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APPROXIMATION OF FUNCTIONS OF LIPSCHITZ CLASS BY $(N,p_n)(E,1)$ SUMMABILITY MEANS OF CONJUGATE SERIES OF FOURIER SERIES

VISHNU NARAYAN MISHRA AND VAISHALI SONAVANE

Abstract. Analysis of signals or time functions are of great importance, because it convey information or attributes of some phenomenon. In this paper, three theorems on degree of approximation of a signals (or functions) $f \in \text{Lip}(\alpha,r)$ and $\text{Lip}(\xi(t),r)$, $(r \geqslant 1)$ have been established.

1. Introduction

The theory of summability arises from the process of summation of series and the significance of the concept of summability has been strikingly demonstrated in various contexts, e.g., in analytic continuation, quantum mechanics, probability theory, Fourier analysis, approximation theory and fixed point theory. The methods of almost summability and statistical summability have become an active area of research in recent years. Positive approximation processes play an important role in Approximation Theory and appear in a very natural way dealing with approximation of continuous functions, especially one, which requires further qualitative properties such as monotonicity, convexity and shape preservation and so on. The degree of approximation of functions belonging to $\text{Lip}\,\alpha$, $\text{Lip}(\alpha,r)$, $\text{Lip}(\xi(t),r)$ $0<\alpha\leqslant 1$, $1\leqslant r<\infty$ classes by Nörlund (N_p) matrices and general summability matrices have been proved by various investigators like Chandra ([1]–[2]), Khan ([4]–[6]), Mohapatra and Russell [7], Leindler [8] and Mishra and Mishra [9], Mishra et al. [10]. But most of these results are not satisfactory for $\alpha = 1$ are not of $O(n^{-1})$. Therefore this deficiency has motivated to investigate the degree of approximation using generalized Minkowski's inequality cases $0<\alpha\leqslant 1$ and $\alpha = 1$ separately. Recently, Lal and Mishra [15] have proved a theorem on the degree of approximation of functions belonging to the class $\text{Lip}(\alpha,r)$ and $\text{Lip}(\xi(t),r)$ by product summability means of the form $(N,p_n)(E,1)$ of Fourier series. In this paper, we obtain new theorems on degree of approximation of the function belonging to $\text{Lip}(\alpha,r)$ and $\text{Lip}(\xi(t),r)$ by $(N,p_n)(E,1)$ means of conjugate series of Fourier series.

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2. Definitions and notations

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of n^{th} partial sums $\{s_n\}$. Let $\{p_n\}$ be a non-negative sequence of constants, real or complex, and let us write $P_n = \sum\limits_{k=0}^{n} p_k \neq 0 \ \forall n \geqslant 0,\ p_{-1} = 0 = P_{-1}$ and $P_n \to \infty$ as $n \to \infty$.

The sequence to sequence transformation $t_n^N = \sum_{k=0}^{n} p_{n-k} s_k / P_n$ defines the sequence of $\{t_n^N\}$ Nörlund means of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$.

The series $\sum_{n=0}^{\infty} u_n$ is said to be N_p summable to the sum s if $\lim\limits_{n\to\infty} t_n^N$ exists and is equal to a finite number s.

The necessary and sufficient condition for the regularity of (N, p_n) method is

$$\frac{p_n}{P_n} \to 0 \quad \text{as} \quad n \to \infty.$$

Let $E_n^{(1)} = \frac{1}{2^n} \sum\limits_{k=0}^{n} \binom{n}{k} s_k$. If $E_n^{(1)} \to s$ as $n \to \infty$, then $\sum_{n=0}^{\infty} u_n$ is said to be summable s to by the Euler Method $(E, 1)$ and this method is regular (Hardy [3]).

The product summability $(N, p_n)(E, 1)$ is obtained by superimposing (N, p_n)-summability on $(E, 1)$-summability.

The (N, p_n) transform of $(E, 1)$ transform defines the $(N, p_n)(E, 1)$ transform t_n^{NE} of the n^{th} partial sum s_n of the series $\sum_{n=0}^{\infty} u_n$ by

$$t_n^{NE} = \frac{1}{P_n} \sum\nolimits_{k=0}^{n} p_{n-k} E_k^{(1)} = \frac{1}{P_n} \sum\nolimits_{k=0}^{n} p_{n-k} \frac{1}{2^k} \sum\nolimits_{v=0}^{k} \binom{k}{v} s_v$$

$$t_n^{NE} = \frac{1}{P_n} \sum\nolimits_{k=0}^{n} p_k \frac{1}{2^{n-k}} \sum\nolimits_{v=0}^{n-k} \binom{n-k}{v} s_v.$$

If $t_n^{NE} \to s$ as $n \to \infty$ then the infinite series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{s_n\}$ is said to be summable $(N, p_n)(E, 1)$ to the sum s if $\lim\limits_{n\to\infty} t_n{}^{NE}$ exists and is equal to s.

$$\begin{aligned} s_n \to s \Rightarrow E_n^{(1)} \to s,\ \text{as}\ n \to \infty, \quad & (E,1)\ \text{method is regular} \\ \Rightarrow t_n^N(E_n^1) = t_n^{NE} = P_n^{-1} \sum_{k=0}^{n} p_{n-k} s_k \to s,\ \text{as}\ n \to \infty, \quad & (N, p_n)\ \text{method is regular} \\ \Rightarrow (N, p_n)(E, 1)\ \text{method is regular} & \end{aligned}$$

For a 2π-periodic function $f \in L^p := L^p[0, 2\pi],\ p \geqslant 1,$ integrable in the sense of Lebesgue, let

$$s_n(f; x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad n \in N \quad \text{with} \quad s_0(f; x) = \frac{a_0}{2}.$$

Denotes $(n+1)^{th}$ partial sums, called trigonometric polynomial of degree (or order) n, of the Fourier series of f. The Fourier series and trigonometric polynomials play an important role in various scientific and engineering fields, e.g., Lo and Hui [17] use the Fourier series expansion in a very nice way. Based upon the Fourier series expansion, they propose a simple and easy-to-use approach for computing accurate estimates of Black-Scholes double barrier option prices with time-dependent parameters.

The conjugate series of Fourier series is given by

$$\sum_{n=1}^{\infty}(b_n\cos nx - a_n\sin nx) \equiv \sum_{n=1}^{\infty} B_n(x). \tag{1}$$

A signal (function) $f \in \mathrm{Lip}\,\alpha$, if

$$f(x+t)-f(x) = O(|t^{\alpha}|) \quad \text{for} \quad 0<\alpha\leqslant 1, \quad t>0$$

and $f\in \mathrm{Lip}(\alpha,r)$, for $0\leqslant x\leqslant 2\pi$, [4] if

$$\left(\int_0^{2\pi}|f(x+t)-f(x)|^r dx\right)^{1/r} = O\left(|t|^{\alpha}\right), \quad 0<\alpha\leqslant 1, \quad r\geqslant 1, \quad t>0.$$

$f\in \mathrm{Lip}(\xi(t),r)$ [6], if

$$\mathrm{Lip}(\xi(t),r) = \left\{f\in L^p[0,2\pi] : \|f(x+t)-f(x)\|_p = O(\xi(t))\right\}, \quad t>0, \quad r\geqslant 1.$$

L_∞-norm of a function $f:R\to R$ is defined by $\|f\|_\infty = \sup\{|f(x)| : x\in R\}$.

L_r-norm of a function is defined by $\|f\|_r = \left(\int_0^{2\pi}|f(x)|^r dx\right)^{1/r}$, $1\leqslant r<\infty$.

The degree of approximation of a function $f:R\to R$ by trigonometric polynomial t_n of order n under sup norm $\|\ \|_\infty$ is defined by ([12])

$$\|t_n - f\|_\infty = \sup\{|t_n(x)-f(x)| : x\in R\}$$

and $E_n(f)$ of a function $f\in L_r$ is given by $E_n(f) = \min_n \|t_n - f\|_r$.

Some interesting results on summability methods and their applications can be seen in Totur and Canak [11], Mursaleen [13] and Bor and Ozarslan [14], Nigam and Sharma [19], Sulaiman [20], Canak et al. [21], Szász [22], Erdem and Totur [23] and Canak [24].

Abel's Transformation: The formula

$$\sum_{k=m}^{n} u_k v_k = \sum_{k=m}^{n-1} U_k(v_k - v_{k+1}) - U_{m-1}v_m + U_n v_n, \tag{2}$$

where $0\leqslant m\leqslant n$, $U_k = u_0+u_1+u_2+\ldots+u_k$, if $k\geqslant 0$, $U_{-1}=0$, which can be verified, is known as Abel's transformation and will be used extensively in what follows.

If $v_m, v_{m+1},\ldots,v_n$ are non-negative and non-increasing, the left hand side of (2) does not exceed

$2v_m \max_{m-1\leqslant k\leqslant n}|U_k|$ in absolute value. In fact,

$$\begin{aligned}\left|\sum_{k=m}^{n} u_k v_k\right| &\leqslant \max|U_k|\left\{\sum_{k=m}^{n-1}(v_k - v_{k+1}) + v_m + v_n\right\}\\ &= 2v_m \max|U_k|. \end{aligned}\tag{3}$$

We use the following notations throughout this paper

$$\psi_x(t) = \psi(t) = f(x+t) - f(x-t), \quad \Delta p_k = p_k - p_{k+1}, \quad k \geqslant 0$$

$$\left(\widetilde{NE}\right)_n(t) = \frac{1}{2\pi P_n} \sum_{k=0}^{n} p_k \frac{\cos^{n-k}\left(\frac{t}{2}\right)\cos\left(n-k+\frac{1}{2}\right)(t)}{\sin\left(\frac{t}{2}\right)}$$

and $\tau := [1/t]$, the integral of $1/t$.

The study of error estimates of the periodic functions in Lipschitz $[\mathrm{Lip}\,\alpha \subseteq \mathrm{Lip}(\alpha,r) \subseteq \mathrm{Lip}(\xi(t),r)]$ spaces through the summability means of Fourier series, referred as Fourier approximation in the literature, has been of a growing interests over the last decades as mentioned in Srivastava and Singh [18]. The engineers and scientists use properties of Fourier approximation for designing digital filters. Especially, Psarakis and Moustakides [16] presented a new L_2 based method for designing the Finite Impulse Response (FIR) digital filters and get corresponding optimum approximations having improved performance. L_p-space in general, L_2 and L_∞ in particular play an important role in the theory of signals and filters. A good amount of work on L_p-boundedness of Cesàro means, a particular type of Hausdorff matrix, of orthonormal expansions for general exponential weights. In the last four decades, a lot of work has been done on the trigonometric Fourier approximation of $f \in \mathrm{Lip}(\xi(t),r)$, $r>1$. The purpose of this research article is to determine the degree of approximation of a signal (function) $f \in \mathrm{Lip}(\xi(t),r)$, $(r \geqslant 1)$ by product summability transforms of conjugate series of its Fourier series under proper (correct) set of conditions. The product transform $(N,p_n)(E,1)$ plays an important role in signal theory as a double digital filter in finite impulse response in particular [25]. This fact shows that in same way our results are very extensive results.

3. Main results

In this paper, we determine the degree of approximation $f \in \mathrm{Lip}(\alpha,r)$ and $\mathrm{Lip}(\xi(t),r)$, $(r \geqslant 1)$, through trigonometric polynomials.

THEOREM 3.1. *Let (N,p_n) be a regular Nörlund defined by a positive generating sequence $\{p_n\}$.*

Let $f:[0,2\pi] \to \mathbb{R}$ be 2π-periodic, integrable in the sense of Lebesgue and belonging to $\mathrm{Lip}(\alpha,r)$, $(r \geqslant 1)$-class. If either

$$(i) \quad (n+1)p_n = O(P_n), \qquad (ii) \quad \sum_{k=0}^{n-1} |\Delta p_k| = O\left(\frac{P_n}{n+1}\right), \tag{4}$$

or

$$(i)' \quad (n+1)p_n = O(P_n), \qquad (ii)' \quad \sum_{k=0}^{n-1} \left|\Delta \frac{P_k}{k+1}\right| = O\left(\frac{P_n}{n+1}\right). \tag{5}$$

Then the degree of approximation of f by $(N,p_n)(E,1)$ transform

$$\tilde{t}_n^{NE} = \frac{1}{P_n} p_k \frac{1}{2^{n-k}} \sum_{v=0}^{n-k} \binom{n-k}{v} \tilde{s}_v$$

of its conjugate series (1)

$$\widetilde{E}_n\left(f,L^r_{(\alpha)}\right)=\left\|\widetilde{t}_n^{NE}-\widetilde{f}\right\|_{L^r_{(\alpha)}}=\begin{cases}O((n+1)^{-\alpha}), & 0<\alpha<1\\ O\left(\frac{\log(n+1)}{(n+1)}\right), & \alpha=1.\end{cases},\quad \text{for } n=0,1,2,3,\ldots.$$

THEOREM 3.2. *Let (N,p_n) be a regular Nörlund method defined by a positive sequence $\{p_n\}$ satisfying equation (4) or (5). Suppose that $\xi(t)$ be a modulus of continuity such that*

$$\int_0^v \frac{\xi(t)}{t}dt=O(\xi(v)),\quad 0<v<\pi. \tag{6}$$

Let $f:[0,2\pi]\to\mathbb{R}$ be a 2π-periodic, integrable in the sense of Lebesgue and belonging to $\text{Lip}(\xi(t),r)$, $(r\geqslant 1)$-class then its degree of approximation by $\widetilde{t}_n^{NE}$ means of its conjugate series (1) is given by

$$\widetilde{E}_n\left(f,L^r_{(\alpha)}\right)=\left\|\widetilde{t}_n^{NE}-\widetilde{f}\right\|_{L^r_{(\alpha)}}=O\left(\frac{1}{n+1}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\xi(t)}{t^2}dt\right).$$

THEOREM 3.3. *Let (N,p_n) be a regular Nörlund method defined by a positive sequence $\{p_n\}$ satisfying equation (4) or (5).*

Let $f:[0,2\pi]\to\mathbb{R}$ be a 2π-periodic, integrable in the sense of Lebesgue and belonging to $\text{Lip}(\xi(t),r)$, $(r\geqslant 1)$-class

$$\left(\frac{\xi(t)}{t}\right)\quad \textit{is monotonic decreasing in}\quad \left(\frac{\pi}{n+1},\pi\right) \tag{7}$$

then its degree of approximation by $\widetilde{t}_n^{NE}$ means of its conjugate series (1) is given by

$$\widetilde{E}_n\left(f,L^r_{(\alpha)}\right)=\left\|\widetilde{t}_n^{NE}-\widetilde{f}\right\|_{L^r_{(\alpha)}}=O\left(\xi\left(\frac{1}{(n+1)}\right)\log(n+1)\right).$$

4. Lemmas

In order to prove our theorems, we need the following Lemmas.

LEMMA 4.1. *For $0<t\leqslant\frac{\pi}{(n+1)}$, $(\widetilde{NE})_n(t)=O(\tau)$.*

Proof. For $0<t\leqslant\frac{\pi}{(n+1)}$, $\cos nt\leqslant 1$, and $\sin(t/2)\geqslant(t/\pi)$, for $0<t\leqslant\frac{\pi}{(n+1)}$, we have

$$\begin{aligned}\left|(\widetilde{NE})_n(t)\right| &=\left|\frac{1}{2\pi P_n}\sum_{k=0}^{n}p_k\frac{\cos^{n-k}\left(\frac{t}{2}\right)\cos\left(n-k+\frac{1}{2}\right)(t)}{\sin\left(\frac{t}{2}\right)}\right|\\ &\leqslant\frac{1}{2\pi P_n}\sum_{k=0}^{n}p_k\frac{1}{t/\pi}\\ &=O(\tau).\end{aligned}$$

This completes the proof of Lemma 4.1. □

LEMMA 4.2. *For* $\frac{\pi}{(n+1)} < t < \pi$, $(\widetilde{NE})_n(t) = O\left(\frac{\tau^2}{(n+1)}\right)$ *under conditions of our theorem* $\{p_n\}$.

Proof. Case 1. If $(n+1)\,p_n = O(P_n)$, $\sum_{k=0}^{n-1} |\Delta p_k| = O\left(\frac{P}{n+1}\right)$.

For $\sin\frac{t}{2} \geqslant \frac{t}{\pi}$, $\left|\cos^{n-k}\frac{t}{2}\right| \leqslant 1$, $0 < k < n$ and using Abel's lemma, we get

$$\begin{aligned}
&\left|(\widetilde{NE})_n(t)\right| \\
&= \left|\frac{1}{2\pi P_n}\sum_{k=0}^{n} p_k \frac{\cos^{n-k}\left(\frac{t}{2}\right)\cos\left(n-k+\frac{1}{2}\right)(t)}{\sin\left(\frac{t}{2}\right)}\right| \\
&\leqslant \frac{1}{2tP_n}\sum_{k=0}^{n}\left|p_k \cos\left(n-k+\frac{1}{2}\right)t\right| \\
&\leqslant \frac{1}{2tP_n}\left[\sum_{k=0}^{n-1}|p_k - p_{k+1}|\sum_{j=0}^{k}\cos\left(n-j+\frac{1}{2}\right)t + p_n\sum_{k=0}^{n}\cos\left(n-k+\frac{1}{2}\right)t\right] \\
&\leqslant \frac{1}{2tP_n}\left[\sum_{k=0}^{n-1}|\Delta p_k| + p_n\right]\max_{0\leqslant k\leqslant n}\left|\sum_{k=0}^{n}\cos\left(n-k+\frac{1}{2}\right)t\right| \\
&= O\left(\frac{\tau^2}{P_n}\right)\left[\sum_{k=0}^{n-1}|\Delta p_k| + p_n\right] \\
&= O\left(\frac{\tau^2}{P_n}\right)\left(\frac{P_n}{n+1} + \frac{P_n}{n+1}\right) \\
&= O\left(\frac{\tau^2}{n+1}\right).
\end{aligned}$$

Case 2. For $\sin\frac{t}{2} \geqslant \frac{t}{\pi}$, $\left|\cos^{n-k}\frac{t}{2}\right| \leqslant 1$, $0 < k < n$, $(n+1)\,p_n = O(P_n)$ and $\sum_{k=0}^{n-1}\left|\Delta\left(\frac{P_k}{k+1}\right)\right| = O\left(\frac{P_n}{n+1}\right)$, using Abel's lemma, we have

$$\begin{aligned}
&\left|(\widetilde{NE})_n(t)\right| \\
&= \left|\frac{1}{2\pi P_n}\sum_{k=0}^{n} p_k \frac{\cos^{n-k}\left(\frac{t}{2}\right)\cos\left(n-k+\frac{1}{2}\right)(t)}{\sin\left(\frac{t}{2}\right)}\right| \\
&\leqslant \frac{1}{2tP_n}\sum_{k=0}^{n} p_k \cos\left(n-k+\frac{1}{2}\right)(t) \\
&= O\left(\frac{1}{2tP_n}\sum_{k=0}^{n}\left(\frac{P_k}{k+1}\right)\cos\left(n-k+\frac{1}{2}\right)(t)\right)
\end{aligned}$$

$$= O\left[\frac{1}{2tP_n}\left\{\sum_{k=0}^{n-1}\left|\Delta\left(\frac{P_k}{k+1}\right)\right|\sum_{v=0}^{k}\cos(n-v+1)\left(\frac{t}{2}\right)+\frac{P_n}{n+1}\sum_{k=0}^{n}\cos\left(n-k+\frac{1}{t}\right)t\right\}\right]$$

$$= O\left(\frac{1}{2tP_n}\right)\left[\sum_{k=0}^{n-1}\left|\Delta\left(\frac{P_k}{k+1}\right)\right|+\frac{P_n}{n+1}\right]\max_{0\leqslant k\leqslant n}\left|\sum_{k=0}^{n}\cos\left(n-k+\frac{1}{2}\right)t\right|$$

$$= O\left(\frac{1}{2tP_n}\right)\left[\sum_{k=0}^{n-1}\left|\Delta\left(\frac{P_k}{k+1}\right)\right|+\frac{P_n}{n+1}\right]\max_{0\leqslant k\leqslant n}\left|\frac{\cos(2n-k+1)\frac{t}{2}\cos(k+1)\frac{t}{2}}{\sin\frac{t}{2}}\right|$$

$$= O\left(\frac{1}{2t^2P_n}\right)\left[\frac{P_n}{n+1}+\frac{P_n}{n+1}\right] = O\left(\frac{\tau^2}{(n+1)}\right).$$

This completes the proof of Lemma 4.2. □

LEMMA 4.3. *Let* $f \in \mathrm{Lip}(\alpha, r),\ 0<\alpha\leqslant 1,\ r\geqslant 1,$ *then*

$$\left[\int_0^{2\pi}|\psi(x,t)|^r\right]^{\frac{1}{r}} = O\left(|t|^{\alpha}\right).$$

Proof. Clearly,

$$\begin{aligned}|\psi(x,t)| &= |f(x+t)-f(x-t)-f(x)+f(x)|\\ &\leqslant |f(x+t)-f(x)|+|f(x-t)-f(x)|.\end{aligned}$$

Then using Minkowski's inequality, we have

$$\begin{aligned}\left[\int_0^{2\pi}|\psi(x,t)|^r dx\right]^{\frac{1}{r}} &\leqslant \left[\int_0^{2\pi}\{|f(x+t)-f(x)|+|f(x-t)-f(x)|\}^r dx\right]^{\frac{1}{r}}\\ &\leqslant \left[\int_0^{2\pi}|f(x+t)-f(x)|^r dx\right]^{\frac{1}{r}}+\left[\int_0^{2\pi}|f(x-t)-f(x)|^r dx\right]^{\frac{1}{r}}\\ &= O\left(|t|^{\alpha}\right)+O\left(|t|^{\alpha}\right)\\ &= O\left(|t|^{\alpha}\right).\end{aligned}$$

This completes the proof of Lemma 4.3. □

LEMMA 4.4. *If* $f\in \mathrm{Lip}(\xi, r),\ r\geqslant 1,$ *then*

$$\left[\int_0^{2\pi}|\psi(x,t)|^r\right]^{\frac{1}{r}} = O(\xi(t)).$$

Proof. The proof of this Lemma is similar to proof of Lemma 4.3. □

5. Proof of the Theorem 3.1

Let $\widetilde{s}_n(f;x)$ denotes the partial sum of series (1), then we have

$$\widetilde{s}_n(f;x)-\widetilde{f}(x)=\frac{1}{2\pi}\int_0^{\pi}\psi(x,t)\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin(t/2)}dt.$$

Denoting $(E,1)$ means of $\widetilde{s}_n(f;x)$ by $\widetilde{E}_n^{(1)}(x)$, we obtain

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left\{\widetilde{s}_n(f;x)-\widetilde{f}(x)\right\}=\frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\psi(x,t)}{\sin(t/2)}\sum_{k=0}^{n}\binom{n}{k}\cos\left(k+\frac{1}{2}\right)t\,dt$$

$$\begin{aligned}\widetilde{E}_n^{(1)}(x)-\widetilde{f}(x)&=\frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\psi(x,t)}{\sin(t/2)}\text{Re}\left\{\sum_{k=0}^{n}\binom{n}{k}e^{i\left(k+\frac{1}{2}\right)t}\right\}dt\\&=\frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\psi(x,t)}{\sin(t/2)}\text{Re}\{e^{it/2}\left(1+e^{it}\right)^n\}dt\\&=\frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\psi(x,t)}{\sin(t/2)}\text{Re}\{2^n\cos^n\left(\frac{t}{2}\right)e^{i(n+1)t/2}\}dt\\&=\frac{1}{2\pi}\int_0^{\pi}\psi(x,t)\frac{\cos^n\left(\frac{t}{2}\right)\cos(n+1)\left(\frac{t}{2}\right)}{\sin(t/2)}dt.\end{aligned}$$

(N,p_n) means of $\widetilde{E}_n^{(1)}(x)$ i.e. $\widetilde{t}_n^{NE}(x)$ is given by

$$\frac{1}{P_n}\sum_{k=0}^{n}p_k\left\{\widetilde{E}_{n-k}^{(1)}(f;x)-\widetilde{f}(x)\right\}=\frac{1}{2\pi P_n}\sum_{k=0}^{n}p_k\int_0^{\pi}\psi(x,t)\frac{\cos^{n-k}\left(\frac{t}{2}\right)\cos\left(n-k+\frac{1}{2}\right)(t)}{\sin(t/2)}dt$$

$$\widetilde{t}_n^{NE}(x)-\widetilde{f}(x)=\int_0^{\pi}\psi(x,t)\left(\widetilde{NE}\right)_n(t)\,dt \tag{8}$$

Hence by generalized Minkowski's inequality ([12], p.p. 18–19) Lemma (4.3), we shall obtain the proof of this theorem in a quite different method as following:

$$\begin{aligned}\left\|\widetilde{t}_n^{NE}-\widetilde{f}\right\|_{L^r_{(\alpha)}}&=\left[\int_0^{2\pi}\left|\widetilde{t}_n^{NE}(f;x)-\widetilde{f}(x)\right|^r dx\right]^{\frac{1}{r}}\\&=\left[\int_0^{2\pi}\left|\int_0^{\pi}\psi(x,t)\left(\widetilde{NE}\right)_n(t)\,dt\right|^r dx\right]^{\frac{1}{r}}\\&\leqslant\int_0^{\pi}\left\{\int_0^{2\pi}|\psi(x,t)|^r dx\right\}^{\frac{1}{r}}\left|\left(\widetilde{NE}\right)_n(t)\right|dt\\&=O\left(\int_0^{\frac{\pi}{(n+1)}}(t^{\alpha})\left(\widetilde{NE}\right)_n(t)\,dt\right)+O\left(\int_{\frac{\pi}{(n+1)}}^{\pi}(t^{\alpha})\left(\widetilde{NE}\right)_n(t)\,dt\right)\\&=I_1+I_2,\quad\text{(say)}.\end{aligned} \tag{9}$$

If $f \in \operatorname{Lip}\alpha,$ then $\psi(x,t) \in \operatorname{Lip}(\alpha,r)$.

Using Lemma 4.1, we have

$$|I_1| = O\left(\int_0^{\frac{\pi}{(n+1)}} t^{\alpha}\frac{1}{t}dt\right) = O\left((n+1)^{-\alpha}\right). \tag{10}$$

Now applying Lemma (4.2), we get

$$\begin{aligned}|I_2| = O\left(\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{\tau^2}{(n+1)}dt\right) &= O\left[\left(\frac{1}{(n+1)}\right)\int_{\frac{\pi}{(n+1)}}^{\pi} t^{\alpha-2}dt\right]\\ &= \begin{cases} O\left[\left(\frac{1}{(n+1)}\right)\left(\frac{1}{1-\alpha}\right)\left(\frac{1}{(n+1)^{\alpha-1}} - \pi^{\alpha-1}\right)\right], & 0<\alpha<1\\ O\left(\frac{\log(n+1)}{n+1}\right), & \alpha = 1\end{cases}\\ &= \begin{cases} O\left((n+1)^{-\alpha}\right), & 0<\alpha<1\\ O\left(\frac{\log(n+1)}{n+1}\right), & \alpha = 1\end{cases}\end{aligned} \tag{11}$$

On combining equations (9) to (11), we have

$$\widetilde{E}_n\left(f, L^r_{(\alpha)}\right) = \left\|\widetilde{t}_n^{NE} - \widetilde{f}\right\|_{L^r_{(\alpha)}} = \begin{cases} O((n+1)^{-\alpha}), & 0<\alpha<1\\ O\left(\frac{\log(n+1)}{(n+1)}\right), & \alpha = 1\end{cases}$$

for $n = 0,1,2,3,\ldots$.

This completes the proof of Theorem 3.1. □

6. Proof of the Theorem 3.2

Using (9), Lemma 4.4 and generalized Minkowski's inequality

$$\begin{aligned}\left\|\widetilde{t}_n^{NE} - \widetilde{f}\right\| &= \left[\int_0^{2\pi}\left|\widetilde{t}_n^{NE}(f;x) - \widetilde{f}(x)\right|^r dx\right]^{\frac{1}{r}}\\ &= \left[\int_0^{2\pi}\left|\int_0^{\pi}\psi(x,t)(\widetilde{NE})_n(t)dt\right|^r dx\right]^{\frac{1}{r}}\\ &\leqslant \int_0^{\pi}\left\{\int_0^{2\pi}|\psi(x,t)|^r dx\right\}^{\frac{1}{r}}\left|(\widetilde{NE})_n(t)\right|dt\\ &= \int_0^{\pi} O(\xi(t))\left|(\widetilde{NE})_n(t)\right|dt\end{aligned}$$

$$
\begin{aligned}
&= O\left(\int_0^{\frac{\pi}{(n+1)}} (\xi(t))(\widetilde{NE})_n(t)dt\right) + O\left(\int_{\frac{\pi}{(n+1)}}^{\pi} (\xi(t))(\widetilde{NE})_n(t)dt\right) \\
&= I_1' + I_2'. \qquad (12)
\end{aligned}
$$

Now, using Lemma 4.1 and (6), we get

$$
\begin{aligned}
I_1' &= O\left(\int_0^{\frac{\pi}{(n+1)}} \frac{\xi(t)}{t}dt\right) \\
&= O\left((n+1)\int_0^{\frac{1}{(n+1)}} \frac{\xi(t)}{t}dt\right) \\
&= O\left(\xi\left(\frac{1}{n+1}\right)\right), \qquad (13)
\end{aligned}
$$

in view of second mean value theorem for integrals and $\xi\left(\frac{\pi}{n+1}\right) \leqslant \pi\xi\left(\frac{1}{n+1}\right)$, for $\frac{\pi}{n+1} \geqslant \frac{1}{n+1}$.

Applying Lemma 4.2, we have

$$
\begin{aligned}
I_2' &= O\left(\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{\xi(t)\tau^2}{(n+1)}dt\right) \\
&= O\left(\frac{1}{(n+1)}\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{\xi(t)}{t^2}dt\right). \qquad (14)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{1}{(n+1)}\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{\xi(t)}{t^2}dt &\geqslant \frac{\pi}{(n+1)}\xi\left(\frac{1}{(n+1)}\right)\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{1}{t^2}dt \\
&= \frac{\pi}{(n+1)}\xi\left(\frac{1}{(n+1)}\right)\left\{-\frac{1}{t}\right\}_{\frac{\pi}{(n+1)}}^{\pi} \\
&= \frac{\pi}{(n+1)}\xi\left(\frac{1}{(n+1)}\right)\left(-\frac{1}{\pi} + \frac{n+1}{\pi}\right)
\end{aligned}
$$

It then follows that

$$
\xi\left(\frac{1}{(n+1)}\right) = O\left(\frac{1}{(n+1)}\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{\xi(t)}{t^2}dt\right). \qquad (15)
$$

On combining equation from (12) to (15), we get

$$\widetilde{E}_n\left(f,L^r_{(\xi)}\right)=\left\|\widetilde{t}_n^{NE}-\widetilde{f}\right\|_{L^r_{(\xi)}}=O\left(\frac{1}{(n+1)}\int_{\frac{\pi}{(n+1)}}^{\pi}\frac{\xi(t)}{t^2}dt\right). \tag{16}$$

This completes the proof of Theorem 3.2. □

7. Proof of the Theorem 3.3

Following the proof of the theorem 3.2, we obtain

$$\left\|\widetilde{t}_n^{NE}-\widetilde{f}\right\|_{L^r_{(\xi)}}=O\left(\frac{1}{(n+1)}\int_{\frac{\pi}{(n+1)}}^{\pi}\frac{\xi(t)}{t^2}dt\right). \tag{17}$$

Applying condition (7) and mean value theorem for integral calculus, (17) becomes

$$\begin{aligned}\left\|\widetilde{t}_n^{NE}-\widetilde{f}\right\|_{L^r_{(\xi)}}&=O\left(\xi\left(\frac{1}{(n+1)}\right)\int_{\frac{\pi}{(n+1)}}^{\pi}\frac{1}{t}dt\right)\\&=O\left(\xi\left(\frac{1}{(n+1)}\right)\log(n+1)\right),\end{aligned}$$

in view of second mean value theorem for integrals and $\xi\left(\frac{\pi}{n+1}\right)\leqslant\pi\xi\left(\frac{1}{n+1}\right)$, for $\frac{\pi}{n+1}\geqslant\frac{1}{n+1}$.

This completes the proof of the Theorem 3.3. □

REMARK 1. In the proof of theorem 3.1 of Lal and Mishra [15], the estimate for the case $\alpha=1$ is $O\left(\frac{\log(n+1)\pi}{n+1}\right)$.

Since $\frac{1}{n+1}\leqslant\frac{\log(n+1)\pi}{n+1}$, the π is not needed in above estimation.

8. Corollaries

In addition, several corollaries are derived from our results as well as those obtained previously by others. The important particular cases of $(N,1)(E,1)$ are

(1) $(C,1)(E,1)$ if $p_n=1$.

(2) $(C,\delta)(E,1)$ if $p_n=\binom{n+\delta-1}{\delta-1}$, $\delta>0$.

1. If $\xi(t)=t^{\alpha-\beta}$, $0\leqslant\beta<\alpha\leqslant1$ in theorem 3.2, then

$$\left\|\widetilde{t}_n^{NE}-\widetilde{f}\right\|=\begin{cases}O\left(\frac{1}{(n+1)^{\alpha-\beta}}\right), & 0\leqslant\beta<\alpha<1.\\ O\left(\frac{\log(n+1)}{n+1}\right), & 0=\beta<\alpha=1.\end{cases}$$

Proof. Putting $\xi(t) = t^{\alpha-\beta}$, $0 \leqslant \beta < \alpha \leqslant 1$ in (16), we have

$$\left\|\widetilde{t}_n^{NE} - \widetilde{f}\right\| = O\left(\frac{1}{(n+1)} \int\limits_{\frac{\pi}{(n+1)}}^{\pi} t^{\alpha-\beta-2} dt\right)$$

$$= \begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha-\beta}}\right), & 0 \leqslant \beta < \alpha < 1. \\ O\left(\dfrac{\log(n+1)}{n+1}\right), & 0 = \beta < \alpha = 1. \end{cases}$$

2. If $p_n = 1$ $\forall n \geqslant 1$ in theorem 3.1, then for $f \in \mathrm{Lip}(\alpha, r)$, the degree of approximation by $(C,1)(E,1)$ means is given by

$$\widetilde{t}_n^{CE} = \frac{1}{n+1} \sum\nolimits_{k=0}^{n} \frac{1}{2^k} \sum\nolimits_{v=0}^{k} \binom{k}{v} \widetilde{s_v}$$

of its conjugate series (1) is given by

$$\left\|\widetilde{t}_n^{CE} - \widetilde{f}\right\|_r = \begin{cases} O\left(\left((n+1)^{-\alpha}\right)\right), & 0 < \alpha < 1. \\ O\left(\dfrac{\log(n+1)}{n+1}\right), & \alpha = 1. \end{cases}$$

3. If $p_n = 1$ $\forall n \geqslant 1$ in theorem 3.2, then for $f \in \mathrm{Lip}(\xi(t), r)$, the degree of approximation by $(C,1)(E,1)$ means t_n^{CE} is given by

$$\left\|\widetilde{t}_n^{CE} - \widetilde{f}\right\|_{L^r_{(\xi)}} = O\left(\frac{1}{(n+1)} \int\limits_{\frac{\pi}{(n+1)}}^{\pi} \frac{\xi(t)}{t^2} dt\right).$$

4. If $\xi(t) = t^{\alpha}$, and $r \to \infty$ in theorem 3.1, then for $f \in \mathrm{Lip}\,\alpha$, $0 < \alpha \leqslant 1$, the degree of approximation under supremem norm is given by

$$\left\|\widetilde{t}_n^{NE} - \widetilde{f}\right\|_{\infty} = ess \sup_{0 \leqslant x \leqslant 2\pi} \left\{\widetilde{t}_n^{NE}(x) - \widetilde{f}(x)\right\} = \begin{cases} O\left(\left((n+1)^{-\alpha}\right)\right), & 0 < \alpha < 1. \\ O\left(\dfrac{\log(n+1)}{n+1}\right), & \alpha = 1. \end{cases}$$

REMARK 2. The independent proofs of Cor 2, Cor 3 and Cor 4 can be derived along the same lines as the theorems.

9. Example

In this example, we see how the ${E_n}^1$ and $t_n^N(f;x)$ (Nörlund (N_p)) summability of partial sums of a Fourier series is better behaved than the sequence of partial sums $s_n(x)$ itself.

Let

$$f(x)=\begin{cases}-1, & -\pi\leqslant x<0,\\ 1, & 0\leqslant x<\pi,\end{cases}$$

with $f(x+2\pi)=f(x)$ for all real x. Fourier series of $f(x)$ is given by

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx,\quad -\pi\leqslant x\leqslant\pi. \tag{18}$$

Then n^{th} partial sum $s_n(x)$ of Fourier series (18) is given by

$$s_n(x)=\frac{4}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\ldots+\frac{1}{n}\sin nx\right). \tag{19}$$

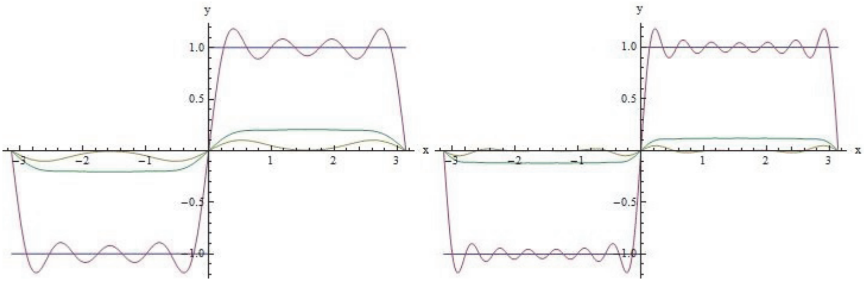

Figure 1: *Graph of $f(x)$ (blue), $s_n(x)$ (pink), $E_n{}^1(f;x)$ (yellow), $t_n^N(f;x)$ (green), $n=7$ and* 14.

The E_n^1 summability is defined as the n^{th} partial sum of E_n^1 summability and we denote it by E_n^1. If

$$E_n^1(f;x)=\frac{1}{(2)^n}\sum_{k=0}^{n}\binom{n}{k}s_k(f;x)\to s,\quad \text{as}\quad n\to\infty \tag{20}$$

Now, N_p take to be the Nörlund matrix generated by $p_n=n+1$, then Nörlund means N_p is given by

$$t_n^N(f;x)=\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)s_k(f;x). \tag{21}$$

In the figure 1, we observe that $E_n^1(f;x)$ and $t_n^N(f;x)$ converges to $f(x)$ faster than $s_n(x)$ in the interval $[-\pi,\pi]$. We further note that near the points of discontinuities i.e. $-\pi$, 0 and π, the graph of s_7 and s_{14} show peaks and move closer the line passing through points of discontinuity as n increases (Gibbs Phenomenon), but in the graph of $E_n^1(f;x)$ and $t_n^N(f;x)$, $n = 7,14$ the peaks become flatter. The Gibbs Phenomenon is an overshoot a peculiarity of the Fourier series and other eigen function series at a simple discontinuity i.e. the convergence of Fourier series is very slow at the point of discontinuity. Thus the product summability means of the Fourier series of $f(x)$ overshoot the Gibbs Phenomenon and show the smoothing effect of the method. Thus $E_n^1(f;x)$ and $t_n^N(f;x)$ are the better approximant than $s_n(x)$.

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Vishnu Narayan Mishra
Applied Mathematics and Humanities Department
Sardar Vallabhbhai National Institute of Technology
Ichchhanath Mahadev Dumas Road
Surat, Surat-395 007 (Gujarat), India
and
L. 1627 Awadh Puri Colony Beniganj, Phase-III
Opposite – Industrial Training Institute (I.T.I.)
Ayodhya Main Road, Faizabad-224 001 (Uttar Pradesh), India
e-mail: vishnunarayanmishra@gmail.com;
vishnu_narayanmishra@yahoo.co.in

Vaishali Sonavane
Applied Mathematics and Humanities Department
Sardar Vallabhbhai National Institute of Technology
Ichchhanath Mahadev Dumas Road
Surat, Surat-395 007 (Gujarat), India
e-mail: vaishalisnvn@gmail.com