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# SOME PROPERTIES OF MEROMORPHIC FUNCTIONS CONCERNING SHARED–SETS

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*Abstract.* Using Nevanlinna's value distribution theory, we study shared-set problems of meromorphic functions and prove that there exist three finite sets $S_1$ $(\sharp(S_1)=1)$, $S_2$ $(\sharp(S_2)=1)$ and $S_3$ $(\sharp(S_3)=5)$ such that any two meromorphic functions $f$ and $g$ sharing $S_j$ $(j=1,2,3)$ must be identical. Our results are improvements of those of former authors and the complement of Ref. [W. Lin and H. X. Yi, Uniqueness theorems for meromorphic functions that share three sets. Complex Variables, **44** (2003), 315–327.]. In addition, we show the accuracy of the results by giving some examples.

## 1. Introduction

In what follows, the term "meromorphic" will always mean meromorphic in the complex plane $\mathbb{C}$. It is also assumed that reader is familiar with the basic concepts and notations of Nevanlinna theory, for instance, $T(r,f)$, $N(r,f)$, $m(r,f)$, $\overline{N}(r,f)$ and so on (see [7], [14], [16]). We denote by $S(r,f)$ any functions satisfying $S(r,f)=o\{T(r,f)\}$, as $r\to+\infty$, possibly outside a set with finite measure.

Let $S$ be a subset of distinct element in $\hat{\mathbb{C}}$. Define

$$E(S,f)=\bigcup_{a\in S}\{z\in\mathbb{C}\mid f_a(z)=0,\ \text{counting multiplicities}\},$$

where $f_a(z)=f(z)-a$ if $a\in\mathbb{C}$ and $f_\infty(z)=\dfrac{1}{f(z)}$.

Let $f$ and $g$ be two non-constant meromorphic functions in $\mathbb{C}$. If $E(S,f)=E(S,g)$, we say that $f$ and $g$ share the set $S$ CM. In particular, when $S=\{a\}$, where $a\in\hat{\mathbb{C}}$, we say that $f$ and $g$ share the value $a$ CM.

The following classical result due to Nevanlinna [13] has prompted research activity on shared value problems up until today.

THE FIVE-POINT THEOREM. *If two meromorphic functions $f$ and $g$ share five distinct values $a_1,a_2,a_3,a_4,a_5$, then $f\equiv g$.*

*The functions $f(z)=e^z$ and $g(z)=e^{-z}$ share the values $0,\pm1,\infty$ CM, and yet $f\not\equiv g$. This shows that the number 5 in the five-point theorem is the best possible.*

In 1976, F. Gross [6] extended the study by considering pre-images of a set and introduced the notion of unique range set. Further, Gross proved that there exist three

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finite sets $S_j$ $(j=1,2,3)$ such that any two non-constant entire functions $f$ and $g$ satisfying $E(S_j,f)=E(S_j,g)$ for $j=1,2,3$ must be identical, and posed the following question (see [5, Question 6]).

QUESTION A. Can one find two finite sets $S_j$ $(j=1,2)$ such that any two entire functions $f$ and $g$ satisfying $E(S_j,f)=E(S_j,g)$ $(j=1,2)$ must be identical?

If the answer to Question A is affirmative, it would be interesting to know how large both sets would have to be.

Many authors have been considering about it, and got a lot of related results. Some of them are due to Yi [13–16], Mues and Reinders [12], Frank and Reinders [4], Li and Yang [9], Fujimoto [5], Yi and Li [15] and so on. We recall the following results given by Yi [19]:

THEOREM A. [19] *Let* $S_1=\{0\}$ *and* $S_2=\{w \mid w^n(w+a)-b=0\}$, *where* $n(\geqslant 2)$ *is an integer,* $a$ *and* $b$ *are two non-zero constants such that the algebraic equation* $w^n(w+a)-b=0$ *has no multiple roots. If* $f$ *and* $g$ *are two non-constant entire functions satisfying* $E(S_j,f)=E(S_j,g)$ *for* $j=1,2$, *then* $f\equiv g$.

THEOREM B. [19] *If* $S_1$ *and* $S_2$ *are two finite sets such that any two non-constant entire functions* $f$ *and* $g$ *satisfying* $E(S_j,f)=E(S_j,g)$ *for* $j=1,2$ *must be identical, then* $\max\{\sharp(S_1),\sharp(S_2)\}\geqslant 3$, *where* $\sharp(S)$ *denotes the cardinality of the set* $S$.

However, Theorem A and Theorem B seem to be invalid for meromorphic functions. In 1994, Yi [17] proved the following result.

THEOREM C. [17] *Let* $S_1=\{a\}$ *or* $\{\infty\}$, $S_2=\{c_1,c_2\}$ *and* $S_3=\{a+b,a+b\omega,\dots,a+b\omega^{n-1}\}$, *where* $n(\geqslant 7)$ *is an integer,* $b\neq 0$, $c_1\neq a$, $c_2\neq a$, $(c_1-a)^n\neq(c_2-a)^n$ *and* $(c_k-a)^n(c_j-a)^n\neq b^{2n}$ $(k,j=1,2)$. *Suppose that* $f$ *and* $g$ *are two non-constant meromorphic functions satisfying* $E(S_j,f)=E(S_j,g)$ *for* $j=1,2,3$, *then* $f\equiv g$.

From Theorem C we immediately obtain that there exist three sets $S_1$ $(\sharp(S_1)=1)$, $S_2$ $(\sharp(S_2)=2)$ and $S_3$ $(\sharp(S_3)=7)$ such that any two meromorphic functions $f$ and $g$ sharing $S_j$ $(j=1,2,3)$ must be identical.

Recently, Yi [20] introduced the polynomial

$$P(z)=az^n-n(n-1)z^2+2n(n-2)z-(n-1)(n-2),\quad (a\neq 0,2) \tag{1}$$

and proved that $P(z)$ has only simple zeros. In fact, we consider the rational function

$$R(z)=\frac{az^n}{n(n-1)(z-\alpha_1)(z-\alpha_2)}, \tag{2}$$

where $\alpha_1$ and $\alpha_2$ are two distinct roots of the equation $n(n-1)z^2-2n(n-2)z+(n-1)(n-2)=0$.

From (2), we have

$$R'(z)=\frac{(n-2)az^{n-1}(z-1)^2}{n(n-1)(z-\alpha_1)^2(z-\alpha_2)^2}, \tag{3}$$

so $z=0$ is one root with multiplicity $n$ of $R(z)=0$ and $z=1$ is one roof with multiplicity 3 of the equation $R(z)-c=0$, where $c=\frac{a}{2}(\neq 1)$. Thus,

$$R(z)-c=\frac{a(z-1)^3Q_{n-3}(z)}{n(n-1)(z-\alpha_1)(z-\alpha_2)},$$

where $Q_{n-3}(z)$ is a polynomial of degree $n-3$. Moreover, we have

$$R(z)-1=\frac{P(z)}{n(n-1)(z-\alpha_1)(z-\alpha_2)}. \tag{4}$$

Therefore, from (2) and (3), we obtain that $P(z)$ has only simple zeros.

In 2003, using the polynomial $P(z)$ defined as (1), Lin and Yi [10] obtained that there exist three finite sets $S_1=\{0\}$ $(\sharp(S_1)=1)$, $S_2=\{\infty\}$ $(\sharp(S_1)=1)$ and $S_3=\{z|P(z)=0\}$ $(\sharp(S_3)=5)$ such that any two meromorphic functions $f$ and $g$ sharing $S_j$ $(j=1,2)$ with the same multiplicities must be identical.

Here, we are interesting what would have to happen when the set $S_2=\{\infty\}$ is replaced by $S_2=\{a\}$, where $a$ is a nonzero finite number. Indeed, we shall give our main result in Section 3, but the proof method is different from [10].

For the convenience, we explain some notations which will be used in the paper.

DEFINITION 1. Let $f$ be a nonconstant meromorphic function and let $a$ be a finite complex number. We denote by $N_{1)}\left(r,\frac{1}{f-a}\right)$ the counting function of simple zeros of $f-a$.

DEFINITION 2. Let $p$ be a positive integer and let $a$ be a finite complex number. We denote by $N_{(p}\left(r,\frac{1}{f-a}\right)$ the counting function of the zeros of $f-a$ with multiplicities at least $p$, and by $\overline{N}_{(p}\left(r,\frac{1}{f-a}\right)$ the corresponding reduced counting function. Moreover, we set $N_p\left(r,\frac{1}{f-a}\right)=\overline{N}_{(1}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-a}\right)+\cdots+\overline{N}_{(p}\left(r,\frac{1}{f-a}\right)$.

## 2. Some auxiliary results

In order to prove our main results, we shall need the following lemmas.

LEMMA 1. [16] *Let $f(z)$ be a meromorphic function in $\mathbb{C}$. Then for all irreducible rational function $R(z,f)$ in $f$ with coefficients meromorphic and small with respect to $f$, we have*

$$T(r,R(z,f))=dT(r,f)+S(r,f),$$

*where $S(r,f)=o(T(r,f))$ for $r\notin E$, $E$ is a set with finite measure and $d$ is the degree of $R(z,f)$ in $f$.*

LEMMA 2. [21] *Let $f(z)$ and $g(z)$ be two meromorphic functions in $\mathbb{C}$. Let*

$$H = \frac{f''}{f'} - \frac{2f'}{f-1} - \left(\frac{g''}{g'} - \frac{2g'}{g-1}\right).$$

*If $f$ and $g$ share the value $1$ CM and $H \not\equiv 0$, then*

$$N_{1)}\left(r, \frac{1}{f-1}\right) \leqslant N(r,f) + S(r,f) + S(r,g).$$

LEMMA 3. [21] *Let $h = \dfrac{f''}{f'} - \dfrac{2f'}{f-1}$, where $f$ is a nonconstant meromorphic function. If $z_0$ is a simple pole of $f$, then $h$ is regular at $z_0$.*

LEMMA 4. [4] *Let $A(z) = (n-1)^2(z^n-1)(z^{n-2}-1) - n(n-2)(z^{n-1}-1)^2$, where $n \geqslant 5$ is an integer, then $A(z) = (z-1)^4(z-\beta_1)\cdots(z-\beta_{2n-6})$, where $\beta_j \in \mathbb{C}\backslash\{0,1\}$ $(j = 1,2,\cdots,2n-6)$, which are distinct respectively.*

LEMMA 5. *Suppose that $R(z)$ is defined as (2), $f$ and $g$ are nonconstant meromorphic functions satisfying $E(\{0,1\},f) = E(\{0,1\},g)$. If $R(f) \equiv R(g)$, then $f \equiv g$.*

*Proof.* We consider the case for $n = 5$, the same argument will be hold for $n \geqslant 6$.

The equation $R(f) \equiv R(g)$ yields that $T(r,f) = T(r,g) + S(r,f)$ and $S(r,f) = S(r,g)$. Moreover, we rewrite it as

$$\frac{f^5}{(f-\alpha_1)(f-\alpha_2)} \equiv \frac{g^5}{(g-\alpha_1)(g-\alpha_2)}. \tag{5}$$

where $\alpha_1$ and $\alpha_2$ are two distinct roots of the equation $10z^2 - 15z + 6 = 0$.

In addition, we have $R'(f) \equiv R'(g)$, that is

$$\frac{f^4(f-1)^2 f'}{(f-\alpha_1)^2(f-\alpha_2)^2} \equiv \frac{g^4(g-1)^2 g'}{(g-\alpha_1)^2(g-\alpha_2)^2}. \tag{6}$$

Combining (5) and (6), we obtain

$$\frac{(f-1)^2 f'}{f^6} \equiv \frac{(g-1)^2 g'}{g^6}. \tag{7}$$

We can rewrite (7) as follows.

$$\left(\frac{1}{f}\right)^2\left(1-\frac{1}{f}\right)^2\left(\frac{1}{f}\right)' \equiv \left(\frac{1}{g}\right)^2\left(1-\frac{1}{g}\right)^2\left(\frac{1}{g}\right)' \tag{8}$$

Set $f^* = 1 - \dfrac{1}{f}$ and $g^* = 1 - \dfrac{1}{g}$, then (8) can be rewritten as

$$(1-f^*)^2(f^*)^2(f^*)' \equiv (1-g^*)^2(g^*)^2(g^*)'$$

Integrating both sides of the equality, we get

$$6(f^*)^5-15(f^*)^4+10(f^*)^3\equiv 6(g^*)^5-15(g^*)^4+10(g^*)^3+c, \tag{9}$$

where $c$ is a constant.

Since $f$ and $g$ share $\{0,1\}$, we can obtain $f$ and $g$ share 0 CM and 1 CM from (5). Therefore, we can deduce that $f^*$ and $g^*$ share $\infty$ CM and 0 CM. Now we consider the following two cases.

*Case I.* If 0 is not a Picard exceptional value of $f^*$, then from (9), we can obtain that $c=0$. Set $h=\frac{f^*}{g^*}$, then $h\neq\infty, z\in\mathbb{C}$. Combining (9), we get

$$6(g^*)^2(h^5-1)-15g^*(h^4-1)+10(h^3-1)\equiv 0. \tag{10}$$

Therefore,

$$\Big(12(h^5-1)g^*-15(h^4-1)\Big)^2\equiv -15\Big[16(h^3-1)(h^5-1)-15(h^4-1)^2\Big].$$

Suppose that $h$ is not a constant, by Lemma 4, we obtain

$$\Big(12(h^5-1)g^*-15(h^4-1)\Big)^2\equiv -15(h-1)^4(h-\beta_1)\cdots(h-\beta_4),$$

where $\beta_j\in\mathbb{C}\backslash\{0,1\}$, $(j=1,\cdots,4)$, which are distinct respectively.

This implies that every zero of $h-\beta_j$ $(j=1,\cdots,4)$ has a multiplicity at least 2. By the second fundamental theorem, we get a contradiction. Therefore, $h$ is a constant, from (10) we get $h\equiv 1$, hence $f^*\equiv g^*$. Furthermore, we have $f\equiv g$.

*Case II.* If 0 is a Picard exceptional value of $f^*$, then it is the Picard exceptional value of $g^*$ too. Similar to Case I, we can get $f\equiv g$ when the case $c=0$. Therefore, we only consider the following equation.

$$6(f^*)^5-15(f^*)^4+10(f^*)^3\equiv 6(g^*)^5-15(g^*)^4+10(g^*)^3+c, \tag{11}$$

where $c\neq 0$.

Set $Q(z)=6z^5-15z^4+10z^3+c$, then $Q'(z)=30z^2(z-1)^2$. We can deduce that $z=0$ is not a multiple zero of $Q(z)$ for $c\neq 0$. In addition, if $z=1$ is a multiple zero of $Q(z)$, then we can obtain that $c=-1$. It follows that the equation $6z^5-15z^4+10z^3+c=0$ has five distinct roots when $c\neq 0,-1$.

Next, we consider the case $c\neq -1$ in (11), we have

$$6(g^*)^5-15(g^*)^4+10(g^*)^3+c\equiv 6(f^*)^3(f^*-\gamma_1)(f^*-\gamma_2),$$

where $\gamma_1,\gamma_2$ are the roots of $z^2-\frac{5}{2}z+\frac{5}{3}=0$.

Noting that $Q(z) = 6z^5 - 15z^4 + 10z^3 + c \ (c \neq 0, -1)$ has five distinct zeros, which we denote as $z_j \ (j = 1 \cdots 5)$, by second fundamental theorem, we have

$$\begin{aligned}3T(r, g^*) &\leqslant \sum_{j=1}^{5} \overline{N}\Big(r, \frac{1}{g^* - z_j}\Big) + S(r, g^*) \\ &\leqslant \overline{N}\Big(r, \frac{1}{f^*}\Big) + \overline{N}\Big(r, \frac{1}{f^* - \gamma_1}\Big) + \overline{N}\Big(r, \frac{1}{f^* - \gamma_2}\Big) + S(r, g^*) \\ &\leqslant 2T(r, f^*) + S(r, g^*),\end{aligned}$$

which gives a contradiction.

If $c = -1$, we rewrite (11) as

$$6(f^*)^5 - 15(f^*)^4 + 10(f^*)^3 + 1 \equiv 6(g^*)^5 - 15(g^*)^4 + 10(g^*)^3 \equiv 6(g^*)^3(g^* - \gamma_1)(g^* - \gamma_2).$$

Therefore, we also can get a contradiction similarly.

This completes the proof of Lemma 5. □

LEMMA 6. *Let $f(z)$ and $g(z)$ be two meromorphic functions in $\mathbb{C}$, and let $F = R(f)$ and $G = R(g)$, where $R(z)$ is defined as (2) and $n \geqslant 5$. If $AF \equiv G + B$, where $A(\neq 0)$ and $B$ are two constants, then one of the following cases holds.*

*(I) $AF \equiv G$;*

*(II) $N\Big(r, \frac{1}{g}\Big) \neq S(r, g)$ and $N\Big(r, \frac{1}{f}\Big) \neq S(r, f)$.*

*Proof.* By the assumption, we obtain that $T(r, f) = T(r, g) + S(r, f)$. Now we shall distinguish the following three cases to discuss when $n = 5$.

*Case I.* If $B = 0$, then $AF \equiv G$, that is Case (I) holds.

*Case II.* If $B = \frac{-a}{2}$, then $AF \equiv G - \frac{a}{2}$, that is

$$\frac{Af^5}{20f^2 - 30f + 12} \equiv \frac{(g-1)^3(g^2 + 3g + 6)}{20g^2 - 30g + 12}. \tag{12}$$

We claim that Case (II) holds, namely $N\Big(r, \frac{1}{f}\Big) \neq S(r, f)$ and $N\Big(r, \frac{1}{g}\Big) \neq S(r, g)$. Otherwise, $N\Big(r, \frac{1}{f}\Big) = S(r, f)$ or $N\Big(r, \frac{1}{g}\Big) = S(r, g)$.

Firstly, we consider the case when $N\Big(r, \frac{1}{g}\Big) = S(r, g)$. By the second fundamental theorem and (12), we get

$$\begin{aligned}2T(r, g) &\leqslant \overline{N}\Big(r, \frac{1}{g-1}\Big) + \overline{N}\Big(r, \frac{1}{g - \beta_1}\Big) + \overline{N}\Big(r, \frac{1}{g - \beta_2}\Big) + \overline{N}\Big(r, \frac{1}{g}\Big) + S(r, g) \\ &\leqslant \overline{N}\Big(\frac{1}{f}\Big) + S(r, g) \leqslant T(r, f) + S(r, g),\end{aligned}$$

where $\beta_1, \beta_2$ are the roots of $z^2+3z+6=0$, it is a contradiction.

Next, we discuss the case when $N\left(r,\dfrac{1}{f}\right)=S(r,f)$, we also can get

$$\begin{aligned} T(r,g) &\leqslant \overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}\left(r,\frac{1}{g-\beta_1}\right)+\overline{N}\left(r,\frac{1}{g-\beta_2}\right)+S(r,g) \\ &\leqslant \overline{N}\left(\frac{1}{f}\right)+S(r,g)=S(r), \end{aligned}$$

a contradiction.

Therefore, Case (II) holds.

*Case III.* $B\neq 0, \dfrac{-a}{2}$, noting that $R(z)-c=0$ $\left(c\neq 0,\ \dfrac{a}{2}\right)$ has five distinct roots, by the second fundamental theorem, we have

$$3T(r,g)\leqslant \sum_{j=1}^{5}\overline{N}\left(r,\frac{1}{g-\gamma_j}\right)+S(r,g)\leqslant \overline{N}\left(r,\frac{1}{f}\right)+S(r,g)\leqslant T(r,f)+S(r,g),$$

where $\gamma_j$ $(j=1,\dots,5)$ are the roots of $G+B=0$, it is a contradiction.

The same argument for $n\geqslant 6$. This completes the proof of Lemma 6. $\square$

REMARK 3. Under the condition of Lemma 6 and "$n\geqslant 6''$, we can deduce the further result that $AF\equiv G$.

LEMMA 7. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in $\mathbb{C}$ such that $\overline{N}\left(r,\dfrac{1}{f-1}\right)+\overline{N}\left(r,\dfrac{1}{g-1}\right)=S(r,f)+S(r,g)$, $R(z)$ is defined as (2) and $\dfrac{1}{R(f)-1}\equiv\dfrac{A}{R(g)-1}+B$, where $A(\neq 0)$ and $B$ are two constants. If $f$ and $g$ share $\{0,1\}$ CM and $n\geqslant 5$, then $B=0$.*

*Proof.* For the convenience, we put $F=R(f)$, $G=R(g)$. Suppose that $B\neq 0$, by the assumption, we have

$$\frac{1}{F-1}\equiv\frac{B\left(G+\dfrac{A-B}{B}\right)}{G-1}. \tag{13}$$

Moreover, we have $T(r,f)=T(r,g)+S(r,f)$.

We distinguish the following three cases to discuss.

*Case I.* $A=B$, the equation (13) is rewritten as

$$\frac{1}{F-1}\equiv\frac{BG}{G-1}. \tag{14}$$

Thus, we have $\overline{N}(r,F)=\overline{N}\left(r,\dfrac{1}{G}\right)$, that is

$$\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-\alpha_1}\right)+\overline{N}\left(r,\frac{1}{f-\alpha_2}\right)=\overline{N}\left(r,\frac{1}{g}\right).$$

From the second fundamental theorem, we obtain

$$\begin{aligned}2T(r,f) &\leqslant \overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-\alpha_1}\right)+\overline{N}\left(r,\frac{1}{f-\alpha_2}\right)+S(r,f)\\ &\leqslant \overline{N}\left(r,\frac{1}{g}\right)+S(r,f)\leqslant T(r,g)+S(r,f),\end{aligned}$$

a contradiction.

*Case II.* $\frac{A-B}{B}=-\frac{a}{2}$, the equation (13) is rewritten as

$$\frac{1}{F-1}\equiv\frac{B\left(G-\frac{a}{2}\right)}{G-1}, \tag{15}$$

We claim that $N\left(r,\frac{1}{f}\right)\neq S(r,f)$. Indeed, if $N\left(r,\frac{1}{f}\right)=S(r,f)$, by (15), we have $\overline{N}(r,F)=\overline{N}\left(r,\frac{1}{G-\frac{a}{2}}\right)$, that is

$$\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-\alpha_1}\right)+\overline{N}\left(r,\frac{1}{f-\alpha_2}\right)=\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}\left(r,\frac{1}{g-\beta_1}\right)+\overline{N}\left(r,\frac{1}{g-\beta_2}\right),$$

where $\beta_1,\beta_2$ are the roots of $z^2+3z+6=0$.

Thus, using the second fundamental theorem, we have

$$\begin{aligned}3T(r,f) &\leqslant \overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-\alpha_1}\right)+\overline{N}\left(r,\frac{1}{f-\alpha_2}\right)+\overline{N}\left(r,\frac{1}{f}\right)+S(r,g)\\ &\leqslant \overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}\left(r,\frac{1}{g-\beta_1}\right)+\overline{N}\left(r,\frac{1}{g-\beta_2}\right)+\overline{N}\left(r,\frac{1}{f}\right)\\ &\quad +S(r,g)\\ &\leqslant 2T(r,g)+S(r,g),\end{aligned}$$

a contradiction.

Therefore, there exists $z_0$ such that $f(z_0)=0$, then $g(z_0)=0$ or $g(z_0)=1$. By (15), it is easy to obtain a contradiction when $g(z_0)=1$. Therefore, we have $g(z_0)=0$. Combing (15), we deduce $B=-\frac{2}{a}$. Rewriting (15) as

$$G\equiv\frac{aF}{2F-2+a}$$

It follows that $\overline{N}(r,G)=\overline{N}\left(r,\frac{1}{F-1+\frac{a}{2}}\right)$. Since $a\neq 1,2$, we have $1-\frac{a}{2}\neq 0,\ \frac{a}{2}$, by the second fundamental theorem, we get

$$\begin{aligned}
4T(r,f) &\leqslant \sum_{j=1}^{5}\overline{N}\Big(r,\frac{1}{f-\gamma_j}\Big)+\overline{N}\Big(r,\frac{1}{f-1}\Big)+S(r,f)\\
&\leqslant \overline{N}(r,g)+\overline{N}\Big(r,\frac{1}{g-\alpha_1}\Big)+\overline{N}\Big(r,\frac{1}{g-\alpha_2}\Big)+S(r,f)\\
&\leqslant 3T(r,g)+S(r,f),
\end{aligned}$$

where $\gamma_j$ $(j=1,\ldots,5)$ are five distinct roots of $2F-2+a=0$ respectively, it is a contradiction.

*Case III.* $A\neq B$ and $\frac{A-B}{B}\neq-\frac{a}{2}$, by (13), we have $\overline{N}(r,F)=\overline{N}\left(r,\frac{1}{G+\frac{A-B}{B}}\right)$. Noting that $BG+A-B=0$ has at least five distinct roots respectively, similar to the case II, we can also deduce a contradiction.

Therefore $B=0$. This completes the proof of Lemma 7. □

## 3. Main results

In this section, we shall give the main result as follows.

THEOREM 1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions in $\mathbb{C}$, and let $S_1=\{0\}$, $S_2=\{1\}$ or $\{\infty\}$, $S_3=\Big\{z|P(z)=0\Big\}$, where $P(z)$ is defined as (1) and $a\neq 0,1,2$. If $E(S_j,f)=E(S_j,g)$ $(j=1,2,3)$ and $n\geqslant 5$, then $f\equiv g$.*

REMARK 1. Suppose that $S_1=\{0\},S_2=\{1\},S_3=\Big\{z|z^5-20z^2+30z-12=0\Big\}$. Let $f(z)=e^z+1$ and $g(z)=e^{-z}+1$. It is easy to see that $f$ and $g$ satisfy $E(S_j,f)=E(S_j,g)$ $(j=1,2,3)$, but $f\not\equiv g$. This shows that the assumption $a\neq 1$ in Theorem 1 can not be omitted.

REMARK 2. Suppose that $a=2$. It is easy to see that $z=1$ is one root with multiplicity 3 of the equation $2z^n-n(n-1)z^2+2n(n-2)z-(n-1)(n-2)=0$, where $n\geqslant 5$ is an integer. Thus, $S_3$ in Theorem 1 is not a subset of distinct element in $\mathbb{C}$ with $\sharp(S_3)=n$ . This shows that the assumption $a\neq 2$ in Theorem 1 is needed.

*Proof of Theorem 1.* When $S_2=\{\infty\}$, by the Ref. [10] we obtain that Theorem 1 is valid. Therefore, we only need to prove Theorem 1 in the case $S_2=\{1\}$.

Firstly, we consider the case for $n=5$.

For the sake of simplicity, we set

$$P(f)=af^5-20f^2+30f-12,\qquad P(g)=ag^5-20g^2+30g-12,$$

$$F=\frac{af^5}{20f^2-30f+12}=\frac{af^5}{20(f-\alpha_1)(f-\alpha_2)},$$

$$G=\frac{ag^5}{20g^2-30g+12}=\frac{af^5}{20(g-\alpha_1)(g-\alpha_2)}.$$

Then we have $F$ and $G$ share 0, 1 CM and

$$F'=\frac{3af^4(f-1)^2f'}{20(f-\alpha_1)^2(f-\alpha_2)^2},\qquad G'=\frac{3ag^4(g-1)^2g'}{20(g-\alpha_1)^2(g-\alpha_2)^2},$$

$$F-1=\frac{P(f)}{20(f-\alpha_1)(f-\alpha_2)},\qquad G-1=\frac{P(g)}{20(g-\alpha_1)(g-\alpha_2)},$$

$$F-\frac{a}{2}=\frac{a(f-1)^3(f-\beta_1)(f-\beta_2)}{20(f-\alpha_1)(f-\alpha_2)},\qquad G-\frac{a}{2}=\frac{a(g-1)^3(g-\beta_1)(g-\beta_2)}{20(g-\alpha_1)(g-\alpha_2)},$$

where $\beta_1,\beta_2$ are the roots of $z^2+3z+6=0$. We consider the following two cases.

*Case I.* $N\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{g-1}\right)\neq S(r)$, where $S(r)=\max\left\{S(r,f),S(r,g)\right\}$. Set

$$\varphi=\frac{F'}{(F-1)F}-\frac{G'}{(G-1)G},$$

that is,

$$\varphi=\frac{(f-1)^2f'}{P(f)f}-\frac{(g-1)^2g'}{P(g)g}.$$

Suppose that $\varphi\not\equiv 0$, since $F$ and $G$ share 0, 1 CM, we obtain

$$m(r,\varphi)=S(r),\qquad N(r,\varphi)=S(r). \tag{16}$$

By the first fundamental theorem, we have

$$2N\left(r,\frac{1}{f-1}\right)\leqslant N\left(r,\frac{1}{\varphi}\right)\leqslant T(r,\varphi)=S(r). \tag{17}$$

Similarly, we have

$$2N\left(r,\frac{1}{g-1}\right)\leqslant T(r,\varphi)=S(r). \tag{18}$$

Combining (17) and (18), we obtain

$$N\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{g-1}\right)=S(r),$$

a contradiction.

Therefore, we have $\varphi\equiv 0$, that is $\frac{F'}{(F-1)F}\equiv\frac{G'}{(G-1)G}$. Integrating both sides of the equation, we get

$$\frac{F}{F-1}\equiv\frac{AG}{G-1}, \tag{19}$$

where $A\neq 0$ is a constant.

In the condition of $N\left(r,\dfrac{1}{f-1}\right)+N\left(r,\dfrac{1}{g-1}\right)\neq S(r)$, we get that there exists $z_0\in\mathbb{C}$, such that $f(z_0)=1$ and $g(z_0)=1$. Therefore, we have $A=1$. Furthermore, we get $F\equiv G$, by Lemma 5, we have $f\equiv g$.

*Case II.* $N\left(r,\dfrac{1}{f-1}\right)+N\left(r,\dfrac{1}{g-1}\right)=S(r)$, where $S(r)=\max\left\{S(r,f),S(r,g)\right\}$.
Set

$$H=\frac{F''}{F'}-\frac{2F'}{F-1}-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right).$$

Suppose that $H\not\equiv 0$, from Lemma 3, we have

$$N(r,H)\leqslant \overline{N}(r,f)+\overline{N}(r,g)+\overline{N}_{(2}\left(r,\frac{1}{f-\alpha_1}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-\alpha_2}\right)+\overline{N}_{(2}\left(r,\frac{1}{g-\alpha_1}\right)$$
$$+\overline{N}_{(2}\left(r,\frac{1}{g-\alpha_2}\right)+\overline{N}^*\left(r,\frac{1}{f'}\right)+\overline{N}^*\left(r,\frac{1}{g'}\right),$$

where $\alpha_1,\alpha_2$ are the roots of $20z^2-30z+12=0$ and $\alpha_1,\alpha_2\neq 0,1,\ 2$, $\overline{N}^*\left(r,\dfrac{1}{f'}\right)$ denotes the reduced counting function of the zeros of $f'$ which are not the zeros of $f(f-1)(f-\alpha_1)(f-\alpha_2)(F-1)$.

By Lemma 2, we have $N_{1)}\left(r,\dfrac{1}{F-1}\right)\leqslant N\left(r,\dfrac{1}{H}\right)$. Therefore, we have

$$N\left(r,\frac{1}{F-1}\right)=N_{1)}\left(r,\frac{1}{F-1}\right)+N_{(2}\left(r,\frac{1}{F-1}\right)\leqslant N\left(r,\frac{1}{H}\right)+N_{(2}\left(r,\frac{1}{F-1}\right)$$
$$\leqslant \overline{N}(r,f)+\overline{N}(r,g)+\overline{N}_{(2}\left(r,\frac{1}{f-\alpha_1}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-\alpha_2}\right)+\overline{N}_{(2}\left(r,\frac{1}{g-\alpha_1}\right)$$
$$+\overline{N}_{(2}\left(r,\frac{1}{g-\alpha_2}\right)+N_0\left(r,\frac{1}{f'}\right)+N_0\left(r,\frac{1}{g'}\right), \tag{20}$$

where $N_0\left(r,\dfrac{1}{f'}\right)$ denotes the counting function of the zeros of $f'$ which are not the zeros of $f(f-1)(f-\alpha_1)(f-\alpha_2)$.

The same argument shows that

$$N\left(r,\frac{1}{G-1}\right)\leqslant \overline{N}(r,f)+\overline{N}(r,g)+\overline{N}_{(2}\left(r,\frac{1}{f-\alpha_1}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-\alpha_2}\right)+\overline{N}_{(2}\left(r,\frac{1}{g-\alpha_1}\right)$$
$$+\overline{N}_{(2}\left(r,\frac{1}{g-\alpha_2}\right)+N_0\left(r,\frac{1}{f'}\right)+N_0\left(r,\frac{1}{g'}\right). \tag{21}$$

Set $\psi=\dfrac{(f-2)f'}{(f-1)(f-\alpha_1)(f-\alpha_2)}$, then it is easy to see that

$$m(r,\psi)=S(r,f),\quad N(r,\psi)\leqslant \overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{f-\alpha_1}\right)+\overline{N}\left(r,\frac{1}{f-\alpha_2}\right).$$

On the other hand, we have

$$N\Big(r,\frac{1}{f-2}\Big)+N_0^*\Big(r,\frac{1}{f'}\Big)\leqslant N\Big(r,\frac{1}{\psi}\Big)\leqslant N(r,\psi)+S(r,f),$$

where $N_0^*\Big(r,\frac{1}{f'}\Big)$ denotes the counting function of the zeros of $f'$ which are not the zeros of $(f-1)(f-\alpha_1)(f-\alpha_2)$. Thus,

$$N\Big(r,\frac{1}{f-2}\Big)+N_0^*\Big(r,\frac{1}{f'}\Big)\leqslant \overline{N}\Big(r,\frac{1}{f-1}\Big)+\overline{N}\Big(r,\frac{1}{f-\alpha_1}\Big)+\overline{N}\Big(r,\frac{1}{f-\alpha_2}\Big)+S(r,f). \tag{22}$$

The same argument shows that

$$N\Big(r,\frac{1}{g-2}\Big)+N_0^*\Big(r,\frac{1}{g'}\Big)\leqslant \overline{N}\Big(r,\frac{1}{g-1}\Big)+\overline{N}\Big(r,\frac{1}{g-\alpha_1}\Big)+\overline{N}\Big(r,\frac{1}{g-\alpha_2}\Big)+S(r,g). \tag{23}$$

Noting that $a\neq 1$, thus $z=2$ is not the roof of $az^5-20z^2+30z-12=0$ and $N_0\Big(r,\frac{1}{f'}\Big)$ $\leqslant N_0^*\Big(r,\frac{1}{f'}\Big)$. Therefore, using the second fundamental theorem and combining with (20)–(23), we deduce

$$\begin{aligned}
8\Big(T(r,f)+T(r,g)\Big)\leqslant\ & \overline{N}\Big(r,\frac{1}{f}\Big)+\overline{N}\Big(r,\frac{1}{f-\alpha_1}\Big)+\overline{N}\Big(r,\frac{1}{f-\alpha_2}\Big)+N\Big(r,\frac{1}{F-1}\Big)\\
&+N\Big(r,\frac{1}{f-2}\Big)+\overline{N}\Big(r,\frac{1}{f-1}\Big)+\overline{N}\Big(r,\frac{1}{g}\Big)+\overline{N}\Big(r,\frac{1}{g-\alpha_1}\Big)\\
&+\overline{N}\Big(r,\frac{1}{g-\alpha_2}\Big)+N\Big(r,\frac{1}{G-1}\Big)+N\Big(r,\frac{1}{g-2}\Big)+\overline{N}\Big(r,\frac{1}{g-1}\Big)\\
&-N_0\Big(r,\frac{1}{f'}\Big)-N_0\Big(r,\frac{1}{g'}\Big)+S(r,f)+S(r,g)\\
\leqslant\ & 2\Big(\overline{N}(r,f)+\overline{N}(r,g)\Big)+2\Big(N_2\Big(r,\frac{1}{f-\alpha_1}\Big)+N_2\Big(r,\frac{1}{g-\alpha_1}\Big)\Big)\\
&+2\Big(N_2\Big(r,\frac{1}{f-\alpha_2}\Big)+N_2\Big(r,\frac{1}{g-\alpha_2}\Big)\Big)+\overline{N}\Big(r,\frac{1}{f}\Big)+\overline{N}\Big(r,\frac{1}{g}\Big)\\
&+2\Big(\overline{N}\Big(r,\frac{1}{f-1}\Big)+\overline{N}\Big(r,\frac{1}{g-1}\Big)\Big)+S(r,f)+S(r,g)\\
\leqslant\ & 7\Big(T(r,f)+T(r,g)\Big)+S(r,f)+S(r,g),
\end{aligned}$$

a contradiction.

Therefore, we have $H\equiv 0$, namely

$$\frac{F''}{F'}-\frac{2F'}{F-1}\equiv\frac{G''}{G'}-\frac{2G'}{G-1}.$$

Integrating both sides of the equality, we get

$$\frac{1}{F-1}\equiv\frac{A}{G-1}+B,$$

where $A \neq 0, B$ are constants.

By Lemma 7, we have $B = 0$, so that $\frac{1}{F-1} \equiv \frac{A}{G-1}$, we may write this as $AF = G + (A-1)$.

Applying Lemma 6, we obtain that either $A = 1$ or $N\left(r, \frac{1}{g}\right) \neq S(r)$ and $N\left(r, \frac{1}{f}\right) \neq S(r)$. When $N\left(r, \frac{1}{g}\right) \neq S(r)$ and $N\left(r, \frac{1}{f}\right) \neq S(r)$, there exists $z_0$ such that $f(z_0) = g(z_0) = 0$, and hence, we also get $A = 1$. Thus, by Lemma 5, we get $f \equiv g$.

Similar to the proceeding of proof for $n = 5$, we can obtain $f \equiv g$ for $n \geqslant 6$. Therefore, we complete the proof of Theorem 1. □

## 4. Further Remarks

In the direction to the question of Gross, Yi [17] also proved that there exists $S_1$ $(\sharp(S_1) = 2)$ and $S_2$ $(\sharp(S_2) = 9)$ such that any two meromorphic functions $f$ and $g$ satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$ must be identical.

THEOREM D. [17] *Let* $S_1 = \{a+b, a+bw, \cdots, a+bw^{n-1}\}$ *and let* $S_2 = \{c_1, c_2\}$, *where* $n > 8$, $(c_1 - a)^n \neq (c_2 - a)^n, (c_k - a)^n (c_j - a)^n \neq b^{2n} (k, j = 1, 2)$. *If* $f$ *and* $g$ *are non-constant meromorphic functions such that* $E(S_j, f) = E(S_j, g)$ *for* $j = 1, 2$, *then* $f \equiv g$.

Whereafter, Li and Yang [9], Yi [18] proved that there exists a set $S$ $(\sharp(S) = 11)$ such that the conditions $E(S, f) = E(S, g)$ and $E(\infty, f) = E(\infty, g)$ imply $f(z) \equiv g(z)$ for any pair of non-constant meromorphic functions $f$ and $g$.

In 1997, Fang and Guo [2] extended the result of Li and Yang [9], Yi [18], and proved that $\sharp(S) = 9$. Afterwards, I. Lahiri [8], Fang and Lahiri [3], H. Yi and the present author [22] also obtained $\sharp(S) = 8$ or $\sharp(S) = 7$ under adding certain condition respectively.

By the polynomial $P(z)$ defined as $(1)$, Yi proved the following result.

THEOREM E. [20] *Let* $f(z)$ *and* $g(z)$ *be two non-constant meromorphic functions in* $\mathbb{C}$, *and let* $S = \left\{z | P(z) = 0\right\}$, *where* $a$ $(\neq 0, 2)$ *is a constant. If* $E(S, f) = E(S, g)$, $E(\infty, f) = E(\infty, g)$ *and* $n \geqslant 8$, *then* $f \equiv g$.

From Theorem E we immediately obtain that there exist two sets $S_1$ $(\sharp(S_1) = 1)$ and $S_2$ $(\sharp(S_2) = 8)$ such that any two meromorphic functions $f$ and $g$ sharing $S_j$ $(j = 1, 2)$ with the same multiplicities must be identical.

Noting that Lemma 5 and Lemma 7 are valid under the condition "$E(\{0,1\}, f) = E(\{0,1\}, g)$", i.e. $(\sharp(S_1) = 2)$ and $n \geqslant 5$, so we are natural to pose an open question.

OPEN QUESTION. *Whether can Theorem 1 hold under the condition* $E(\{0,1\}, f) = E(\{0,1\}, g)$ *?*

In recent years, along with the value distribution theory of meromorphic functions, many authors introduced and investigated the approximations of functions in various fields [1]. Furthermore, V. N. Mishra constructed and investigated various properties on the approximations of functions in Banach spaces [11]. Naturally, we are interesting to know what happen on the subject of difference and $q$-difference under the sharing-set conditions of Theorem 1. Unfortunately, we do not find the effective method to resolve it.

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