum_{k\in\mathbb{N}_0} \left|\langle f, \omega_{\ell,j,k}\rangle\right|^2 = \sum_{t=(L+1)^j\ell}^{(L+1)^j(\ell+1)-1} \sum_{k\in\mathbb{N}_0} \left|\langle f, \omega_{t,0,k}\rangle\right|^2. \tag{3.21}$$

From equation (3.21), it follows that each wavelet space $W_{j,\ell}$, $j \in \mathbb{N}_0$, $\ell = 1,\dots,L$ can be further splitted into $(L+1)^j$ subspaces U_t, $t \in [(L+1)^j\ell,\ (L+1)^j(\ell+1)-1]$. If we keep the parameter j fixed, say $J > 0$, we will obtain

$$L^2(K) = \sum_{t=0}^{(L+1)^J-1} U_t + \sum_{\ell=1}^{L}\sum_{j\geqslant J} W_{j,\ell}. \tag{3.22}$$

THEOREM 3.2. *Let $X(\Psi)$ be a tight wavelet frame constructed via UEP in an MRA and $h_1,h_2,\dots,h_L$ are the framelet symbols satisfying the UEP condition (2.14). Let $\{\omega_n : n \in \mathbb{N}_0\}$ be defined as in (3.13). Then for any fixed $J > 0$, the family of functions*

$$\mathscr{F} = \Big\{\omega_{n,0,k} : 0 \leqslant n \leqslant (L+1)^J - 1,\ k \in \mathbb{N}_0\Big\} \bigcup \Big\{\psi^{\ell}_{j,k} : 1 \leqslant \ell \leqslant L,\ j \geqslant J,\ k \in \mathbb{N}_0\Big\}$$

forms a tight frame for $L^2(K)$.

Proof. By Theorem 2.1, the wavelet system $X(\Psi)$ constitutes a tight wavelet frame for $L^2(K)$. Therefore by equation (3.18), we have for any $f \in L^2(K)$

$$\begin{aligned}
||f||^2 &= \sum_{k\in\mathbb{N}_0} \big|\langle f,\varphi_{0,k}\rangle\big|^2 + \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\Big|\Big\langle f,\psi^{\ell}_{j,k}\Big\rangle\Big|^2 \\
&= \sum_{k\in\mathbb{N}_0} \big|\langle f,\varphi_{J,k}\rangle\big|^2 + \sum_{\ell=1}^{L}\sum_{j\geqslant J}\sum_{k\in\mathbb{N}_0}\Big|\Big\langle f,\psi^{\ell}_{j,k}\Big\rangle\Big|^2 \\
&= \sum_{n=0}^{(L+1)^J-1}\sum_{n\in\mathbb{N}_0} \big|\langle f,\omega_{n,0,k}\rangle\big|^2 + \sum_{\ell=1}^{L}\sum_{j\geqslant J}\sum_{k\in\mathbb{N}_0}\Big|\Big\langle f,\psi^{\ell}_{j,k}\Big\rangle\Big|^2. \qquad \square
\end{aligned}$$

DEFINITION 3.3. The functions $\{\omega_n : n \in \mathbb{N}_0\}$ are called as the *basic framelet packets* on the local field K of positive characteristic associated with the refinable function φ.

With the help of basic framelet packets, we are now in a position to construct a class of tight frames for $L^2(K)$ by choosing other $L^2(K)$ space decompositions. For simplicity, let us consider a disjoint partition Υ_J of a finite set of non-negative integers

$$\Omega_J = \Big\{r \in \mathbb{N}_0 : 0 \leqslant r \leqslant (L+1)^J - 1\Big\} \tag{3.23}$$

into disjoint of the form

$$\Lambda_{j,n} = \Big\{(L+1)^j n,\dots,(L+1)^j(n+1)-1\Big\}, \quad j,n \in \mathbb{N}_0,$$

i.e.,

$$\Upsilon_J = \Big\{\Lambda_{j,n} : \bigcup \Lambda_{j,n} = \Omega_J\Big\}, \tag{3.24}$$

Then, it follows from (3.16) and (3.21) that

$$\begin{aligned} L^2(K) &= \sum_{t=0}^{(L+1)^J-1} U_t + \sum_{\ell=1}^{L} \sum_{j \geqslant J} W_{j,\ell} \\ &= \sum_{\Lambda_{J,n} \in \Upsilon_J} \sum_{t=(L+1)^j n}^{(L+1)^J(n+1)-1} U_t + \sum_{\ell=1}^{L} \sum_{j \geqslant J} W_{j,\ell} \\ &= \sum_{\Lambda_{J,n} \in \Upsilon_J} U_t^j + \sum_{\ell=1}^{L} \sum_{j \geqslant J} W_{j,\ell}. \end{aligned}$$

THEOREM 3.4. *Suppose $X(\Psi)$ is a tight wavelet frame constructed via UEP in an MRA and $h_1, h_2, \ldots, h_L$ are the framelet symbols satisfying the UEP condition (2.14). Let $\{\omega_n : n \in \mathbb{N}_0\}$ be defined as in equation (3.13). For any fixed $J > 0$, Υ_J is a partition of Ω_J, where Ω_J and Υ_J are defined in (3.23) and (3.24), respectively. Then the family of functions*

$$\mathscr{F}_{\Upsilon_J} = \Big\{ \omega_{n,0,k} : \Lambda_{J,n} \in \Upsilon_J, \ \ k \in \mathbb{N}_0 \Big\} \bigcup \Big\{ \psi_{\ell,j,k} : 1 \leqslant \ell \leqslant L, \ \ j \geqslant J, \ \ k \in \mathbb{N}_0 \Big\}$$

constitutes a tight frame for $L^2(K)$.

Proof. For any arbitrary $f \in L^2(K)$, we have

$$\begin{aligned} \sum_{\Lambda_{J,n} \in \Upsilon_J} \sum_{k \in \mathbb{N}_0} \left| \langle f, \omega_{n,j,k} \rangle \right|^2 &= \sum_{\Lambda_{J,n} \in \Upsilon_J} \sum_{n=(L+1)^j n}^{(L+1)^j(n+1)-1} \sum_{k \in \mathbb{N}_0} \left| \langle f, \omega_{n,0,k} \rangle \right|^2 \\ &= \sum_{n=0}^{(L+1)^J-1} \left| \langle f, \omega_{n,0,k} \rangle \right|^2. \end{aligned}$$

By invoking Theorem 3.2, we get the desired result. □

4. Tight framelet packets on local fields via MRA space V_J

Besides the recursive derivation of tight framelet packets introduced in Section 3, tight framelet packets can also be constructed by decomposing the MRA space V_J directly for a fixed level $J > 0$ to the level 0.

At the first level of decomposition, by Lemma 3.1, V_J is decomposed into the $L+1$ spaces $W_{J-1,\mathbf{r}}, \mathbf{r} \in \Delta_1$ where

$$\Delta_1 = \Big\{ \mathbf{r} = (r_J, r_{J-1}, \ldots, r_1) : 0 \leqslant r_J \leqslant L, \, r_{J-1} = \ldots = r_1 = 0 \Big\}.$$

For this choice of $\mathbf{r} = (r_J, r_{J-1}, \ldots, r_1)$, we define

$$\mathbf{r}(n) = r_n, \quad n = 1, 2, \ldots, J,$$

$$\omega_{\mathbf{r}}(x) = q^{1/2} \sum_{n\in\mathbb{N}_0} a_n^{\mathbf{r}(1)} \varphi\big(\mathfrak{p}^{-1}x - u(n)\big),$$

and

$$W_{J-1,\mathbf{r}} := \overline{\text{span}}\big\{\omega_{\mathbf{r},J-1,k} : k \in \mathbb{N}_0\big\}.$$

Therefore, for any $f \in L^2(K)$, we have

$$\sum_{k\in\mathbb{N}_0} \big|\langle f, \varphi_{J,k}\rangle\big|^2 = \sum_{\mathbf{r}\in\Delta_1} \sum_{k\in\mathbb{N}_0} \big|\big\langle f, \omega_{\mathbf{r},J-1,k}\big\rangle\big|^2.$$

At the second level of decomposition, by Lemma 3.1, each space $W_{J-1,\mathbf{r}}, \mathbf{r} \in \Delta_1$ is decomposed with the constructed mask $\mathbf{h}$ into spaces $W_{J-2,\mathbf{r}'}, \mathbf{r}' \in \Delta_2^{\mathbf{r}}$, where $\Delta_2^{\mathbf{r}}$ is a subset of Δ_2 defined by

$$\Delta_2^{\mathbf{r}} = \big\{\mathbf{r}' \in \Delta_2 : \mathbf{r}'(1) = \mathbf{r}(1)\big\}$$

and Δ_2 is a J-tuple index set defined by

$$\Delta_2 = \Big\{\mathbf{r} = (r_J, r_{J-1}, \ldots, r_1) : 0 \leqslant r_{J-1}, r_J \leqslant L,\, r_{J-2} = \ldots = r_1 = 0\Big\},$$

$$\omega_{\mathbf{r}'}(x) = q^{1/2} \sum_{n\in\mathbb{N}_0} a_n^{\mathbf{r}'(2)} \varphi\big(\mathfrak{p}^{-1}x - u(n)\big),$$

$$W_{J-2,\mathbf{r}'} := \overline{\text{span}}\big\{\omega_{\mathbf{r}',J-2,k} : k \in \mathbb{N}_0\big\}.$$

Thus, for any $f \in L^2(K)$, we have

$$\sum_{k\in\mathbb{N}_0} \big|\langle f, \omega_{\mathbf{r},J-1,k}\rangle\big|^2 = \sum_{\mathbf{r}'\in\Delta_2^{\mathbf{r}}} \sum_{k\in\mathbb{N}_0} \big|\langle f, \omega_{\mathbf{r}',J-2,k}\rangle\big|^2.$$

Finally, at the m-th level $(2 \leqslant m \leqslant J)$ of decomposition, by Lemma 3.1, each space $W_{J-m+1,\mathbf{r}}$, $\mathbf{r} \in \Delta_{m-1}$ is decomposed with the constructed mask $\mathbf{h}$ into spaces $W_{J-m,\mathbf{r}'}, \mathbf{r}'$ $\in \Delta_m^{\mathbf{r}}$, where $\Delta_m^{\mathbf{r}}$ is a subset of Δ_m defined by

$$\Delta_m^{\mathbf{r}} = \Big\{\mathbf{r}' \in \Lambda_m : \mathbf{r}'(n) = \mathbf{r}(n), \ \text{ for } \ 1 \leqslant n \leqslant m-1\Big\}$$

and Δ_m is a J-tuple index set defined by

$$\Delta_m = \Big\{\mathbf{r} = (r_J, r_{J-1}, \ldots, r_1) : 0 \leqslant r_{J-m} \leqslant L,\, r_{J-m} = \ldots = r_1 = 0\Big\},$$

$$\omega_{\mathbf{r}'}(x) = q^{1/2} \sum_{n\in\mathbb{N}_0} a_n^{\mathbf{r}'(m)} \varphi\big(\mathfrak{p}^{-1}x - u(n)\big),$$

$$W_{J-m,\mathbf{r}'} := \overline{\text{span}}\big\{\omega_{\mathbf{r}',J-m,k} : k \in \mathbb{N}_0\big\}.$$

Therefore for any $f \in L^2(K)$, we have

$$\sum_{k\in\mathbb{N}_0} \big|\big\langle f, \omega_{\mathbf{r},J-m+1,k}\big\rangle\big|^2 = \sum_{\mathbf{r}'\in\Delta_m^{\mathbf{r}}} \sum_{k\in\mathbb{N}_0} \big|\big\langle f, \omega_{\mathbf{r}',J-m,k}\big\rangle\big|^2.$$

In particular, at the J-th level of decomposition, by Lemma 3.1, each space $W_{1,\mathbf{r}}, \mathbf{r} \in \Delta_{J-1}$ is decomposed with $\mathbf{h}$ into spaces $W_{0,\mathbf{r}'}, \mathbf{r}' \in \Delta_J^{\mathbf{r}}$, where $\Delta_J^{\mathbf{r}}$ is a subset of Δ_J defined by

$$\Delta_J^{\mathbf{r}} = \left\{ \mathbf{r}' \in \Delta_J : \mathbf{r}'(n) = \mathbf{r}(n), \text{ for } 1 \leqslant n \leqslant J-1 \right\}$$

and Δ_J is a J-tuple index set defined by

$$\Delta_J = \left\{ \mathbf{r} = (r_J, r_{J-1}, \ldots, r_1) : 0 \leqslant r_t \leqslant L, 1 \leqslant t \leqslant J \right\}, \tag{4.1}$$

$$\omega_{\mathbf{r}'}(x) = q^{1/2} \sum_{n \in \mathbb{N}_0} a_n^{\mathbf{r}'(J)} \varphi\big(\mathfrak{p}^{-1}x - u(n)\big),$$

$$W_{0,\mathbf{r}'} := \overline{\operatorname{span}}\big\{ \omega_{\mathbf{r}',0,k} : k \in \mathbb{N}_0 \big\}.$$

Thus, for any $f \in L^2(K)$, we have

$$\sum_{k \in \mathbb{N}_0} \left| \langle f, \omega_{\mathbf{r},1,k} \rangle \right|^2 = \sum_{\mathbf{r}' \in \Lambda_J^{\mathbf{r}}} \sum_{k \in \mathbb{N}_0} \left| \langle f, \omega_{\mathbf{r}',0,k} \rangle \right|^2.$$

Combining all the inner product equations in the above construction, we get

$$\sum_{k \in \mathbb{N}_0} \left| \langle f, \varphi_{J,k} \rangle \right|^2 = \sum_{\mathbf{r} \in \Lambda_J} \sum_{k \in \mathbb{N}_0} \left| \langle f, \omega_{\mathbf{r},0,k} \rangle \right|^2, \quad \text{for any } f \in L^2(K). \tag{4.2}$$

In other words, we obtain another representation of V_J as

$$V_J := \overline{\operatorname{span}}\left\{ \omega_{\mathbf{r},0,k} : \mathbf{r} \in \Delta_J, k \in \mathbb{N}_0 \right\}.$$

THEOREM 4.1. *Suppose $X(\Psi)$ is a tight wavelet frame constructed via UEP in an MRA and $\mathbf{h} = [h_0, h_1, \ldots, h_L]$ is the combined mask satisfying the UEP condition (2.14). Then for any fixed $J > 0$, the family of functions*

$$\mathscr{F} = \left\{ \omega_{\mathbf{r},0,k} : \mathbf{r} \in \Delta_J \right\} \bigcup \left\{ \psi_{\ell,j,k} : \ell = 1, \ldots, L, j \geqslant J, k \in \mathbb{N}_0 \right\}$$

forms a tight frame for $L^2(K)$, where Δ_J is a index set defined in (4.1).

Proof. Since $X(\Psi)$ is a tight wavelet frame of $L^2(K)$, then by (4.2), we have

$$\begin{aligned} \|f\|_2^2 &= \sum_{k \in \mathbb{N}_0} \left| \langle f, \varphi_{J,k} \rangle \right|^2 + \sum_{\ell=1}^{L} \sum_{j \geqslant J} \sum_{k \in \mathbb{N}_0} \left| \left\langle f, \psi_{j,k}^{\ell} \right\rangle \right|^2 \\ &= \sum_{\mathbf{r} \in \Delta_J} \sum_{k \in \mathbb{N}_0} \left| \langle f, \omega_{\mathbf{r},0,k} \rangle \right|^2 + \sum_{\ell=1}^{L} \sum_{j \geqslant J} \sum_{k \in \mathbb{N}_0} \left| \left\langle f, \psi_{j,k}^{\ell} \right\rangle \right|^2 \end{aligned}$$

for any $f \in L^2(K)$. $\square$

Similar to the recursive construction of tight framelet packets on local fields of positive characteristic, we can obtain tight framelet packets by performing various disjoint partitions Γ_J of Δ_J with each partition separating Δ_J into disjoint subsets of the form

$$I_{j,\mathbf{r}} = \Big\{(r_J,\ldots,r_{j+1},r'_j,\ldots,r'_1) \in \Delta_J : \mathbf{r} = (r_J,\ldots,r_{j+1},0,\ldots,0) \in \Delta_{J-j}\Big\},$$

i.e.,

$$\Gamma_J = \Big\{I_{j,\mathbf{r}} : \bigcup I_{j,\mathbf{r}} = \Delta_J\Big\}. \tag{4.3}$$

THEOREM 4.2. *Suppose $X(\Psi)$ is a tight wavelet frame constructed via UEP in an MRA and $\mathbf{h} = [h_0,h_1,\ldots,h_L]$ is the combined mask satisfying the UEP condition (2.14). Let Γ_J be a disjoint partition of Δ_J, where Δ_J and Γ_J are defined in (4.1) and (4.3), respectively. Then the collection*

$$\mathscr{F}_{\Gamma_J} = \Big\{\omega_{\mathbf{r},j,k} : I_{j,\mathbf{r}} \in \Gamma_J, k \in \mathbb{N}_0\Big\} \bigcup \Big\{\psi^{\ell}_{j,k} : \ell = 1,\ldots,L, j \geqslant J \in \mathbb{Z}, k \in \mathbb{N}_0\Big\}$$

generates a tight frame for $L^2(K)$.

Proof. Since Γ_J is a disjoint partition of Δ_J, for any $f \in L^2(K)$, we have

$$\begin{aligned}\sum_{I_{j,\mathbf{r}}\in\Gamma_J}\sum_{k\in\mathbb{N}_0}\big|\langle f,\omega_{\mathbf{r},j,k}\rangle\big|^2 &= \sum_{I_{j,\mathbf{r}}\in\Gamma_J}\sum_{\mathbf{r}'\in I_{j,\mathbf{r}}}\sum_{k\in\mathbb{N}_0}\big|\langle f,\omega_{\mathbf{r}',0,k}\rangle\big|^2\\ &= \sum_{\mathbf{r}\in\Delta_J}\sum_{k\in\mathbb{N}_0}\big|\langle f,\omega_{\mathbf{r},0,k}\rangle\big|^2.\end{aligned}$$

By applying Theorem 4.1, Theorem 4.2 is proved. □

REFERENCES

[1] B. BEHERA AND Q. JAHAN, *Wavelet packets and wavelet frame packets on local fields of positive characteristic*, J. Math. Anal. Appl. **395** (2012), 1–14.

[2] J. J. BENEDETTO AND R. L. BENEDETTO, *A wavelet theory for local fields and related groups*, J. Geom. Anal. **14** (2004), 423–456.

[3] Q. CHEN AND Z. CHANG, *A study on compactly supported orthogonal vector-valued wavelets and wavelet packets*, Chaos, Solit. Fractals. **31** (2007), 1024–1034.

[4] C. K. CHUI AND C. LI, *Non-orthogonal wavelet packets*, SIAM J. Math. Anal. **24** (3) (1993), 712–738.

[5] R. R. COIFMAN, Y. MEYER, S. QUAKE AND M. V. WICKERHAUSER, *Signal processing and compression with wavelet packets*, Technical Report, Yale University, 1990.

[6] L. DEBNATH AND F. A. SHAH, *Wavelet Transforms and Their Applications*, Birkhäuser, New York, 2015.

[7] YU. A. FARKOV, *Orthogonal wavelets with compact support on locally compact Abelian groups*, Izv. Math. **69** (3) (2005), 623–650.

[8] H. K. JIANG, D. F. LI AND N. JIN, *Multiresolution analysis on local fields*, J. Math. Anal. Appl. **294** (2004), 523–532.

[9] A. YU. KHRENNIKOV, V. M. SHELKOVICH AND M. SKOPINA, *p-Adic refinable functions and MRA-based wavelets*, J. Approx. Theory. **161** (2009), 226–238.

[10] W. C. LANG, *Orthogonal wavelets on the Cantor dyadic group*, SIAM J. Math. Anal. **27** (1996), 305–312.

[11] D. LU AND Q. B. FAN, *A class of tight framelet packets*, Czechoslovak Math. J. **61** (2011), 623–639.

[12] S. F. LUKOMSKII, *Multiresolution analysis on product of zero-dimensional Abelian groups*, J. Math. Anal. Appl. **385** (2012), 1162–1178.

[13] S. F. LUKOMSKII, *Step refinable functions and orthogonal MRA on Vilenkin groups*, J. Fourier Anal. Appl. **20** (2014), 42–65.

[14] D. RAMAKRISHNAN AND R. J. VALENZA, *Fourier Analysis on Number Fields*, Graduate Texts in Mathematics 186, Springer-Verlag, New York, 1999.

[15] A. RON AND Z. SHEN, *Affine systems in* $L^2(\mathbb{R}^d)$*: the analysis of the analysis operator*, J. Funct. Anal. **148** (1997), 408–447.

[16] F. A. SHAH, *Construction of wavelet packets on* p*-adic field*, Int. J. Wavelets Multiresolut. Inf. Process. **7** (5) (2009), 553–565.

[17] F. A. SHAH, *Biorthogonal* p*-wavelet packets related to the Walsh polynomials*, J. Classical Anal. **2** (2012), 135–146.

[18] F. A. SHAH AND ABDULLAH, *Nonuniform multiresolution analysis on local fields of positive characteristic*, Complex Anal. Opert. Theory. (2014) DOI 10.1007/s11785-014-0412-0.

[19] F. A. SHAH AND ABDULLAH, *Wave packet frames on local fields of positive characteristic*, Appl. Math. Comput. **249** (2014), 133–141.

[20] F. A. SHAH AND ABDULLAH, *A characterization of tight wavelet frames on local fields of positive characteristic*, J. Contemp. Math. Anal. **49** (6) (2014), 251–259.

[21] F. A. SHAH AND L. DEBNATH, p*-Wavelet frame packets on a half-line using the Walsh-Fourier transform*, Int. Transf. Spec. Funct. **22** (12) (2011), 907–917.

[22] F. A. SHAH AND L. DEBNATH, *Tight wavelet frames on local fields*, Analysis. **33** (2013), 293–307.

[23] Z. SHEN, *Non-tensor product wavelet packets in* $L^2(\mathbb{R}^s)$, SIAM J. Math. Anal. **26** (4) (1995), 1061–1074.

[24] T. STAVROPOULOS AND M. PAPADAKIS, *On the multiresolution analyses of abstract Hilbert spaces*, Bull. Greek Math. Soc. **40** (1998), 79–92.

[25] M. H. TAIBLESON, *Fourier Analysis on Local Fields*, Princeton University Press, 1975.

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