CERTAIN NON–LINEAR DIFFERENTIAL POLYNOMIALS
HAVING COMMON POLES SHARING A NON ZERO POLYNOMIAL WITH FINITE WEIGHT

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Abstract. With the notion of weighted sharing we study the uniqueness property of meromorphic functions having common poles when certain non-linear differential polynomials share a non zero polynomial function. Our theorems in the paper will improve, extend and supplement a number of recent results in a more compact and convenient way.

1. Introduction, definitions and results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane.

Let \( f \) and \( g \) be two non-constant meromorphic functions and let \( a \) be a finite complex number. We say that \( f \) and \( g \) share \( a \) CM, provided that \( f - a \) and \( g - a \) have the same zeros with the same multiplicities. Similarly, we say that \( f \) and \( g \) share \( a \) IM, provided that \( f - a \) and \( g - a \) have the same zeros ignoring multiplicities. In addition we say that \( f \) and \( g \) share \( \infty \) CM, if \( 1/f \) and \( 1/g \) share 0 CM, and we say that \( f \) and \( g \) share \( \infty \) IM, if \( 1/f \) and \( 1/g \) share 0 IM.

We adopt the standard notations of value distribution theory (see [6]). We denote by \( T(r) \), the maximum of \( T(r, f) \) and \( T(r, g) \). The notation \( S(r) \) denotes any quantity satisfying \( S(r) = o(T(r)) \) as \( r \to \infty \), outside of a possible exceptional set of finite linear measure.

A finite value \( z_0 \) is said to be a fixed point of \( f(z) \) if \( f(z_0) = z_0 \). For a positive integer \( m \) and a number \( \mu \), let \( m^* = \chi_\mu m \), where \( \chi_\mu = 0 \) if \( \mu = 0 \) and \( \chi_\mu = 1 \) if \( \mu \neq 0 \). Throughout this paper, we need the following definition.

\[
\Theta(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)},
\]

where \( a \) is a value in the extended complex plane.

We start with the following famous theorem of W.K. Hayman (see [5], Corollary of Theorem 9) obtained in 1959.

**Theorem A.** Let \( f \) be a transcendental meromorphic function and \( n(\geq 3) \) is an integer. Then \( f^n f' = 1 \) has infinitely many solutions.


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In 1997, Yang and Hua obtained the following uniqueness result corresponding to Theorem A:

**Theorem B.** [17] Let $f$ and $g$ be two non-constant meromorphic functions, $n \geq 11$ be a positive integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share a CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where $c_1$, $c_2$ and $c$ are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant $t$ such that $t^{n+1} = 1$.

Using the idea of sharing fixed points, in 2002, M.L. Fang and H.L. Qiu further extended Theorem B in the following manner.

**Theorem C.** [4] Let $f$ and $g$ be two non-constant meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where $c_1$, $c_2$ and $c$ are three nonzero complex numbers satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a complex number $t$ such that $t^{n+1} = 1$.

For the past few years researchers have become more interested in the value sharing of nonlinear differential polynomials which are the $k$-th derivative of some linear expression of $f$ and $g$.

In 2010, J.F. Xu, F. Lu and H.X. Yi proved the following results.

**Theorem D.** [15] Let $f$ and $g$ be two non-constant meromorphic functions, and let $n$, $k$ be two positive integers with $n > 3k + 10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share $z$ CM, $f$ and $g$ share $\infty$ IM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where $c_1$, $c_2$ and $c$ are three constants satisfying $4n^2(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant $t$ such that $t^n = 1$.

**Theorem E.** [15] Let $f$ and $g$ be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n}$, and let $n$, $k$ be two positive integers with $n > 3k + 12$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share $z$ CM, $f$ and $g$ share $\infty$ IM, then $f \equiv g$.

In the mean time in 2008 Zhang and Lin [21, 22] obtained a more generalised result for entire function as follows.

**Theorem F.** [21, 22] Let $f$ and $g$ be two non-constant entire functions, and $n$, $m$, $k$ be three positive integers with $n > 2k + m^* + 4$. Suppose $(f^n(\mu f^m + \lambda))^{(k)}$, $(g^n(\mu g^m + \lambda))^{(k)}$ share 1 CM, where $\lambda$, $\mu$ are constants such that $|\lambda| + |\mu| \neq 0$. If

(i) $\lambda \mu \neq 0$, and gcd $(n,m) = d$, then $f^d \equiv g^d$; especially when $d = 1$, $f \equiv g$. or while $m = 1$ and $\Theta(\infty, f) > \frac{2}{n}$, then $f \equiv g$;

(ii) if $\lambda \mu = 0$, then either $f = tg$, where $t$ is a constant satisfying $t^{n+m^*} = 1$ or $f = c_1 e^{cz}$, $g = c_2 e^{-cz}$, where $c_1$, $c_2$ and $c$ are three constants such that $(-1)^k \lambda^2(c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1$ or $(-1)^k \mu^2(c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1$.

In 2001 an idea of gradation of sharing of values was introduced in {[8], [9]} which measures how close a shared value is to being share CM or to being shared IM. This notion is known as weighted sharing and is defined as follows.

**Definition 1.** [8, 9] Let $k$ be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f$, $g$ share the value $a$ with weight $k$. 
The definition implies that if \( f, g \) share a value \( a \) with weight \( k \) then \( z_0 \) is an \( a \)-point of \( f \) with multiplicity \( m (\leq k) \) and only if it is an \( a \)-point of \( g \) with multiplicity \( m (\leq k) \) and \( z_0 \) is an \( a \)-point of \( f \) with multiplicity \( m (\leq k) \) and only if it is an \( a \)-point of \( g \) with multiplicity \( n (> k) \), where \( m \) is not necessarily equal to \( n \).

We write \( f, g \) share \((a, k)\) to mean that \( f, g \) share the value \( a \) with weight \( k \). Clearly if \( f, g \) share \((a, k)\), then \( f, g \) share \((a, p)\) for any integer \( p, 0 \leq p < k \). Also we note that \( f, g \) share a value \( a \) IM or CM if and only if \( f, g \) share \((a, 0)\) or \((a, \infty)\) respectively. If \( a \) is a small function we define that \( f, g \) share \((a, l)\) which means \( f \) and \( g \) share \( a \) with weight \( l \) if \( f - a \) and \( g - a \) share \((0, l)\).

With the notion of weighted sharing in 2011, X. Q. Lin [12] improved Theorem F as follows.

**Theorem G.** [12] Let \( f \) and \( g \) be two non-constant entire functions, and let \( n, m, \) and \( k \) be three positive integers. Suppose \( (f^n(\mu f^m + \lambda))^k, (g^n(\mu g^m + \lambda))^k \) share \((1, l)\), where \( \lambda, \mu \) are constants such that \(|\lambda| + |\mu| \neq 0 \) and one of the following conditions holds:

(i) \( l = 2 \) and \( n > 2k + m^* + 4 \);
(ii) \( l = 1 \) and \( n > \frac{5k+3m^*+9}{2} \);
(iii) \( l = 0 \) and \( n > 5k + 4m^* + 7 \).

then conclusion of Theorem F holds.

In 2012 Wang and Luo [13] investigated Theorem F for meromorphic functions and replaced value sharing by fixed point sharing.

**Theorem H.** [13] Let \( f \) and \( g \) be two transcendental meromorphic functions and \( n, m, k \) be three positive integers with \( n > 3k + m^* + 7 \). Suppose \( (f^n(\mu f^m + \lambda))^k, (g^n(\mu g^m + \lambda))^k \) share \((z, \infty)\), \( f, g \) share \((\infty, 0)\); where \( \lambda(\neq 0), \mu \) be constants. then one of the following results holds:

(i) if \( \mu = 0 \), then either \( f = tg \), where \( t \) is a constant satisfying \( t^n = 1 \), or \( k = 1, f = c_1e^{cz^2}, g = c_2e^{-cz^2} \), where \( c_1, c_2 \) and \( c \) are three constants such that \( 4\lambda^2(c_1c_2)^{n|nc|^2} = -1 \).
(ii) \( \mu \neq 0 \) and \( m \geq 2 \) and \( \text{gcd} (n,m) = 1 \), then \( f \equiv g \).
(iii) If \( \mu \neq 0 \) and \( m = 1 \) then either \( f \equiv g \) or \( f = -\frac{\lambda(h^n - 1)}{\mu(h^n + 1 - 1)}, \ f = -\frac{\lambda(h^n - 1)}{\mu(h^n + 1 - 1)}; \)

where \( h \) is a non-constant meromorphic function.

Also J. Wang, W. Lu and Y. Chen [14] investigated the IM value sharing counterpart of Theorem H as follows.

**Theorem I.** [14] Let \( f \) and \( g \) be two non-constant meromorphic functions, and \( n, k, m \) be three positive integers with \( n > 9k + 6m^* + 13 \). Suppose \( (f^n(\mu f^m + \lambda))^k, (g^n(\mu g^m + \lambda))^k \) share \((1, 0)\), where \( \lambda, \mu \) are constants such that \(|\lambda| + |\mu| \neq 0 \), and \( f, g \) share \((\infty, 0)\).
If \( \lambda \mu \neq 0 \), \( m > 1 \) and \( (n,n+m) = 1 \), or while \( m = 1 \) and \( \Theta(\infty, f) > 2/n \), then \( f \equiv g \);

(ii) if \( \lambda \mu = 0 \), then either \( f = tg \), where \( t \) is a constant satisfying \( t^{n+m^*} = 1 \) or \( f = c_1 e^{cz^2}, \ g = c_2 e^{-cz^2} \), where \( c_1, c_2 \) and \( c \) are three constants such that

\[
[(n+m^*)c]^{2k} = 1 \text{ or } (-1)^k \lambda^2 (c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1.
\]

The purpose of the paper is to unify all the above mentioned theorems into a single result under relaxed sharing hypothesis, which will improve, extend and generalize all the results discussed above in a large extent. We present the main result as follows.

**THEOREM 1.** Let \( f \) and \( g \) be two transcendental meromorphic functions sharing \( (\infty,0) \); \( (f^m(\mu f^m + \lambda))^{(k)}, \ (g^n(\mu g^m + \lambda))^{(k)} \) share \( (p(z),l) \), where \( p(z) \) be a nonzero polynomial with \( \deg(p) = r, \ \lambda, \ \mu \) are constants such that \( |\lambda| + |\mu| \neq 0 \) and \( n, m, k \) be three positive integers. Suppose one of the following conditions hold:

(a) \( l \geq 3 \text{ and } n > \max\{3k+m^*+6,k+2r\} \);

(b) \( l = 2 \text{ and } n > \max\{3k+m^*+8,k+2r\} \);

(c) \( l = 1 \text{ and } n > \max\{4k+\frac{3m^*}{2}+9,k+2r\} \);

(d) \( l = 0 \text{ and } n > \max\{9k+4m^*+14,k+2r\} \).

Then

(i) if \( \lambda \mu \neq 0 \) and (a) \( m = 1, \ \Theta(\infty, f) + \Theta(\infty, g) > 4/n \); or (b) \( m \geq 2 \) and for some constant \( t \), satisfying \( t^d \equiv 1 \), we have \( f \equiv tg \), where \( d = (n+m,n) \).

(ii) if \( \lambda \mu = 0 \), then either \( f = tg \), where \( t \) is a constant satisfying \( t^{n+m^*} = 1 \) or if \( p(z) \) is not a constant, then \( f = c_1 e^{Q(z)}, \ g = c_2 e^{-Q(z)} \), where \( Q(z) = \int_0^z p(z)dz \), \( c_1, c_2 \) and \( c \) are constants such that \( a^{2k}_m(c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = -1 \);

if \( p(z) \) is a nonzero constant \( b \), then \( f = c_3 e^{cz^2}, \ g = c_4 e^{-cz^2} \), where \( c_3, c_4 \) and \( c \) are constants such that \( (-1)^k a^{2k}_m(c_3 c_4)^{n+m^*} [(n+m^*)c]^{2k} = 2^2 \), when \( m^* = m \) and \( a_{m^*} = \mu \), when \( m^* = 0 \).

**THEOREM 2.** Let \( f \) and \( g \) be two transcendental entire functions sharing \( (\infty,0) \); \( (f^m(\mu f^m + \lambda))^{(k)}, \ (g^n(\mu g^m + \lambda))^{(k)} \) share \( (p(z),l) \), where \( p(z) \) be a nonzero polynomial with \( \deg(p) = r, \ \lambda, \ \mu \) are constants such that \( |\lambda| + |\mu| \neq 0 \) and \( n, m, k \) be three positive integers. Suppose one of the following conditions holds:

(a) \( l \geq 2 \text{ and } n > \max\{2k+m^*+4,k+2r\} \);

(b) \( l = 1 \text{ and } n > \max\{\frac{5k+3m^*+9}{2},k+2r\} \);

(c) \( l = 0 \text{ and } n > \max\{4k+3m^*+6,k+2r\} \).

Then

(i) if \( \lambda \mu \neq 0 \) and (a) \( m = 1, \ \Theta(\infty, f) \geq 2/n \), we have \( f \equiv tg \), where \( d = (n+m,n) \).

(ii) if \( \lambda \mu = 0 \), then either \( f = tg \), where \( t \) is a constant satisfying \( t^{n+m^*} = 1 \) or
if \( p(z) \) is not a constant, then \( f = c_1e^{cQ(z)}, \ g = c_2e^{-cQ(z)} \), where \( Q(z) = \int_0^z p(z)dz \), \( c_1, c_2 \) and \( c \) are constants such that \( a_{m^*}^2(c_1c_2)^{n+m^*}[(n+m^*)c]^2 = -1 \);

if \( p(z) \) is a nonzero constant \( b \), then \( f = c_3e^{cz}, \ g = c_4e^{-cz} \), where \( c_3, c_4 \) and \( c \) are constants such that \((-1)^k a_{m^*}^2(c_3c_4)^{n+m^*}[(n+m^*)c]^2k = b^2 \), where \( a_{m^*} = \mu \), when \( m^* = m \) and \( a_{m^*} = \lambda \), when \( m^* = 0 \).

**Remark 1.** In both the theorems when \( p(z) \) is a constant \( f \) and \( g \) can be taken as non-constant instead of transcendental.

We now explain following definitions and notations which are used in the paper.

**Definition 2.** [7] Let \( p \) be a positive integer and \( a \in \mathbb{C} \cup \{\infty\} \).

(i) \( N(r,a;f \geq p) \) (\( N(r,a;f \geq p) \)) denotes the counting function (reduced counting function) of those \( a \)-points of \( f \) whose multiplicities are not less than \( p \).

(ii) \( N(r,a;f \leq p) \) (\( N(r,a;f \leq p) \)) denotes the counting function (reduced counting function) of those \( a \)-points of \( f \) whose multiplicities are not greater than \( p \).

**Definition 3.** [11, cf.[18]] For \( a \in \mathbb{C} \cup \{\infty\} \) and a positive integer \( p \) we denote by \( N_p(r,a;f) \) the sum \( N(r,a;f) + N(r,a;f \geq 2) + \ldots N(r,a;f \geq p) \). Clearly \( N_1(r,a;f) = N(r,a;f) \).

**Definition 4.** Let \( a, b \in \mathbb{C} \cup \{\infty\} \). Let \( p \) be a positive integer. We denote by \( N(r,a;f \geq p \mid g = b) \) (\( N(r,a;f \geq p \mid g \neq b) \)) the reduced counting function of those \( a \)-points of \( f \) with multiplicities \( \geq p \), which are the \( b \)-points (not the \( b \)-points) of \( g \).

**Definition 5.** [cf.[1], 2] Let \( f \) and \( g \) be two non-constant meromorphic functions such that \( f \) and \( g \) share the value 1 IM. Let \( z_0 \) be a 1-point of \( f \) with multiplicity \( p \), a 1-point of \( g \) with multiplicity \( q \). We denote by \( N_L(r,1;f) \) the counting function of those 1-points of \( f \) and \( g \) where \( p > q \), by \( N_E^{(1)}(r,1;f) \) the counting function of those 1-points of \( f \) and \( g \) where \( p = q = 1 \) and by \( N_E^{(2)}(r,1;f) \) the counting function of those 1-points of \( f \) and \( g \) where \( p = q \geq 2 \), each point in these counting functions is counted only once. In the same way we can define \( N_L(r,1;g), N_E^{(1)}(r,1;g), N_E^{(2)}(r,1;g) \).

**Definition 6.** [cf.[1], 2] Let \( k \) be a positive integer. Let \( f \) and \( g \) be two non-constant meromorphic functions such that \( f \) and \( g \) share the value 1 IM. Let \( z_0 \) be a 1-point of \( f \) with multiplicity \( p \), a 1-point of \( g \) with multiplicity \( q \). We denote by \( N_{f,k}(r,1;g) \) the reduced counting function of those 1-points of \( f \) and \( g \) such that \( p > q = k \). \( N_{g,k}(r,1;f) \) is defined analogously.

**Definition 7.** [8, 9] Let \( f, g \) share a value \( a \) IM. We denote by \( N_*(r,a;f,g) \) the reduced counting function of those \( a \)-points of \( f \) whose multiplicities differ from the multiplicities of the corresponding \( a \)-points of \( g \).

Clearly \( N_*(r,a;f,g) \equiv N_*(r,a;g,f) \) and \( N_*(r,a;f,g) = N_L(r,a;f) + N_L(r,a;g) \).

**Definition 8.** Let \( a,b_1,b_2,\ldots,b_q \in \mathbb{C} \cup \{\infty\} \). We denote by \( N(r,a;f \mid g \neq b_1,b_2,\ldots,b_q) \) the counting function of those \( a \)-points of \( f \), counted according to multiplicity, which are not the \( b_i \)-points of \( g \) for \( i = 1,2,\ldots,q \).
2. Lemmas

Let $F$ and $G$ be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by $H$ the function as follows:

$$H = \left( \frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G-1} \right),$$

and

$$V = \left( \frac{F'}{F - 1} - \frac{F'}{F} \right) - \left( \frac{G'}{G - 1} - \frac{G'}{G} \right).$$

**Lemma 1.** [13] Let $f$ be a non-constant meromorphic function and let $a_n(z)(\neq 0)$, $a_{n-1}(z)$, ..., $a_0(z)$ be meromorphic functions such that $T(r,a_i(z)) = S(r,f)$ for $i = 0, 1, 2, \ldots, n$. Then

$$T(r,a_nf^n + a_{n-1}f^{n-1} + \ldots + a_1f + a_0) = nT(r,f) + S(r,f).$$

**Lemma 2.** [20] Let $f$ be a non-constant meromorphic function, and $p, k$ be positive integers. Then

$$N_p \left( r, 0; f^{(k)} \right) \leq T \left( r, f^{(k)} \right) - T(r,f) + N_{p+k}(r,0;f) + S(r,f),$$

and

$$N_p \left( r, 0; f^{(k)} \right) \leq k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f).$$

**Lemma 3.** [10] If $N(r,0;f^{(k)} | f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of $f$, where a zero of $f^{(k)}$ is counted according to its multiplicity then

$$N(r,0;f^{(k)} | f \neq 0) \leq k\overline{N}(r,\infty;f) + N(r,0;f | < k) + k\overline{N}(r,0;f | \geq k) + S(r,f).$$

**Lemma 4.** Suppose that $f$ and $g$ be two non-constant meromorphic functions. Let $F = \left[ f^n(\mu f^m + \lambda) \right]^{(k)}$, $G = \left[ g^n(\mu g^m + \lambda) \right]^{(k)}$, where $n, k, m$ are positive integers. If $f$, $g$ share $\infty$ IM and $V \equiv 0$, then $F \equiv G$.

**Proof.** Suppose $V \equiv 0$. Then by integration we obtain

$$1 - \frac{1}{F} \equiv A(1 - \frac{1}{G}).$$

If $z_0$ is a pole of $f$ then it is a pole of $g$. Hence from the definition of $F$ and $G$ we have $\frac{1}{F(z_0)} = 0$ and $\frac{1}{G(z_0)} = 0$. So $A = 1$ and hence $F \equiv G$. \(\square\)

**Lemma 5.** [11] Let $f_1$ and $f_2$ be two non-constant meromorphic functions satisfying $\overline{N}(r,0;f_i) + \overline{N}(r,\infty;f_i) = S(r,f_1,f_2)$ for $i = 1, 2$. If $f_1^s f_2^t - 1$ is not identically zero for arbitrary integers $s$ and $t(|s| + |t| > 0)$, then for any positive $\varepsilon$, we have

$$N_0(r,1;f_1,f_2) \leq \varepsilon T(r) + S(r,f_1,f_2),$$
where \( N_0(r, 1; f_1, f_2) \) denotes the reduced counting function related to the common 1-points of \( f_1 \) and \( f_2 \) and \( T(r) = T(r, f_1) + T(r, f_2) \), \( S(r, f_1, f_2) = o(T(r)) \) as \( r \to \infty \) possibly outside a set of finite linear measure.

**Lemma 6.** [6] Suppose that \( f \) is a non-constant meromorphic function, \( k \geq 2 \) is an integer. If

\[
N(r, \infty, f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S(r, \frac{f'}{f}),
\]

then \( f = e^{az+b} \), where \( a \neq 0 \), \( b \) are constants.

**Lemma 7.** Let \( f \) and \( g \) be two non-constant meromorphic functions and \( k, m, n > 3k + m^* \) be three positive integers. If \( [f^n(\mu f^m + \lambda)]^{(k)} = [g^n(\mu g^m + \lambda)]^{(k)} \), then \( f^n(\mu f^m + \lambda) \equiv g^n(\mu g^m + \lambda) \).

**Proof.** We have \( [f^n(\mu f^m + \lambda)]^{(k)} = [g^n(\mu g^m + \lambda)]^{(k)} \).

When \( k \geq 2 \), integrating we get

\[
[f^n(\mu f^m + \lambda)]^{(k-1)} = [g^n(\mu g^m + \lambda)]^{(k-1)} + c_{k-1}.
\]

If possible suppose \( c_{k-1} \neq 0 \).

Now in view of Lemma 2 for \( p = 1 \) and using the second fundamental theorem we get

\[
(n + m^*) T(r, f)
\leq T(r, [f^n(\mu f^m + \lambda)]^{(k-1)}) + \overline{N}(r, \infty; f) + \overline{N}(r, c_{k-1}; [f^n(\mu f^m + \lambda)]^{(k-1)}) + N_k(r, 0; f^n(\mu f^m + \lambda)) + S(r, f)
\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; [g^n(\mu g^m + \lambda)]^{(k-1)}) + k \overline{N}(r, 0; f) + N(r, 0; \mu f^m + \lambda) + S(r, f)
\leq \{k + 1 + m^*\} T(r, f) + \{k - 1\} \overline{N}(r, \infty; g) + N_k(r, 0; g^n(\mu g^m + \lambda)) + S(r, f)
\leq \{k + 1 + m^*\} T(r, f) + k \overline{N}(r, 0; g) + N(r, 0; \mu g^m + \lambda)
\leq \{k + 1 + m^*\} T(r, f) + \{2k - 1 + m^*\} T(r, g) + S(r, f) + S(r, g)
\leq \{3k + 2m^*\} T(r) + S(r).
\]

Similarly we get

\[
(n + m^*) T(r, g) \leq \{3k + 2m^*\} T(r) + S(r).
\]

Combining these we get

\[
(n - m^* - 3k) T(r) \leq S(r),
\]

which is a contradiction since \( n > 3k + m^* \).
Therefore \( c_{k-1} = 0 \) and so \([f^n(\mu f^m + \lambda)]^{(k-1)} \equiv [g^n(\mu g^m + \lambda)]^{(k-1)}\). Repeating \( k-1 \) times, we obtain

\[ f^n(\mu f^m + \lambda) \equiv g^n(\mu g^m + \lambda) + c_0. \]

If \( k = 1 \), clearly integrating once we obtain the above. If possible suppose \( c_0 \neq 0 \).

Now using the second fundamental theorem we get

\[
(n + m^*) T(r, f) \\
\leq \overline{N}(r, 0; f^n(\mu f^m + \lambda)) + \overline{N}(r, \infty; f^n(\mu f^m + \lambda)) + \overline{N}(r, c_0; f^n(\mu f^m + \lambda)) \\
\leq (m^* + 2) T(r, f) + \overline{N}(r, 0; g^n(\mu g^m + \lambda)) \\
\leq (m^* + 2) T(r, f) + (m^* + 1) T(r, g) + S(r, f) + S(r, g) \\
\leq (3 + 2m^*) T(r) + S(r).
\]

Similarly we get

\[
(n + m^*) T(r, g) \leq (3 + 2m^*) T(r) + S(r).
\]

Combining these we get

\[
(n - m^* - 3) T(r) \leq S(r),
\]

which is a contradiction since \( n > 3 + m^* \).

Therefore \( c_0 = 0 \) and so

\[ f^n(\mu f^m + \lambda) \equiv g^n(\mu f^m + \lambda). \]

This completes the Lemma. \( \square \)

**Lemma 8.** Suppose that \( f \) and \( g \) be two non-constant meromorphic functions. \( F, G \) be defined as in Lemma 4 and \( H \neq 0 \). If \( f, g \) share \((\infty, 0)\) and \( F, G \) share \((1, k_1)\), then

\[
(n + m^* - k - 1) \overline{N}(r; \infty; f) \leq (k + m^* + 1) \{T(r, f) + T(r, g)\} + \overline{N}_s(r; 1; F, G) \\
+ S(r, f) + S(r, g).
\]

Similar result holds for \( g \) also.

**Proof.** Suppose \( \infty \) is an e.v.P. of \( f \) and \( g \) then the lemma follows immediately.

Next suppose \( \infty \) is not an e.v.P of \( f \) and \( g \). Since \( H \neq 0 \) from Lemma 4 we have \( V \neq 0 \). We suppose that \( z_0 \) is a pole of \( f \) with multiplicity \( q \) and a pole of \( g \) with multiplicity \( r \). Clearly \( z_0 \) is a pole of \( F \) with multiplicity \( (n + m)q + k \) and a pole of \( G \) with multiplicity \( (n + m)r + k \). Noting that \( f, g \) share \((\infty, 0)\) from the definition of \( V \) it is clear that \( z_0 \) is a zero of \( V \) with multiplicity at least \( n + m + k - 1 \). Now using the Milloux theorem [6], p. 55, and Lemma 1, we obtain from the definition of \( V \) that

\[ m(r, V) = S(r, f) + S(r, g). \]
Thus using Lemma 1 and (2.4) we get
\[(n + m^* + k - 1) N(r, \infty; f) \leq N(r, 0; V) \leq T(r, V) + O(1) \leq N(r, \infty; V) + m(r, V) + O(1)
\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}_*(r, 1; F, G)
\]
\[+ S(r, f) + S(r, g) \leq N_{k+1}(r, 0; f^n(\mu f^m + \lambda)) + N_{k+1}(r, 0; g^n(\mu g^m + \lambda)) + k\overline{N}(r, \infty; f)
\]
\[+ k\overline{N}(r, \infty; g) + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \leq (k + 1) \overline{N}(r, 0; f) + N(r, 0; (\mu f^m + \lambda)) + (k + 1) \overline{N}(r, 0; g)
\]
\[+ N(r, 0; (\mu g^m + \lambda)) + 2k\overline{N}(r, \infty; f) + \overline{N}_*(r, 1; F, G)
\]
\[+ S(r, f) + S(r, g).
\]

This gives
\[(n + m^* - k - 1) \overline{N}(r, \infty; f) \leq (k + m^* + 1) \{T(r, f) + T(r, g)\} + \overline{N}_*(r, 1; F, G)
\]
\[+ S(r, f) + S(r, g).
\]

This completes the proof of the lemma. □

**Lemma 9.** Let \(f, g\) be two transcendental meromorphic functions and \(F = \frac{f^n(\mu f^m + \lambda)^{(k)}}{p(z)^{(k)}}\), \(G = \frac{g^n(\mu g^m + \lambda)^{(k)}}{p(z)^{(k)}}\), where \(p(z)\) is a non zero polynomial with \(\text{deg}(p) = r, n(\geq 1), k(\geq 1), m(\geq 2)\) are positive integers such that \(n > 3k + m^* + 3\). If \(f, g\) share \((\infty, 0)\) and \(H \equiv 0\) then either \([f^n(\mu f^m + \lambda)^{(k)}][g^n(\mu g^m + \lambda)^{(k)}] \equiv p^2\) or \(f^n(\mu f^m + \lambda) \equiv g^n(\mu g^m + \lambda)\).

**Proof.** Since \(H \equiv 0\), on integration we get
\[
\frac{1}{F - 1} \equiv \frac{bG + a - b}{G - 1},
\]
where \(a, b\) are constants and \(a \neq 0\). From (2.5) it is clear that \(F\) and \(G\) share \((1, \infty)\).

We now consider the following cases.

**Case 1.** Let \(b \neq 0\) and \(a \neq b\).

If \(b = -1\), then from (2.5) we have
\[
F \equiv \frac{-a}{G - a - 1}.
\]
Therefore
\[
\overline{N}(r, a + 1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + S(r, f).
\]
So in view of Lemma 2 and the second fundamental theorem we get
\[
(n + m^*) \, T(r, g)
\leq T(r, G) + N_{k+1}(r, 0; g^n(\mu g^m + \lambda)) - \overline{N}(r, 0; G) + S(r, g)
\leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, a + 1; G) + N_{k+1}(r, 0; g^n(\mu g^m + \lambda)) - \overline{N}(r, 0; G) + S(r, g)
\leq \overline{N}(r, \infty; g) + N_{k+1}(r, 0; g^n(\mu g^m + \lambda)) + \overline{N}(r, \infty; f) + S(r, f) + S(r, g)
\leq 2\overline{N}(r, \infty; g) + (k + 1)\overline{N}(r, 0; g) + T(r, (\mu g^m + \lambda)) + S(r, f) + S(r, g)
\leq (k + m^* + 3) \, T(r, g) + S(r, f) + S(r, g).
\]

Without loss of generality, we suppose that there exists a set \( I \) with infinite measure such that \( T(r, f) \leq T(r, g) \) for \( r \in I \). So for \( r \in I \), \( S(r, f) \) can be replaced by \( S(r, g) \). So for \( r \in I \), we get a contradiction from above since \( n > 3k + m^* + 3 \).

If \( b \neq -1 \), from (2.5) we obtain that
\[
F - (1 + \frac{1}{b}) \equiv \frac{-a}{b^2[G + \frac{a}{b}]}. 
\]
So
\[
\overline{N}(r, \frac{(b-a)}{b}; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + S(r, f).
\]

Using Lemma 2 and the same argument as used in the case when \( b = -1 \) we can get a contradiction.

**Case 2.** Let \( b \neq 0 \) and \( a = b \).

If \( b = -1 \), then from (2.5) we have
\[
FG \equiv p^2, 
\]
that is
\[
[f^n(\mu f^m + \lambda)](k)g^n(\mu g^m + \lambda)](k) \equiv p^2. 
\]
If \( b \neq -1 \), from (2.5) we have
\[
\frac{1}{F} \equiv \frac{bG}{(1 + b)G - 1}. 
\]
Therefore
\[
\overline{N}(r, \frac{1}{1+b}; G) = \overline{N}(r, 0; F). 
\]

So in view of Lemma 2 and the second fundamental theorem we get
\[
(n + m^*) \, T(r, g)
\leq T(r, G) + N_{k+1}(r, 0; g^n(\mu g^m + \lambda)) - \overline{N}(r, 0; G) + S(r, g)
\leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, \frac{1}{1+b}; G) + N_{k+1}(r, 0; g^n(\mu g^m + \lambda)) - \overline{N}(r, 0; G) + S(r, g)
\leq (k + m^* + 3) \, T(r, g) + S(r, f) + S(r, g) 
\]
\[
\begin{align*}
&\leq N(r, \infty; g) + (k + 1)N(r, 0; g) + T(r, (\mu g^m + \lambda)) + N(r, 0; F) + S(r, g) \\
&\leq N(r, \infty; g) + (k + 1)N(r, 0; g) + T(r, (\mu g^m + \lambda)) + (k + 1)N(r, 0; f) \\
&\quad + T(r, (\mu f^m + \lambda)) + kN(r, \infty; f) + S(r, f) + S(r, g) \\
&\leq (k + m^*) + 2) T(r, g) + (2k + m^* + 1) T(r, f) + S(r, f) + S(r, g).
\end{align*}
\]

So for \( r \in I \) we have
\[
(n + m^*) T(r, g) \leq (3k + 2m^* + 3) T(r, g) + S(r, g),
\]
which is a contradiction since \( n > 3k + m^* + 3 \).

**Case 3.** Let \( b = 0 \). From (2.5) we obtain
\[
F \equiv \frac{G + a - 1}{a}.
\] (2.6)

If \( a \neq 1 \) then from (2.6) we obtain
\[
N(r, 1 - a; G) = N(r, 0; F).
\]

We can similarly deduce a contradiction as in Case 2. Therefore \( a = 1 \) and from (2.6) we obtain
\[
F \equiv G.
\]

Then by the Lemma 7 we have
\[
f^n P(f) \equiv g^n P(g).
\]

\[\square\]

**Lemma 10.** Let \( f, g \) be two transcendental meromorphic functions and \( p(z) \) be a non-constant polynomial, where \( n \) and \( k \geq 2 \) be two positive integers. If \( f = e^{\alpha z} \) and \( g = e^{\beta z} \), where \( \alpha, \beta \) are non-constant entire functions such that \([f^n]^{(k)} - p(z)\) and \([g^n]^{(k)} - p(z)\) share 0 CM, then \([f^n]^{(k)}[g^n]^{(k)} \neq p^2\).

**Proof.** Suppose
\[
[f^n]^{(k)}[g^n]^{(k)} \equiv p^2. \tag{2.7}
\]

From (2.7) we have
\[
N(r, 0; [f^n]^{(k)}) = S(r, f) \quad \text{and} \quad N(r, 0; [g^n]^{(k)}) = S(r, g).
\]

Let
\[
F_1 = \frac{[f^n]^{(k)}}{p} \quad \text{and} \quad G_1 = \frac{[g^n]^{(k)}}{p}. \tag{2.8}
\]

We note that \( T(r, F_1) \leq n(k + 1)T(r, f) + S(r, f) \) and so \( T(r, F_1) = O(T(r, f)) \). By Lemma 2, one can obtain \( T(r, F_1) = O(T(r, F_1)) \). Hence \( S(r, F_1) = S(r, f) \). Similarly we get \( S(r, G_1) = S(r, g) \). From (2.7) we get
\[
F_1 G_1 \equiv 1. \tag{2.9}
\]
It is clear that $T(r, F_1) = T(r, G_1) + O(1)$. So $S(r, F_1) = S(r, G_1)$. If $F_1 \equiv cG_1$, where $c$ is a nonzero constant, then $F_1$ is a constant and so $f$ is a polynomial, which contradicts our assumption. Hence $F_1 \not\equiv cG_1$ and so in the view of (2.9) we see that $F_1$ and $G_1$ share $-1$ IM.

Now by Lemma 2 we have

$$N(r, 0; F_1) \leq nN(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f) \leq S(r, F_1).$$

Similarly we have

$$N(r, 0; G_1) \leq nN(r, 0; g) + k\overline{N}(r, \infty; g) + S(r, g) \leq S(r, G_1).$$

Also we see that

$$N(r, \infty; F_1) = S(r, F_1), \quad N(r, \infty; G_1) = S(r, G_1).$$

Let

$$f_1 = \frac{F_1}{G_1}.$$ 

and

$$f_2 = \frac{F_1 - 1}{G_1 - 1}.$$ 

Clearly $f_1$ is non-constant. If $f_2$ is a nonzero constant then $F_1$ and $G_1$ share $\infty$ CM and so from (2.9) we conclude that $F_1$ and $G_1$ have no poles.

Next we suppose that $f_2$ is non-constant. We see that

$$F_1 = \frac{f_1(1 - f_2)}{f_1 - f_2}, \quad G_1 = \frac{1 - f_2}{f_1 - f_2}.$$ 

Clearly

$$T(r, F_1) \leq 2[T(r, f_1) + T(r, f_2)] + O(1)$$

and

$$T(r, f_1) + T(r, f_2) \leq 4T(r, F_1) + O(1).$$

These give $S(r, F_1) = S(r, f_1, f_2)$. Also we note that

$$\overline{N}(r, 0; f_i) + \overline{N}(r, \infty; f_i) = S(r; f_1, f_2)$$

for $i = 1, 2$.

We note that $\overline{N}(r, -1; F_1) \neq S(r, F_1)$, since otherwise by the second fundamental theorem $F_1$ will be a constant.

Also we see that

$$\overline{N}(r, -1; F_1) \leq N_0(r, 1; f_1, f_2).$$
Thus we have
\[ T(r,f_1) + T(r,f_2) \leq 4 N_0(r,1; f_1, f_2) + S(r,F_1). \]
Then by Lemma 5 there exist two mutually prime integers \( s \) and \( t(|s| + |t| > 0) \) such that
\[ f_1^s f_2^t \equiv 1, \]
i.e.,
\[ \left[ \frac{F_1}{G_1} \right]^s \left[ \frac{F_1 - 1}{G_1 - 1} \right]^t \equiv 1. \quad (2.10) \]
If either \( s \) or \( t \) is zero then we arrive at a contradiction and so \( st \neq 0 \).

We now consider following cases:

**Case (i):** Suppose \( s > 0 \) and \( t = -t_1 \), where \( t_1 > 0 \). Then we have
\[ \left[ \frac{F_1}{G_1} \right]^{s t_1} \equiv \left[ \frac{F_1 - 1}{G_1 - 1} \right]^{t_1} . \quad (2.11) \]
Let \( z_1 \) be a pole of \( F_1 \) of multiplicity \( p \). Then from (2.11) we see that \( z_1 \) must be a zero of \( G_1 \) of multiplicity \( p \). Now from (2.11) we get \( 2s = t_1 \), which is impossible. Hence \( F_1 \) has no pole. Similarly we can prove that \( G_1 \) also has no poles.

**Case (ii):** Suppose either \( s > 0 \) and \( t > 0 \) or \( s < 0 \) and \( t < 0 \). Then from (2.11) one can easily prove that \( F_1 \) and \( G_1 \) have no poles.

Consequently from (2.9) we see that \( F_1 \) and \( G_1 \) have no zeros.

Since \( F_1 \) and \( G_1 \) have no zeros and poles, we have
\[ F_1 \equiv e^{\gamma_1} G_1, \]
i.e.,
\[ \left[ f^n \right]^{(k)} \equiv e^{\gamma_1} \left[ g^n \right]^{(k)} , \quad (2.12) \]
where \( \gamma_1 \) is a non-constant entire function.

First suppose that \( \alpha \) and \( \beta \) both are both transcendental entire functions. Moreover from (2.7) we see that we see that
\[ N(r,0; \left[ f^n \right]^{(k)}) \leq N(r,0; p^2) = O(\log r) \]
and we see that
\[ N(r,0; \left[ g^n \right]^{(k)}) \leq N(r,0; p^2) = O(\log r). \]
From this we get
\[ N(r,\infty; f^s) + N(r,0; f^s) + N(r,0; \left[ f^s \right]^{(k)}) = S(r,n\alpha') = S(r, \frac{\left[ f^n \right]'}{f^n} ) \quad (2.13) \]
and
\[ N(r,\infty; g^s) + N(r,0; g^s) + N(r,0; \left[ g^s \right]^{(k)}) = S(r,n\beta') = S(r, \frac{\left[ g^n \right]'}{g^n} ) . \quad (2.14) \]
Then from (2.13), (2.14) and Lemma 6 we must have
\[ f = e^{ax + b}, \quad g = e^{cz + d}, \]  
(2.15)
where \( a \neq 0, \ b, \ c \neq 0 \) and \( d \) are constants. But these types of \( f \) and \( g \) do not agree with the relation (2.7).

Next suppose \( \alpha, \beta \) both are polynomials. Since \( f = e^{\alpha} \) and \( g = e^{\beta} \), it follows that
\[ [f^n]^{(k)} = A[(\alpha')^k + P_{k-1}(\alpha')]e^{n\alpha}, \quad [g^n]^{(k)} = B[(\beta')^k + P_{k-1}(\beta')]e^{n\beta}, \]
where \( A, B \) are non-zero constants, \( P_{k-1}(\alpha'), P_{k-1}(\beta') \) are differential polynomials in \( \alpha' \) and \( \beta' \) of degree at most \( k - 1 \) respectively. From (2.7) we see that \( \alpha + \beta = C \), i.e., \( \alpha' = \beta' \). So \( \deg(\alpha) = \deg(\beta) \).

If \( \deg(\alpha) = \deg(\beta) = 1 \), then from (2.7) we again get a contradiction. So we suppose \( \deg(\alpha) = \deg(\beta) = 2 \). From (2.12) we see that \( [f^n]^{(k)} \) and \( [g^n]^{(k)} \) share \( 0 \) CM. So we have for some non-zero constant \( D \)
\[ [(\alpha')^k + P_{k-1}(\alpha')] \equiv D[(\beta')^k + Q_{k-1}(\beta')], \]
which is impossible as \( k \geq 2 \).

Actually \( [(\alpha')^k + P_{k-1}(\alpha')] \) and \( [(\beta')^k + Q_{k-1}(\beta')] \) contain the terms \( (\alpha')^k + K(\alpha')^{k-2}\alpha'' \) and \( (\beta')^k + K(\beta')^{k-2}\beta'' \) respectively, where \( K \) is a suitably chosen positive integer. But these two terms are not identical. \( \square \)

**Lemma 11.** ([19], Lemma 6) If \( H \equiv 0 \), then \( F, G \) share \( 1 \) CM. If further \( F, G \) share \( \infty \) IM then \( F, G \) share \( \infty \) CM.

**Lemma 12.** Let \( f \) and \( g \) be two transcendental meromorphic functions, let \( p(z) \) be a nonzero polynomial with \( \deg(p) = r; \ n, k \) and \( m \) be three positive integers with \( n > k + 2r \). Suppose that \( H \equiv 0 \). If \( [f^n(\mu f^m + \lambda)]^{(k)} [g^n(\mu g^m + \lambda)]^{(k)} \equiv p^2 \), where \( \lambda, \mu \) are constants such that \( |\lambda| + |\mu| \neq 0 \), \( f \) and \( g \) share \((\infty, 0)\); if \( p(z) \) is not a constant, then \( f = c_1 e^{Q(z)}, g = c_2 e^{-Q(z)} \), where \( Q(z) = \int_0^z p(z)dz \), \( c_1, c_2 \) and \( c \) are constants such that \( a_{m^*}^2 (c_1 c_2)^{n+m^*}[ (n+m^*)c]^{2k} = -1 \), if \( p(z) \) is a nonzero constant \( b \), then \( f = c_3 e^{cz}, g = c_4 e^{-cz} \), where \( c_3, c_4 \) and \( c \) are constants such that \((-1)^k a_{m^*}^2 (c_3 c_4)^{n+m^*}[ (n+m^*)c]^{2k} = b^2 \), where \( a_{m^*} = \mu \), when \( m^* = m \) and \( a_{m^*} = \lambda \), when \( m^* = 0 \). Also when \( p(z) \) is a nonzero constant \( b \), then \( f \) and \( g \) can be taken as non-constant.

**Proof.** Since \( H \equiv 0 \). It follows from Lemma 11 that \( F, G \) share \( 1 \) CM. So \( [f^n]^{(k)} - p(z) \) and \( [g^n]^{(k)} - p(z) \) share \( 0 \) CM except the zeros of \( p(z) \). Let
\[ [f^n(\mu f^m + \lambda)]^{(k)} [g^n(\mu g^m + \lambda)]^{(k)} \equiv p^2. \]  
(2.16)
First suppose that \( \lambda \mu \neq 0 \)

Note that \( f \) and \( g \) share \((\infty, 0)\), we have \( f \neq \infty, g \neq \infty \). Suppose that \( z_0 \) is a zero of \( f \) of order \( p \), then \( z_0 \) will be a zero of \( [f^n(\mu f^m + \lambda)]^{(k)} \) of order \( np - k \). Since \( n > k + 2r \), we can deduce that \( z_0 \) must be a zero of \( p^2(z) \) with order at least
2r + 1. This is impossible. Thus \( f \) has no zero. Similarly \( g \) has no zero. So \( f = e^{\alpha(z)} \), \( g = e^{\beta(z)} \), where \( \alpha(z) \) and \( \beta(z) \) are two non constant entire functions. Then we get

\[
(\mu f^{n+m})^{(k)} = t_2(\alpha', \alpha'', ..., \alpha^{(k)})e^{(n+m)\alpha},
\]

\[
(\lambda f^n)^{(k)} = t_1(\alpha', \alpha'', ..., \alpha^{(k)})e^{n\alpha},
\]

where \( t_i(\alpha', \alpha'', ..., \alpha^{(k)}) \) (\( i = 1, 2 \)) are differential polynomials in \( \alpha', \alpha'', ..., \alpha^{(k)} \). Obviously

\[
t_i(\alpha', \alpha'', ..., \alpha^{(k)}) \neq 0
\]

for \( i = 1, 2 \). From (2.16) and (2.17) we obtain

\[
N(r, 0; t_2(\alpha', \alpha'', ..., \alpha^{(k)})e^{m\alpha(z)} + t_1(\alpha', \alpha'', ..., \alpha^{(k)}))
\]

\[
\leq N(r, 0; p^2(z)) = S(r, f).
\]

Since \( \alpha \) is an entire function, we obtain \( T(r, \alpha^{(j)}) = S(r, f) \) for \( j = 1, 2 \). Hence \( T(r, t_i) = S(r, f) \) for \( i = 1, 2 \).

So from (2.19) we obtain

\[
mT(r, f) = T(r, t_2e^{m\alpha}) + S(r, f)
\]

\[
\leq N(r, 0; t_2e^{m\alpha}) + N(r, 0; t_2e^{m\alpha} + t_1) + S(r, f)
\]

\[
= S(r, f),
\]

which is a contradiction.

Hence we have \( \lambda \mu = 0 \). Here also \( f = e^{\alpha} \) and \( g = e^{\beta} \), where \( \alpha \) and \( \beta \) are two non constant entire function. Then from (2.16) we have

\[
a_m^2[f^{n+m}]^{(k)}g^{n+m} \equiv p^2.
\]

Let \( s = n + m^* \).

**Case 1:** Let \( \text{deg}(p(z)) = r(\geq 1) \). First suppose \( k \geq 2 \). Then from Lemma 10 we get a contradiction.

Next suppose \( k = 1 \). Suppose that \( \alpha \) and \( \beta \) are transcendental. Then from (2.20) we get

\[
AB \alpha' \beta' e^{s(\alpha + \beta)} \equiv p^2(z),
\]

where \( AB = (n + m^*)^2a_m^2 \).

Let \( \alpha + \beta = \gamma \). From (2.21) we know that \( \gamma \) is not a constant since in that case we get a contradiction. Now from (2.21) we get

\[
AB \alpha' (\gamma' - \alpha')e^{n\gamma} \equiv p^2(z).
\]

We have \( T(r, \gamma') = m(r, \gamma') = m(r, \frac{(n^*)'}{\gamma'}) = S(r, e^{n\gamma}) \). Thus from (2.22) we get

\[
T(r, e^{n\gamma}) \leq T(r, \frac{p^2}{\alpha' (\gamma' - \alpha)}) + O(1)
\]
\[
\begin{align*}
\leq T(r, \alpha') + T(r, \gamma' - \alpha') + O(\log r) + O(1) \\
\leq 2 T(r, \alpha') + S(r, \alpha') + S(r, e^{n\gamma}),
\end{align*}
\]
which implies that \( T(r, e^{n\gamma}) = O(T(r, \alpha')) \) and so \( S(r, e^{n\gamma}) \) can be replaced by \( S(r, \alpha') \). Thus we get \( T(r, \gamma') = S(r, \alpha') \) and so \( \gamma' \) is a small with respect to \( \alpha' \). In view of (2.22) and by the second fundamental theorem for small functions we get
\[
T(r, \alpha') \leq \mathcal{N}(r, \infty; \alpha') + \mathcal{N}(r, 0; \alpha') + \mathcal{N}(r, 0; \alpha' - \gamma') + S(r, \alpha')
\leq O(\log r) + S(r, \alpha'),
\]
which shows that \( \alpha' \) is a polynomial and so \( \alpha \) is a polynomial. Similarly we can prove that \( \beta \) is also a polynomial. This contradicts the fact that \( \alpha \) and \( \beta \) are transcendental.

Next suppose without loss of generality that \( \alpha \) is a polynomial and \( \beta \) is a transcendental entire function. Then \( \gamma \) is transcendental. So in view of (2.22) we can obtain
\[
nT(r, e^{\gamma}) \leq T(r, \frac{p^2}{\alpha' (\gamma' - \alpha')}) + O(1)
\leq T(r, \alpha') + T(r, \gamma' - \alpha') + S(r, \gamma)
\leq T(r, \gamma') + S(r, e^{\gamma}) = S(r, e^{\gamma}),
\]
which leads to a contradiction. Thus \( \alpha \) and \( \beta \) both are polynomials. Also from (2.21) we can conclude that \( \gamma(z) = \alpha(z) + \beta(z) \equiv C \) for a constant \( C \) and so \( \alpha'(z) + \beta'(z) \equiv 0 \).

Again from (2.21) we get \( a_m^2 (n + m^*)^2 e^{\gamma} \alpha' \beta' \equiv p^2(z) \). By computation we get
\[
\alpha' = cp(z), \beta' = -cp(z).
\tag{2.23}
\]
Hence
\[
\alpha = cQ(z) + l_1, \beta = -cQ(z) + l_2,
\tag{2.24}
\]
where \( Q(z) = \int_0^z p(z)dz \) and \( l_1, l_2 \) are constants. Finally we take \( f \) and \( g \) as
\[
f(z) = c_1 e^{cQ(z)}, g(z) = c_2 e^{-cQ(z)},
\]
where \( c_1, c_2 \) and \( c \) are constants such that \( a_m^2 [(n + m^*)c]^2 (c_1 c_2)^{n+i} = -1 \).

**Case 2:** Let \( p(z) \) be a nonzero constant \( b \). Obviously we get \( f = e^{cz} \) and \( g = e^{-cz} \), where \( \alpha \) and \( \beta \) are two non-constant entire functions. Proceeding in the same as above we get in view of (2.20), \( \alpha = cz + l_3, \beta = -cz + l_4 \). We can rewrite \( f \) and \( g \) as
\[
f = c_3 e^{cz}, g = c_4 e^{-cz},
\]
where \( c_3, c_4 \) and \( c \) are nonzero constants such that \( (-1)^k a_m^2 (c_3 c_4)^{n+m^*} [(n + m^*)c]^{2k} = b^2 \).

This completes the proof of the lemma. \( \square \)
Lemma 13. Let $f$ and $g$ be two non-constant meromorphic (entire) functions and $n(\geq 2)$, $m$ be two distinct integers satisfying $n+m \geq d+7$ $(n+m \geq d+3)$. Then for two constants $\lambda$, $\mu$, with $|\lambda| + |\mu| \neq 0$,

$$f^n (\mu f^m + \lambda) \equiv g^n (\mu g^m + \lambda) \quad (2.25)$$

implies the following.

(i) if $\lambda \mu \neq 0$ and

(a) $m = 1$, $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$; or

(b) $m \geq 2$ and for some constant $t$, satisfying $t^d \equiv 1$, we have $f \equiv tg$, where $d = (n+m,n)$.

(ii) if $\lambda \mu = 0$, then $f = tg$, where $t$ is a constant satisfying $t^{n+m} = 1$.

Proof. First suppose $\lambda \mu \neq 0$.

Let $m = 1$. In this case noting that $d = 1 = (n+1,n)$, proceeding in the same way as done in Lemma 6 of [10] we can show when $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$, we have $f \equiv g$.

Next suppose $m \geq 2$. Let $f \neq tg$ for a constant $t$ satisfying $t^d = 1$. We put $h = \frac{f}{g}$. Then $h^d \neq 1$, i.e., $(h-v_0)(h-v_1)\ldots(h-v_{d-1}) \neq 0$, where $v_k = \exp \left(\frac{2k\pi i}{d}\right)$, $k = 0, 1, 2, \ldots, d-1$. First suppose that $h$ is constant. Now (2.25) implies

$$\mu g^m(h^{n+m}-1) \equiv -\lambda(h^n-1).$$

Since $gcd(n+m,n) = d$, eliminating $d$ common factors namely $h-v_k$, $k = 0, 1, \ldots, d-1$ from both sides we are left with

$$ag^m(h-\alpha_1)(h-\alpha_2)\ldots(h-\alpha_{n+m-d}) \equiv (h-\beta_1)(h-\beta_2)\ldots(h-\beta_{n-d}),$$

where $\alpha_i$ and $\beta_j$ are those zeros of $h^{n+m}-1$ and $h^n-1$ which are not the zeros of $h^d-1$, $i = 1, 2, \ldots, n+m-d$ and $j = 1, 2, \ldots, n-d$. Also we note that none of the $\alpha_i$’s coincides with $\beta_j$’s. So if $h = \alpha_i$ or $\beta_j$, then we have either $(h-\beta_1)(h-\beta_2)\ldots(h-\beta_{n-d}) \equiv 0$ or $g \equiv 0$ and in both case we get contradictions. Consequently we assume neither $h^{n+m} \equiv 1$ nor $h^n \equiv 1$. Hence we may write

$$g^m = -\frac{\lambda}{\mu} \frac{h^n-1}{h^{n+m}-1}. \quad (2.26)$$

It follows from (2.26) that $g$ is a constant, which is impossible. So $h$ is non-constant. We observe that since a non-constant meromorphic function can not have more than two Picard exceptional values $h$ can take at least $n+m-d-2$ values among $u_j = \exp \left(\frac{2\pi i j}{n+m}\right)$, where $j = 0, 1, 2, \ldots, n+m-1$. Since $f^m$ has no simple pole $h-u_j$ has no simple zero for at least $n+m-d-2$ values of $u_j$, for $j = 0, 1, 2, \ldots, n+m-1$ and for these $n+m-d-2$ values of $j$ within $j = 0, 1, 2, \ldots, n+m-1$, we have $\Theta(u_j; h) \geq \frac{1}{2}$. So by the maximum deficiency sum we have \( \frac{n+m-d-2}{2} \leq 2 \) i.e., $n+m \leq d+6$, which leads to a contradiction as $n+m > d+7$.

When $f$ and $g$ are entire functions, proceeding in the same way we can obtain (2.26) where $h$ is non-constant. Since $g$ has no pole and $h$ can omit at most 2 values, we must have $n+m \leq d+2$, which is a contradiction.
Next suppose $\lambda \mu \neq 0$. Then from the give condition either $\lambda$ or $\mu$ will be zero. So we get $f = tg$, where $t$ is a constant satisfying $t^{n+m'} = 1$. This proves the lemma.

**Lemma 14.** [3] Let $f$ and $g$ be two non-constant meromorphic functions sharing $(1, k_1)$, where $2 \leq k_1 \leq \infty$. Then

$$N(r; f) = 2 + 2N(r; f| = 3) + \ldots + (k_1 - 1) N(r; f| = k_1) + k_1 N_L(r; f) + (k_1 + 1) N_L(r; g) + k_1 \mathcal{N}^{k_1+1}_E(r; g) \leq N(r; g) - N(r; g).$$

**Lemma 15.** [2] Let $f$, $g$ share $(1, 1)$. Then

$$N_{f > 2}(r; g) \leq \frac{1}{2} N(r; f) + \frac{1}{2} N(r; f) - \frac{1}{2} N_0(r; f') + S(r, f),$$

where $N_0(r; f')$ is the counting function of those zeros of $f'$ which are not the zeros of $f(f - 1)$.

**Lemma 16.** [2] Let $f$ and $g$ be two non-constant meromorphic functions sharing $(1, 0)$. Then

$$N_L(r; f) + 2N_L(r; g) + \mathcal{N}^2_L(r; f) - N_{f > 1}(r; g) - N_{g > 1}(r; f) \leq N(r; g) - N(r; g).$$

**Lemma 17.** [2] Let $f$, $g$ share $(1, 0)$. Then

$$N_L(r; f) \leq N(r; f) + N(r; f) + S(r, f)$$

**Lemma 18.** [2] Let $f$, $g$ share $(1, 0)$. Then

(i) $N_{f > 1}(r; g) \leq N(r; f) + N(r; f) - N_0(r; f') + S(r, f)$

(ii) $N_{g > 1}(r; f) \leq N(r; f) + N(r; f) - N_0(r; g') + S(r, g)$.

### 3. Proofs of the theorems

**Proof of Theorem 1.** Let $F = [f^n p(f)]^{(k)} / p(z)$ and $G = [g^n p(g)]^{(k)} / p(z)$, where $P(w) = \mu w^m + \lambda$. It follows that $F$ and $G$ share $(1, l)$ except the zeros of $p(z)$ and $f$, $g$ share $(\infty, 0)$.

**Case 1.** Let $H \neq 0$.

**Subcase 1.1. $l \geq 1$**

From (2.1) it can be easily calculated that the possible poles of $H$ occur at (i) multiple zeros of $F$ and $G$, (ii) those 1 points of $F$ and $G$ whose multiplicities are different, (iii) poles of $F$ and $G$ with different multiplicities, (iv) zeros of $F' (G')$ which are not the zeros of $F(F - 1)(G(G - 1))$.

Since $H$ has only simple poles we get

$$N(r; H) \quad (3.1)$$
\[ \leq \mathcal{N}_*(r, \infty; f, g) + \mathcal{N}_*(r, 1; F, G) + \mathcal{N}(r, 0; F \mid 2) + \mathcal{N}(r, 0; G \mid 2) + \mathcal{N}_0(r, 0; F') + \mathcal{N}_0(r, 0; G') , \]

where \( \mathcal{N}_0(r, 0; F') \) is the reduced counting function of those zeros of \( F' \) which are not the zeros of \( F(F - 1) \) and \( \mathcal{N}_0(r, 0; G') \) is similarly defined.

Let \( z_0 \) be a simple zero of \( F - 1 \) but \( a(z_0) \neq 0, \infty \). Then \( z_0 \) is a simple zero of \( G - 1 \) and a zero of \( H \). So

\[ N(r, 1; F \mid 1) \leq N(r, 0; H) \leq N(r, \infty; H) + S(r, f) + S(r, g). \tag{3.2} \]

While \( l \geq 3 \), using (3.1) and (3.2) we get

\[ \begin{aligned}
\mathcal{N}(r, 1; F) &
\leq N(r, 1; F \mid 1) + \mathcal{N}(r, 1; F \mid 2) \\
&\leq \mathcal{N}(r, \infty; f) + \mathcal{N}(r, 0; F \mid 2) + \mathcal{N}(r, 0; G \mid 2) + \mathcal{N}_*(r, 1; F, G) \\
&+ \mathcal{N}(r, 1; F \mid 2) + \mathcal{N}_0(r, 0; F') + \mathcal{N}_0(r, 0; G') + S(r, f) + S(r, g).
\end{aligned} \tag{3.3} \]

Now in view of Lemmas 14 and 3 we get

\[ \begin{aligned}
\mathcal{N}_0(r, 0; G') + \mathcal{N}_*(r, 1; F \mid 2) + \mathcal{N}_*(r, 1; F, G) &
\leq \mathcal{N}_0(r, 0; G') + N(r, 1; F \mid 2) + \mathcal{N}(r, 1; F \mid 3) + \ldots + \mathcal{N}(r, 1; F \mid l)

+ \mathcal{N}_{+1}^E(r, 1; F) + \mathcal{N}_L(r, 1; F) + \mathcal{N}_L(r, 1; G) + \mathcal{N}_*(r, 1; F, G)

\leq \mathcal{N}_0(r, 0; G') - \mathcal{N}(r, 1; F \mid 1) - \ldots - (l - 2)\mathcal{N}(r, 1; F \mid l) - (l - 1)\mathcal{N}_L(r, 1; F)

- l\mathcal{N}_L(r, 1; G) - (l - 1)\mathcal{N}_{+1}^E(r, 1; F) + N(r, 1; G) - \mathcal{N}(r, 1; G) + \mathcal{N}_*(r, 1; F, G)

\leq \mathcal{N}_0(r, 0; G') + N(r, 1; G) - \mathcal{N}(r, 1; G) - (l - 2)\mathcal{N}_L(r, 1; F) - (l - 1)\mathcal{N}_L(r, 1; G)

\leq N(r, 0; G' \mid G \neq 0) - (l - 2)\mathcal{N}_L(r, 1; F) - (l - 1)\mathcal{N}_L(r, 1; G)

\leq \mathcal{N}(r, 0; G) + \mathcal{N}(r, \infty; g) - (l - 2)\mathcal{N}_*(r, 1; F, G) - \mathcal{N}_L(r, 1; G)

\leq N(r, 0; G) + \mathcal{N}(r, \infty; g) - \mathcal{N}_*(r, 1; F, G) - \mathcal{N}_L(r, 1; G).
\end{aligned} \tag{3.4} \]

Hence using (3.3), (3.4), Lemmas 2 and 8 we get from the second fundamental theorem that

\[ \begin{aligned} (n + m^*)T(r, f) &
\leq T(r, F) + N_{k+1}(r, 0; f^n P(f)) - N_2(r, 0; F) + S(r, f)

\leq \mathcal{N}(r, 0; F) + \mathcal{N}(r, \infty; F) + \mathcal{N}(r, 1; F) + N_{k+2}(r, 0; f^n P(f)) - N_2(r, 0; F)

\leq \mathcal{N}(r, 0; F')

\leq \mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g) + \mathcal{N}(r, 0; F) + N_{k+2}(r, 0; f^n P(f)) + \mathcal{N}(r, 0; F \mid 2)

+ \mathcal{N}(r, 0; G \mid 2) + \mathcal{N}(r, 1; F \mid 2) + \mathcal{N}_*(r, 1; F, G) + \mathcal{N}_0(r, 0; G') - N_2(r, 0; F)

+ S(r, f) + S(r, g)

\leq 3 \mathcal{N}(r, \infty; f) + N_{k+2}(r, 0; f^n P(f)) + N_2(r, 0; G) - \mathcal{N}_*(r, 1; F, G) - \mathcal{N}_L(r, 1; G). \tag{3.5} \]
In a similar way we can obtain

\[
\begin{align*}
&+ S(r, f) + S(r, g) \\
&\leq 3 \overline{N}(r, \infty; f) + N_{k+2}(r, 0; f^n P(f)) + k \overline{N}(r, \infty; g) + N_{k+2}(r, 0; g^n P(g)) \\
&- \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
&\leq (3 + k) \overline{N}(r, \infty; f) + (k + 2) \overline{N}(r, 0; f) + T(r, P(f)) + (k + 2) \overline{N}(r, 0; g) \\
&+ T(r, P(g)) - \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
&\leq (k + m^* + 2) \{T(r, f) + T(r, g)\} + (3 + k) \overline{N}(r, \infty; f) - \overline{N}_*(r, 1; F, G) \\
&+ S(r, f) + S(r, g) \\
&\leq (k + m^* + 2) \{T(r, f) + T(r, g)\} + \frac{(3 + k)(k + m^* + 1)}{n + m^* - k - 1} \{T(r, f) + T(r, g)\} \\
&\leq \left[ k + m^* + 2 + \frac{(3 + k)(k + m^* + 1)}{n + m^* - k - 1} \right] \{T(r, f) + T(r, g)\} + S(r, f) + S(r, g),
\end{align*}
\]

In a similar way we can obtain

\[
(n + m^*) T(r, g) \\
\leq \left[ k + m^* + 2 + \frac{(3 + k)(k + m^* + 1)}{n + m^* - k - 1} \right] \{T(r, f) + T(r, g)\} + S(r, f) + S(r, g).
\]

Adding (3.5) and (3.6) we get

\[
\left[ n - m^* - 2k - 4 - \frac{(6 + 2k)(k + m^* + 1)}{n + m^* - k - 1} \right] \{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g).
\]

Since the quantity in the third bracket can be written as

\[
\left[ \frac{(n + m^* - k - 1)^2 - (2m^* + k + 3)(n + m^* - k - 1) - 2(k + 3)(k + m^* + 1)}{n + m^* - k - 1} \right], \quad (3.7)
\]

by a simple computation one can easily verify that when

\[
n + m^* - k - 1 > 2m^* + 2k + 5 > \frac{2m^* + k + 3 + \sqrt{(2m^* + k + 3)^2 + 8(k + 3)(k + m^* + 1)}}{2},
\]

i.e., when \( n > 3k + m^* + 6 \) we get a contradiction from (3.7).

While \( l \geq 2 \), like (3.3), (3.4) and not using Lemma 8 in (3.5) we can deduce a contradiction when \( n > 3k + m^* + 7 \). So we omit the detail.

While \( l = 1 \), using Lemmas 3, 14, 15, (3.1) and (3.2) we get

\[
\overline{N}(r, 1; F) \\
\leq N(r, 1; F) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_E(r, 1; F) \\
\leq \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}(r, 0; F) \geq 2 \geq \overline{N}(r, 0; G) \geq 2 + \overline{N}_*(r, 1; F, G) \\
+ \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_E(r, 1; F) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') \\
+ S(r, f) + S(r, g),
\]

(3.8)
\[ \begin{align*}
&\leq \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}(r, 0; F) \geq 2 + 2N_L(r, 1; F) \\
&\quad + 2N_L(r, 1; G) + N_E^2(r, 1; F) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') \\
&\quad + S(r, f) + S(r, g) \\
&\leq \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}(r, 0; F) \geq 2 + 2N_L(r, 1; G) \\
&\quad + N(r, 1; G) - \overline{N}(r, 1; G) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') \\
&\quad + S(r, f) + S(r, g) \\
&\leq \frac{3}{2} \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}(r, 0; F) \geq 2 + \frac{1}{2} \overline{N}(r, 0; F) + \overline{N}(r, 0; G) \geq 2 \\
&\quad + N(r, 0; G' | G \neq 0) + \overline{N}_0(r, 0; F') + S(r, f) + S(r, g) \\
&\leq \frac{3}{2} \overline{N}(r, \infty; f) + 2\overline{N}(r, \infty; g) + \overline{N}(r, 0; F) \geq 2 + \frac{1}{2} \overline{N}(r, 0; F) + N_2(r, 0; G) \\
&\quad + \overline{N}_0(r, 0; F') + S(r, f) + S(r, g).
\end{align*} \]

Hence using (3.8), Lemmas 1 and 2 we get from second fundamental theorem that

\[ (n + m^*) T(r, f) \]
\[ \leq T(r, F) + N_{k+2}(r, 0; f^n P(f)) - N_2(r, 0; F) + S(r, f) \]
\[ \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, 1; F) + N_{k+2}(r, 0; f^n P(f)) - N_2(r, 0; F) \\
- N_0(r, 0; F') \]
\[ \leq \frac{5}{2} \overline{N}(r, \infty; f) + 2\overline{N}(r, \infty; g) + N_2(r, 0; F) + \frac{1}{2} \overline{N}(r, 0; F) + N_{k+2}(r, 0; f^n P(f)) \\
+ N_2(r, 0; G) - N_2(r, 0; F) + S(r, f) + S(r, g) \]
\[ \leq \frac{5}{2} \overline{N}(r, \infty; f) + 2\overline{N}(r, \infty; g) + N_{k+2}(r, 0; f^n P(f)) + \frac{1}{2} \overline{N}(r, 0; F) + N_2(r, 0; G) \\
+ S(r, f) + S(r, g) \]
\[ \leq \frac{5}{2} \overline{N}(r, \infty; f) + 2\overline{N}(r, \infty; g) + N_{k+2}(r, 0; f^n P(f)) + \frac{k}{2} \overline{N}(r, \infty; g) + N_{k+2}(r, 0; g^n P(g)) \\
+ \frac{1}{2} \{ k \overline{N}(r, \infty; f) + \overline{N}_{k+1}(r, 0; f^n P(f)) \} + S(r, f) + S(r, g) \]
\[ \leq \frac{5 + k}{2} \overline{N}(r, \infty; f) + (k + 2)\overline{N}(r, \infty; g) + \frac{3k + 5}{2} \overline{N}(r, 0; f) + \frac{3}{2} T(r, P(f)) \\
+ (k + 2) \overline{N}(r, 0; g) + T(r, P(g)) + S(r, f) + S(r, g) \]
\[ \leq (2k + 5 + \frac{3m^*}{2}) T(r, f) + (2k + 4 + m^*) T(r, g) + S(r, f) + S(r, g) \]
\[ \leq (4k + 9 + \frac{5m^*}{2}) T(r) + S(r). \]
In a similar way we can obtain

\[(n + m^*) T(r, g) \leq \left( 4k + 9 + \frac{5m^*}{2} \right) T(r) + S(r). \tag{3.10} \]

Combining (3.9) and (3.10) we see that

\[(n + m^*) T(r) \leq \left( 4k + 9 + \frac{5m^*}{2} \right) T(r) + S(r), \]

i.e

\[
\left( n - 4k + 9 - \frac{3m^*}{2} \right) T(r) \leq S(r). \tag{3.11}
\]

Since \( n > 4k + 9 + \frac{3m^*}{2} \), (3.11) leads to a contradiction.

**Subcase 1.2.** \( l = 0 \). Here (3.2) changes to

\[ N^1_E \left( r, 1; F^{(k)} \mid = 1 \right) \leq N(r, 0; H) \leq N(r, \infty; H) + S(r, F) + S(r, G) \tag{3.12} \]

using Lemmas 3, 16, 17, 18, (3.1) and (3.12) we get

\[
\overline{N}(r, 1; F) \\
\leq N^1_E(r, 1; F) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_E^2(r, 1; F) \\
\leq \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}(r, 0; F) + 2 + \overline{N}(r, 0; G) \geq 2 + 2\overline{N}_L(r, 1; F) \\
+ 2\overline{N}_L(r, 1; G) + \overline{N}_E^2(r, 1; F) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + S(r, f) + S(r, g) \\
\leq \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}(r, 0; F) + 2 + \overline{N}(r, 0; G) \geq 2 + \overline{N}_{F > 1}(r, 1; G) \\
+ \overline{N}_{G > 1}(r, 1; F) + \overline{N}_L(r, 1; F) + N(r, 1; G) - \overline{N}(r, 1; G) + \overline{N}_0(r, 0; F') \\
+ \overline{N}_0(r, 0; G') + S(r, f) + S(r, g) \\
\leq 3 \overline{N}(r, \infty; f) + 2\overline{N}(r, \infty; g) + N_2(r, 0; F) + \overline{N}(r, 0; F) + N_2(r, 0; G) \\
+ N(r, 1; G) - \overline{N}(r, 1; G) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + S(r, f) + S(r, g) \\
\leq 3 \overline{N}(r, \infty; f) + 2\overline{N}(r, \infty; g) + N_2(r, 0; F) + \overline{N}(r, 0; F) + N_2(r, 0; G) \\
+ N(r, 0; G') G \neq 0 \) + \overline{N}_0(r, 0; F') + S(r, f) + S(r, g) \\
\leq 3 \overline{N}(r, \infty; f) + 3\overline{N}(r, \infty; g) + N_2(r, 0; F) + \overline{N}(r, 0; F) + N_2(r, 0; G) \\
+ \overline{N}(r, 0; G) + \overline{N}_0(r, 0; F') + S(r, f) + S(r, g).
Hence using (3.13), Lemmas 1 and 2 we get from second fundamental theorem that

\[(n + m^*) T(r, f) \leq T(r, F) + N_{k+2}(r, 0; f^n P(f)) - N_2(r, 0; F) + S(r, f)\]

\[\leq N(r, 0; F) + N(r, 1; F) + N_{k+2}(r, 0; f^n P(f)) - N_2(r, 0; F) + N_0(r, 0; F') + S(r, f)\]

\[\leq 4N(r, 0; F) + 3N(r, 0; F) + 2N_2(r, 0; F) + N_{k+2}(r, 0; f^n P(f)) + 2N_2(r, 0; G) + S(r, f) + S(r, g)\]

\[\leq 4N(r, 0; F) + 3N(r, 0; F) + 2N_{k+2}(r, 0; f^n P(f)) + 2N_2(r, 0; G) + S(r, f) + S(r, g)\]

\[\leq (2k + 4) N(r, 0; F) + (2k + 3) N(r, 0; F) + 3T(r, P(f)) + 2T(r, P(g)) + S(r, f) + S(r, g)\]

\[\leq (5k + 8 + 3m^*) T(r, f) + (4k + 6 + 2m^*) T(r, g) + S(r, f) + S(r, g)\]

\[\leq (9k + 14 + 5m^*) T(r) + S(r),\]

where \(T(r) = \max\{T(r, f), T(r, g)\}\). In a similar way we can obtain

\[(n + m^*) T(r, g) \leq (9k + 14 + 5m^*) T(r) + S(r).\]  (3.15)

Combining (3.14) and (3.15) we see that

\[(n + m^*) T(r) \leq (9k + 14 + 5m^*) T(r) + S(r),\]

i.e

\[(n - 9k - 14 - 4m^*) T(r) \leq S(r).\]  (3.16)

Since \(n > 9k + 14 + 4m^*, (3.16)\) leads to a contradiction.

**Case 2.** Let \(H \equiv 0\). Then by Lemma 9 we obtain either

\[f^n(\mu f^m + \lambda)]^{(k)}[g^n(\mu g^m + \lambda)]^{(k)} \equiv p^2\]

or

\[f^n(\mu f^m + \lambda) \equiv g^n(\mu g^m + \lambda).\]

So the theorem follows from Lemma 12 and 13. \(\square\)

**Proof of Theorem 1.** Proceeding in the same way the proof of Theorem 2 can be carried out in the line of proof of Theorem 1. \(\square\)
REFERENCES


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