SOME PROPERTIES OF MEROMORPHIC FUNCTIONS CONCERNING SHARED–SETS

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Abstract. Using Nevanlinna’s value distribution theory, we study shared-set problems of meromorphic functions and prove that there exist three finite sets $S_1 (\sharp(S_1) = 1)$, $S_2 (\sharp(S_2) = 1)$ and $S_3 (\sharp(S_3) = 5)$ such that any two meromorphic functions $f$ and $g$ sharing $S_j$ ($j = 1, 2, 3$) must be identical. Our results are improvements of those of former authors and the complement of Ref. [W. Lin and H. X. Yi, Uniqueness theorems for meromorphic functions that share three sets. Complex Variables, 44 (2003), 315–327]. In addition, we show the accuracy of the results by giving some examples.

1. Introduction

In what follows, the term “meromorphic” will always mean meromorphic in the complex plane $\mathbb{C}$. It is also assumed that reader is familiar with the basic concepts and notations of Nevanlinna theory, for instance, $T(r, f)$, $N(r, f)$, $m(r, f)$, $\overline{N}(r, f)$ and so on (see [7], [14], [16]). We denote by $S(r, f)$ any functions satisfying $S(r, f) = o\{T(r, f)\}$, as $r \to +\infty$, possibly outside a set with finite measure.

Let $S$ be a subset of distinct element in $\hat{\mathbb{C}}$. Define

$$E(S, f) = \bigcup_{a \in S} \{z \in \mathbb{C} \mid f_a(z) = 0, \text{ counting multiplicities}\},$$

where $f_a(z) = f(z) - a$ if $a \in \mathbb{C}$ and $f_\infty(z) = \frac{1}{f(z)}$.

Let $f$ and $g$ be two non-constant meromorphic functions in $\mathbb{C}$. If $E(S, f) = E(S, g)$, we say that $f$ and $g$ share the set $S$ CM. In particular, when $S = \{a\}$, where $a \in \hat{\mathbb{C}}$, we say that $f$ and $g$ share the value $a$ CM.

The following classical result due to Nevanlinna [13] has prompted research activity on shared value problems up until today.

THE FIVE-POINT THEOREM. If two meromorphic functions $f$ and $g$ share five distinct values $a_1, a_2, a_3, a_4, a_5$, then $f \equiv g$.

The functions $f(z) = e^z$ and $g(z) = e^{-z}$ share the values $0, \pm 1, \infty$ CM, and yet $f \not\equiv g$. This shows that the number 5 in the five-point theorem is the best possible.

In 1976, F. Gross [6] extended the study by considering pre-images of a set and introduced the notion of unique range set. Further, Gross proved that there exist three
finite sets $S_j$ ($j = 1, 2, 3$) such that any two non-constant entire functions $f$ and $g$ satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2, 3$ must be identical, and posed the following question (see [5, Question 6]).

**QUESTION A.** Can one find two finite sets $S_j$ ($j = 1, 2$) such that any two entire functions $f$ and $g$ satisfying $E(S_j, f) = E(S_j, g)$ ($j = 1, 2$) must be identical?

If the answer to Question A is affirmative, it would be interesting to know how large both sets would have to be.

Many authors have been considering about it, and got a lot of related results. Some of them are due to Yi [13–16], Mues and Reinders [12], Frank and Reinders [4], Li and Yang [9], Fujimoto [5], Yi and Li [15] and so on. We recall the following results given by Yi [19]:

**THEOREM A.** [19] Let $S_1 = \{0\}$ and $S_2 = \{w \mid w^n (w + a) - b = 0\}$, where $n(\geq 2)$ is an integer, $a$ and $b$ are two non-zero constants such that the algebraic equation $w^n (w + a) - b = 0$ has no multiple roots. If $f$ and $g$ are two non-constant entire functions satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$, then $f \equiv g$.

**THEOREM B.** [19] If $S_1$ and $S_2$ are two finite sets such that any two non-constant entire functions $f$ and $g$ satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$ must be identical, then $\max\{\sharp(S_1), \sharp(S_2)\} \geq 3$, where $\sharp(S)$ denotes the cardinality of the set $S$.

However, Theorem A and Theorem B seem to be invalid for meromorphic functions. In 1994, Yi [17] proved the following result.

**THEOREM C.** [17] Let $S_1 = \{a\}$ or $\{\infty\}$, $S_2 = \{c_1, c_2\}$ and $S_3 = \{a + b, a + b\omega, \ldots, a + b\omega^{n-1}\}$, where $n(\geq 7)$ is an integer, $b \neq 0$, $c_1 \neq a$, $c_2 \neq a$, $(c_1 - a)^n \neq (c_2 - a)^n$ and $(c_k - a)^n (c_j - a)^n \neq b^{2n}$ ($k, j = 1, 2$). Suppose that $f$ and $g$ are two non-constant meromorphic functions satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2, 3$, then $f \equiv g$.

From Theorem C we immediately obtain that there exist three sets $S_1$ ($\sharp(S_1) = 1$), $S_2$ ($\sharp(S_2) = 2$) and $S_3$ ($\sharp(S_3) = 7$) such that any two meromorphic functions $f$ and $g$ sharing $S_j$ ($j = 1, 2, 3$) must be identical.

Recently, Yi [20] introduced the polynomial

$$P(z) = az^n - n(n-1)z^2 + 2n(n-2)z - (n-1)(n-2), \quad (a \neq 0, 2) \quad (1)$$

and proved that $P(z)$ has only simple zeros. In fact, we consider the rational function

$$R(z) = \frac{az^n}{n(n-1)(z - \alpha_1)(z - \alpha_2)}, \quad (2)$$

where $\alpha_1$ and $\alpha_2$ are two distinct roots of the equation $n(n-1)z^2 - 2n(n-2)z + (n-1)(n-2) = 0$.

From (2), we have

$$R'(z) = \frac{(n-2)az^{n-1}(z-1)^2}{n(n-1)(z - \alpha_1)^2(z - \alpha_2)^2},$$

(3)
so \( z = 0 \) is one root with multiplicity \( n \) of \( R(z) = 0 \) and \( z = 1 \) is one root with multiplicity 3 of the equation \( R(z) - c = 0 \), where \( c = \frac{a}{2}(\neq 1) \). Thus,

\[
R(z) - c = \frac{a(z-1)^3Q_{n-3}(z)}{n(n-1)(z-\alpha_1)(z-\alpha_2)},
\]

where \( Q_{n-3}(z) \) is a polynomial of degree \( n - 3 \). Moreover, we have

\[
R(z) - 1 = \frac{P(z)}{n(n-1)(z-\alpha_1)(z-\alpha_2)}.
\]

Therefore, from (2) and (3), we obtain that \( P(z) \) has only simple zeros.

In 2003, using the polynomial \( P(z) \) defined as (1), Lin and Yi [10] obtained that there exist three finite sets \( S_1 = \{0\} \ (\#(S_1) = 1) \), \( S_2 = \{\infty\} \ (\#(S_1) = 1) \) and \( S_3 = \{z|P(z) = 0\} \ (\#(S_3) = 5) \) such that any two meromorphic functions \( f \) and \( g \) sharing \( S_j \ (j = 1,2) \) with the same multiplicities must be identical.

Here, we are interested what would have to happen when the set \( S_2 = \{\infty\} \) is replaced by \( S_2 = \{a\} \), where \( a \) is a nonzero finite number. Indeed, we shall give our main result in Section 3, but the proof method is different from [10].

For the convenience, we explain some notations which will be used in the paper.

**DEFINITION 1.** Let \( f \) be a nonconstant meromorphic function and let \( a \) be a finite complex number. We denote by \( N_{(1)}\left(r, \frac{1}{f-a}\right) \) the counting function of simple zeros of \( f-a \).

**DEFINITION 2.** Let \( p \) be a positive integer and let \( a \) be a finite complex number. We denote by \( N_{(p)}\left(r, \frac{1}{f-a}\right) \) the counting function of the zeros of \( f-a \) with multiplicities at least \( p \), and by \( \overline{N}_{(p)}\left(r, \frac{1}{f-a}\right) \) the corresponding reduced counting function. Moreover, we set \( N_{p}\left(r, \frac{1}{f-a}\right) = N_{(1)}\left(r, \frac{1}{f-a}\right) + N_{(2)}\left(r, \frac{1}{f-a}\right) + \cdots + N_{(p)}\left(r, \frac{1}{f-a}\right) \).

**2. Some auxiliary results**

In order to prove our main results, we shall need the following lemmas.

**LEMMA 1.** [16] Let \( f(z) \) be a meromorphic function in \( \mathbb{C} \). Then for all irreducible rational function \( R(z,f) \) in \( f \) with coefficients meromorphic and small with respect to \( f \), we have

\[
T(r,R(z,f)) = dT(r,f) + S(r,f),
\]

where \( S(r,f) = o(T(r,f)) \) for \( r \not\in E \), \( E \) is a set with finite measure and \( d \) is the degree of \( R(z,f) \) in \( f \).
Lemma 2. [21] Let $f(z)$ and $g(z)$ be two meromorphic functions in $\mathbb{C}$. Let
\[ H = \frac{f''}{f'} - \frac{2f'}{f-1} - \left( \frac{g''}{g'} - \frac{2g'}{g-1} \right). \]
If $f$ and $g$ share the value 1 CM and $H \neq 0$, then
\[ N_1\left(r, \frac{1}{f-1}\right) \leq N(r, f) + S(r, f) + S(r, g). \]

Lemma 3. [21] Let $h = \frac{f''}{f'} - \frac{2f'}{f-1}$, where $f$ is a nonconstant meromorphic function. If $z_0$ is a simple pole of $f$, then $h$ is regular at $z_0$.

Lemma 4. [4] Let $A(z) = (n-1)^2(z^n - 1)(z^{n-2} - 1) - n(n-2)(z^{n-1} - 1)^2$, where $n \geq 5$ is an integer, then $A(z) = (z-1)^4(z - \beta_1) \cdots (z - \beta_{2n-6})$, where $\beta_j \in \mathbb{C} \setminus \{0, 1\}$ ($j = 1, 2, \cdots, 2n-6$), which are distinct respectively.

Lemma 5. Suppose that $R(z)$ is defined as (2), $f$ and $g$ are nonconstant meromorphic functions satisfying $E(\{0, 1\}, f) = E(\{0, 1\}, g)$. If $R(f) \equiv R(g)$, then $f \equiv g$.

Proof. We consider the case for $n = 5$, the same argument will be hold for $n \geq 6$. The equation $R(f) \equiv R(g)$ yields that $T(r, f) = T(r, g) + S(r, f)$ and $S(r, f) = S(r, g)$. Moreover, we rewrite it as
\[
\frac{f^5}{(f - \alpha_1)(f - \alpha_2)} \equiv \frac{g^5}{(g - \alpha_1)(g - \alpha_2)}.
\]  
where $\alpha_1$ and $\alpha_2$ are two distinct roots of the equation $10z^2 - 15z + 6 = 0$.

In addition, we have $R'(f) \equiv R'(g)$, that is
\[
\frac{f^4(f-1)^2f'}{(f - \alpha_1)^2(f - \alpha_2)^2} \equiv \frac{g^4(g-1)^2g'}{(g - \alpha_1)^2(g - \alpha_2)^2}.
\]
Combining (5) and (6), we obtain
\[
\frac{(f-1)^2f'}{f^6} \equiv \frac{(g-1)^2g'}{g^6}.
\]
We can rewrite (7) as follows.
\[
\left( \frac{1}{f} \right)^2 \left( 1 - \frac{1}{f} \right)^2 \left( \frac{1}{f} \right)' \equiv \left( \frac{1}{g} \right)^2 \left( 1 - \frac{1}{g} \right)^2 \left( \frac{1}{g} \right)'.
\]
Set $f^* = 1 - \frac{1}{f}$ and $g^* = 1 - \frac{1}{g}$, then (8) can be rewritten as
\[
(1 - f^*)^2(f^*)^2(f^*)_f' \equiv (1 - g^*)^2(g^*)^2(g^*)_f'.
\]
Integrating both sides of the equality, we get
\[ 6(f^*)^5 - 15(f^*)^4 + 10(f^*)^3 \equiv 6(g^*)^5 - 15(g^*)^4 + 10(g^*)^3 + c, \tag{9} \]
where \( c \) is a constant.

Since \( f \) and \( g \) share \( \{0, 1\} \), we can obtain \( f \) and \( g \) share 0 CM and 1 CM from (5). Therefore, we can deduce that \( f^* \) and \( g^* \) share \( \infty \) CM and 0 CM. Now we consider the following two cases.

**Case I.** If 0 is not a Picard exceptional value of \( f^* \), then from (9), we can obtain that \( c = 0 \). Set \( h = \frac{f^*}{g^*} \), then \( h \neq \infty, z \in \mathbb{C} \). Combining (9), we get
\[ 6(g^*)^2(h^5 - 1) - 15g^*(h^4 - 1) + 10(h^3 - 1) \equiv 0. \tag{10} \]
Therefore,
\[ \left( 12(h^5 - 1)g^* - 15(h^4 - 1) \right)^2 \equiv -15\left[ 16(h^3 - 1)(h^5 - 1) - 15(h^4 - 1)^2 \right]. \]
Suppose that \( h \) is not a constant, by Lemma 4, we obtain
\[ \left( 12(h^5 - 1)g^* - 15(h^4 - 1) \right)^2 \equiv -15(h - 1)^4(h - \beta_1) \cdots (h - \beta_4), \]
where \( \beta_j \in \mathbb{C} \setminus \{0, 1\} \), \( (j = 1, \cdots, 4) \), which are distinct respectively.

This implies that every zero of \( h - \beta_j \ (j = 1, \cdots, 4) \) has a multiplicity at least 2. By the second fundamental theorem, we get a contradiction. Therefore, \( h \) is a constant, from (10) we get \( h \equiv 1 \), hence \( f^* \equiv g^* \). Furthermore, we have \( f \equiv g \).

**Case II.** If 0 is a Picard exceptional value of \( f^* \), then it is the Picard exceptional value of \( g^* \) too. Similar to Case I, we can get \( f \equiv g \) when the case \( c = 0 \). Therefore, we only consider the following equation.
\[ 6(f^*)^5 - 15(f^*)^4 + 10(f^*)^3 \equiv 6(g^*)^5 - 15(g^*)^4 + 10(g^*)^3 + c, \tag{11} \]
where \( c \neq 0 \).

Set \( Q(z) = 6z^5 - 15z^4 + 10z^3 + c \), then \( Q'(z) = 30z^2(z - 1)^2 \). We can deduce that \( z = 0 \) is not a multiple zero of \( Q(z) \) for \( c \neq 0 \). In addition, if \( z = 1 \) is a multiple zero of \( Q(z) \), then we can obtain that \( c = -1 \). It follows that the equation \( 6z^5 - 15z^4 + 10z^3 + c = 0 \) has five distinct roots when \( c \neq 0, -1 \).

Next, we consider the case \( c \neq -1 \) in (11), we have
\[ 6(g^*)^5 - 15(g^*)^4 + 10(g^*)^3 + c \equiv 6(f^*)^3(f^* - \gamma_1)(f^* - \gamma_2), \]
where \( \gamma_1, \gamma_2 \) are the roots of \( z^2 - \frac{5}{2}z + \frac{5}{3} = 0 \).
Noting that \( Q(z) = 6z^5 - 15z^4 + 10z^3 + c \) \((c \neq 0, -1)\) has five distinct zeros, which we denote as \( z_j \ (j = 1 \cdots 5) \), by second fundamental theorem, we have

\[
3T(r,g^*) \leq \sum_{j=1}^{5} N\left(r, \frac{1}{g^*-z_j}\right) + S(r,g^*)
\]

\[
\leq N\left(r, \frac{1}{f^*}\right) + N\left(r, \frac{1}{f^*-\gamma_1}\right) + N\left(r, \frac{1}{f^*-\gamma_2}\right) + S(r,g^*)
\]

\[
\leq 2T(r,f^*) + S(r,g^*),
\]

which gives a contradiction.

If \( c = -1 \), we rewrite (11) as

\[
6(f^*)^5 - 15(f^*)^4 + 10(f^*)^3 + 1 = 6(g^*)^5 - 15(g^*)^4 + 10(g^*)^3 = 6(g^*)^3(g^* - \gamma_1)(g^* - \gamma_2).
\]

Therefore, we also can get a contradiction similarly.

This completes the proof of Lemma 5. \( \square \)

**Lemma 6.** Let \( f(z) \) and \( g(z) \) be two meromorphic functions in \( \mathbb{C} \), and let \( F = R(f) \) and \( G = R(g) \), where \( R(z) \) is defined as (2) and \( n \geq 5 \). If \( AF \equiv G + B \), where \( A(\neq 0) \) and \( B \) are two constants, then one of the following cases holds.

1. \( AF \equiv G; \)
2. \( N\left(r, \frac{1}{g}\right) \neq S(r,g) \) and \( N\left(r, \frac{1}{f}\right) \neq S(r,f) \).

**Proof.** By the assumption, we obtain that \( T(r,f) = T(r,g) + S(r,f) \). Now we shall distinguish the following three cases to discuss when \( n = 5 \).

**Case I.** If \( B = 0 \), then \( AF \equiv G \), that is Case (I) holds.

**Case II.** If \( B = \frac{-a}{2} \), then \( AF \equiv G - \frac{a}{2} \), that is

\[
\frac{Af^5}{20f^2 - 30f + 12} = \frac{(g - 1)^3(g^2 + 3g + 6)}{20g^2 - 30g + 12}.
\]

We claim that Case (II) holds, namely \( N\left(r, \frac{1}{f}\right) \neq S(r,f) \) and \( N\left(r, \frac{1}{g}\right) \neq S(r,g) \). Otherwise, \( N\left(r, \frac{1}{f}\right) = S(r,f) \) or \( N\left(r, \frac{1}{g}\right) = S(r,g) \).

Firstly, we consider the case when \( N\left(r, \frac{1}{g}\right) = S(r,g) \). By the second fundamental theorem and (12), we get

\[
2T(r,g) \leq N\left(r, \frac{1}{g-1}\right) + N\left(r, \frac{1}{g-\beta_1}\right) + N\left(r, \frac{1}{g-\beta_2}\right) + N\left(r, \frac{1}{g}\right) + S(r,g)
\]
where $\beta_1, \beta_2$ are the roots of $\gamma^2 + 3\gamma + 6 = 0$, it is a contradiction.

Next, we discuss the case when $N\left(r, \frac{1}{f}\right) = S(r, f)$, we also can get

$$T(r, g) \leq \mathcal{N}\left(r, \frac{1}{g - 1}\right) + \mathcal{N}\left(r, \frac{1}{g - \beta_1}\right) + \mathcal{N}\left(r, \frac{1}{g - \beta_2}\right) + S(r, g)$$

$$\leq \mathcal{N}\left(r, \frac{1}{f}\right) + S(r, g) = S(r),$$

a contradiction.

Therefore, Case (II) holds.

**Case III.** $B \neq 0$, noting that $R(z) - c = 0 \quad (c \neq 0, \frac{a}{2})$ has five distinct roots, by the second fundamental theorem, we have

$$3T(r, g) \leq \sum_{j=1}^{5} \mathcal{N}\left(r, \frac{1}{g - \gamma_j}\right) + S(r, g) \leq \mathcal{N}\left(r, \frac{1}{f}\right) + S(r, g) \leq T(r, f) + S(r, g),$$

where $\gamma_j \ (j = 1, \ldots, 5)$ are the roots of $G + B = 0$, it is a contradiction.

The same argument for $n \geq 6$, This completes the proof of Lemma 6.

**Remark 3.** Under the condition of Lemma 6 and “$n \geq 6$”, we can deduce the further result that $AF \equiv G$.

**Lemma 7.** Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in $\mathbb{C}$ such that $\mathcal{N}\left(r, \frac{1}{f - 1}\right) + \mathcal{N}\left(r, \frac{1}{g - 1}\right) = S(r, f) + S(r, g)$, $R(z)$ is defined as (2) and $\frac{1}{R(f) - 1} \equiv \frac{A}{R(g) - 1} + B$, where $A(\neq 0)$ and $B$ are two constants. If $f$ and $g$ share $\{0, 1\} \ CM$ and $n \geq 5$, then $B = 0$.

**Proof.** For the convenience, we put $F = R(f), G = R(g)$. Suppose that $B \neq 0$, by the assumption, we have

$$\frac{1}{F - 1} \equiv \frac{B\left(G + A - B\right)}{G - 1}.$$  \hspace{1cm} (13)

Moreover, we have $T(r, f) = T(r, g) + S(r, f)$.

We distinguish the following three cases to discuss.

**Case I.** $A = B$, the equation (13) is rewritten as

$$\frac{1}{F - 1} \equiv \frac{BG}{G - 1}.$$ \hspace{1cm} (14)

Thus, we have $\mathcal{N}(r, F) = \mathcal{N}\left(r, \frac{1}{G}\right)$, that is

$$\mathcal{N}(r, f) + \mathcal{N}\left(r, \frac{1}{f - \alpha_1}\right) + \mathcal{N}\left(r, \frac{1}{f - \alpha_2}\right) = \mathcal{N}\left(r, \frac{1}{g}\right).$$
From the second fundamental theorem, we obtain
\[
2T(r, f) \leq \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f-\alpha_1}\right) + \overline{N}\left(r, \frac{1}{f-\alpha_2}\right) + S(r, f)
\]
\[
\leq \overline{N}\left(r, \frac{1}{g}\right) + S(r, f) \leq T(r, g) + S(r, f),
\]
a contradiction.

**Case II.** \( \frac{A-B}{B} = -\frac{a}{2} \), the equation (13) is rewritten as
\[
\frac{1}{F-1} \equiv \frac{B(G-a)}{G-1}, \quad (15)
\]
We claim that \( \overline{N}\left(r, \frac{1}{f}\right) \neq S(r, f) \). Indeed, if \( \overline{N}\left(r, \frac{1}{f}\right) = S(r, f) \), by (15), we have
\[
\overline{N}(r, F) = \overline{N}\left(r, \frac{1}{G-a/2}\right),
\]
that is
\[
\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f-\alpha_1}\right) + \overline{N}\left(r, \frac{1}{f-\alpha_2}\right) = \overline{N}\left(r, \frac{1}{g-1}\right) + \overline{N}\left(r, \frac{1}{g-\beta_1}\right) + \overline{N}\left(r, \frac{1}{g-\beta_2}\right),
\]
where \( \beta_1, \beta_2 \) are the roots of \( z^2 + 3z + 6 = 0 \).

Thus, using the second fundamental theorem, we have
\[
3T(r, f) \leq \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f-\alpha_1}\right) + \overline{N}\left(r, \frac{1}{f-\alpha_2}\right) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, g)
\]
\[
\leq \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}\left(r, \frac{1}{g-1}\right) + \overline{N}\left(r, \frac{1}{g-\beta_1}\right) + \overline{N}\left(r, \frac{1}{g-\beta_2}\right) + \overline{N}\left(r, \frac{1}{f}\right)
\]
\[
+ S(r, g)
\]
\[
\leq 2T(r, g) + S(r, g),
\]
a contradiction.

Therefore, there exists \( z_0 \) such that \( f(z_0) = 0 \), then \( g(z_0) = 0 \) or \( g(z_0) = 1 \). By (15), it is easy to obtain a contradiction when \( g(z_0) = 1 \). Therefore, we have \( g(z_0) = 0 \).

Combing (15), we deduce \( B = -\frac{2}{a} \). Rewriting (15) as
\[
G \equiv \frac{aF}{2F - 2 + a}
\]
It follows that \( \overline{N}(r, G) = \overline{N}\left(r, \frac{1}{F-1 + a/2}\right) \). Since \( a \neq 1, 2 \), we have \( 1 - \frac{a}{2} \neq 0, \frac{a}{2} \), by
the second fundamental theorem, we get

\[
4T(r,f) \leq \sum_{j=1}^{5} N\left(r, \frac{1}{f-x_j}\right) + N\left(r, \frac{1}{f^{\prime}-1}\right) + S(r,f) \\
\leq N(r,g) + N\left(r, \frac{1}{g-x_1}\right) + N\left(r, \frac{1}{g-x_2}\right) + S(r,f) \\
\leq 3T(r,g) + S(r,f),
\]

where \(x_j\ (j = 1, \ldots, 5)\) are five distinct roots of \(2F - 2 + a = 0\) respectively, it is a contradiction.

**Case III.** \(A \neq B\) and \(\frac{A-B}{B} \neq -\frac{a}{2}\), by (13), we have \(N(r,F) = N\left(r, \frac{1}{G+\frac{A-B}{B}}\right)\).

Noting that \(BG + A - B = 0\) has at least five distinct roots respectively, similar to the case II, we can also deduce a contradiction.

Therefore \(B = 0\). This completes the proof of Lemma 7. \(\Box\)

### 3. Main results

In this section, we shall give the main result as follows.

**THEOREM 1.** Let \(f(z)\) and \(g(z)\) be two non-constant meromorphic functions in \(\mathbb{C}\), and let \(S_1 = \{0\}\), \(S_2 = \{1\}\) or \(\{\infty\}\), \(S_3 = \{z|P(z) = 0\}\), where \(P(z)\) is defined as (1) and \(a \neq 0, 1, 2\). If \(E(S_j,f) = E(S_j,g)\) \((j = 1, 2, 3)\) and \(n \geq 5\), then \(f \equiv g\).

**REMARK 1.** Suppose that \(S_1 = \{0\}, S_2 = \{1\}, S_3 = \{z|z^5 - 20z^2 + 30z - 12 = 0\}\). Let \(f(z) = e^z + 1\) and \(g(z) = e^{-z} + 1\). It is easy to see that \(f\) and \(g\) satisfy \(E(S_j,f) = E(S_j,g)\) \((j = 1, 2, 3)\), but \(f \not\equiv g\). This shows that the assumption \(a \neq 1\) in Theorem 1 cannot be omitted.

**REMARK 2.** Suppose that \(a = 2\). It is easy to see that \(z = 1\) is one root with multiplicity 3 of the equation \(2z^n - n(n-1)z^2 + 2n(n-2)z - (n-1)(n-2) = 0\), where \(n \geq 5\) is an integer. Thus, \(S_3\) in Theorem 1 is not a subset of distinct element in \(\mathbb{C}\) with \(\#(S_3) = n\). This shows that the assumption \(a \neq 2\) in Theorem 1 is needed.

**Proof of Theorem 1.** When \(S_2 = \{\infty\}\), by the Ref. [10] we obtain that Theorem 1 is valid. Therefore, we only need to prove Theorem 1 in the case \(S_2 = \{1\}\).

Firstly, we consider the case for \(n = 5\).

For the sake of simplicity, we set

\[
P(f) = af^5 - 20f^3 + 30f - 12, \quad P(g) = ag^5 - 20g^3 + 30g - 12,
\]

\[
F = \frac{af^5}{20f^3 - 30f + 12} = \frac{af^5}{20(f - \alpha_1)(f - \alpha_2)}.
\]
\[ G = \frac{ag^5}{20g^2 - 30g + 12} = \frac{af^5}{20(g - \alpha_1)(g - \alpha_2)}. \]

Then we have \( F \) and \( G \) share 0, 1 CM and

\[ F' = \frac{3af^4(f - 1)^2 f'}{20(f - \alpha_1)^2(f - \alpha_2)^2}, \quad G' = \frac{3ag^4(g - 1)^2 g'}{20(g - \alpha_1)^2(g - \alpha_2)^2}, \]

\[ F - 1 = \frac{P(f)}{20(f - \alpha_1)(f - \alpha_2)}, \quad G - 1 = \frac{P(g)}{20(g - \alpha_1)(g - \alpha_2)}, \]

\[ F - \frac{a}{2} = \frac{a(f - 1)^3(f - \beta_1)(f - \beta_2)}{20(f - \alpha_1)(f - \alpha_2)}, \quad G - \frac{a}{2} = \frac{a(g - 1)^3(g - \beta_1)(g - \beta_2)}{20(g - \alpha_1)(g - \alpha_2)}, \]

where \( \beta_1, \beta_2 \) are the roots of \( z^2 + 3z + 6 = 0 \). We consider the following two cases.

**Case I.** \( N\left( r, \frac{1}{f - 1} \right) + N\left( r, \frac{1}{g - 1} \right) \neq S(r) \), where \( S(r) = \max \{ S(r, f), S(r, g) \} \).

Set

\[ \varphi = \frac{F'}{(F - 1)f} - \frac{G'}{(G - 1)g}, \]

that is,

\[ \varphi = \frac{(f - 1)^2 f'}{P(f)f} - \frac{(g - 1)^2 g'}{P(g)g}. \]

Suppose that \( \varphi \neq 0 \), since \( F \) and \( G \) share 0, 1 CM, we obtain

\[ m(r, \varphi) = S(r), \quad N(r, \varphi) = S(r). \]  \hspace{1cm} (16)

By the first fundamental theorem, we have

\[ 2N\left( r, \frac{1}{f - 1} \right) \leq N\left( r, \frac{1}{\varphi} \right) \leq T(r, \varphi) = S(r). \]  \hspace{1cm} (17)

Similarly, we have

\[ 2N\left( r, \frac{1}{g - 1} \right) \leq T(r, \varphi) = S(r). \]  \hspace{1cm} (18)

Combining (17) and (18), we obtain

\[ N\left( r, \frac{1}{f - 1} \right) + N\left( r, \frac{1}{g - 1} \right) = S(r), \]

a contradiction.

Therefore, we have \( \varphi \equiv 0 \), that is \( \frac{F'}{(F - 1)f} \equiv \frac{G'}{(G - 1)g} \). Integrating both sides of the equation, we get

\[ \frac{F}{F - 1} \equiv \frac{AG}{G - 1}, \]  \hspace{1cm} (19)

where \( A \neq 0 \) is a constant.
In the condition of \( N\left(r, \frac{1}{f-1}\right) + N\left(r, \frac{1}{g-1}\right) \neq S(r) \), we get that there exists \( z_0 \in \mathbb{C} \), such that \( f(z_0) = 1 \) and \( g(z_0) = 1 \). Therefore, we have \( A = 1 \). Furthermore, we get \( F \equiv G \), by Lemma 5, we have \( f \equiv g \).

**Case II.** \( N\left(r, \frac{1}{f-1}\right) + N\left(r, \frac{1}{g-1}\right) = S(r) \), where \( S(r) = \max\left\{ S(r, f), S(r, g) \right\} \).

Set

\[
H = \frac{F''}{F'} - \frac{2F'}{F-1} - \left( \frac{G''}{G'} - \frac{2G'}{G-1} \right).
\]

Suppose that \( H \not\equiv 0 \), from Lemma 3, we have

\[
N(r, H) \leq \overline{N}(r, f) + \overline{N}(r, g) + \overline{N}_2\left(r, \frac{1}{f - \alpha_1}\right) + \overline{N}_2\left(r, \frac{1}{f - \alpha_2}\right) + \overline{N}_2\left(r, \frac{1}{g - \alpha_1}\right)
\]

\[
+ \overline{N}_2\left(r, \frac{1}{g - \alpha_2}\right) + \overline{N}\left(r, \frac{1}{f'}\right) + \overline{N}'\left(r, \frac{1}{g'}\right),
\]

where \( \alpha_1, \alpha_2 \) are the roots of \( 20z^2 - 30z + 12 = 0 \) and \( \alpha_1, \alpha_2 \neq 0, 1, 2, \overline{N}^+(r, \frac{1}{f'}) \) denotes the reduced counting function of the zeros of \( f' \) which are not the zeros of \( f(f-1)(f - \alpha_1)(f - \alpha_2)(F-1) \).

By Lemma 2, we have \( N_1\left(r, \frac{1}{F-1}\right) \leq N\left(r, \frac{1}{H}\right) \). Therefore, we have

\[
N\left(r, \frac{1}{F-1}\right) = N_1\left(r, \frac{1}{F-1}\right) + N_2\left(r, \frac{1}{F-1}\right) \leq N\left(r, \frac{1}{H}\right) + N_2\left(r, \frac{1}{F-1}\right)
\]

\[
\leq \overline{N}(r, f) + \overline{N}(r, g) + \overline{N}_2\left(r, \frac{1}{f - \alpha_1}\right) + \overline{N}_2\left(r, \frac{1}{f - \alpha_2}\right) + \overline{N}_2\left(r, \frac{1}{g - \alpha_1}\right)
\]

\[
+ \overline{N}_2\left(r, \frac{1}{g - \alpha_2}\right) + N_0\left(r, \frac{1}{f'}\right) + N_0\left(r, \frac{1}{g'}\right),
\]

(20)

where \( N_0\left(r, \frac{1}{f'}\right) \) denotes the counting function of the zeros of \( f' \) which are not the zeros of \( f(f-1)(f - \alpha_1)(f - \alpha_2) \).

The same argument shows that

\[
N\left(r, \frac{1}{G-1}\right) \leq \overline{N}(r, f) + \overline{N}(r, g) + \overline{N}_2\left(r, \frac{1}{f - \alpha_1}\right) + \overline{N}_2\left(r, \frac{1}{f - \alpha_2}\right) + \overline{N}_2\left(r, \frac{1}{g - \alpha_1}\right)
\]

\[
+ \overline{N}_2\left(r, \frac{1}{g - \alpha_2}\right) + N_0\left(r, \frac{1}{f'}\right) + N_0\left(r, \frac{1}{g'}\right).
\]

(21)

Set \( \psi = \frac{(f-2)f'}{(f-1)(f - \alpha_1)(f - \alpha_2)} \), then it is easy to see that

\[
m(r, \psi) = S(r, f), \quad N(r, \psi) \leq \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}\left(r, \frac{1}{f - \alpha_1}\right) + \overline{N}\left(r, \frac{1}{f - \alpha_2}\right).
\]
On the other hand, we have

$$N\left(r, \frac{1}{f' - 2}\right) + N_0^+\left(r, \frac{1}{f'}\right) \leq N\left(r, \frac{1}{\psi}\right) \leq N(r, \psi) + S(r, f),$$

where $N^+_0\left(r, \frac{1}{f'}\right)$ denotes the counting function of the zeros of $f'$ which are not the zeros of $(f - 1)(f - \alpha_1)(f - \alpha_2)$. Thus,

$$N\left(r, \frac{1}{f - 2}\right) + N^+_0\left(r, \frac{1}{f'}\right) \leq N\left(r, \frac{1}{f - 1}\right) + N\left(r, \frac{1}{f - \alpha_1}\right) + N\left(r, \frac{1}{f - \alpha_2}\right) + S(r, f).$$

The same argument shows that

$$N\left(r, \frac{1}{g - 2}\right) + N^+_0\left(r, \frac{1}{g'}\right) \leq N\left(r, \frac{1}{g - 1}\right) + N\left(r, \frac{1}{g - \alpha_1}\right) + N\left(r, \frac{1}{g - \alpha_2}\right) + S(r, g).$$

Noting that $a \neq 1$, thus $z = 2$ is not the roof of $az^5 - 20z^2 + 30z - 12 = 0$ and $N_0\left(r, \frac{1}{f'}\right) \leq N^+_0\left(r, \frac{1}{f'}\right)$. Therefore, using the second fundamental theorem and combining with (20)–(23), we deduce

$$8\left(T(r, f) + T(r, g)\right) \leq N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f - \alpha_1}\right) + N\left(r, \frac{1}{f - \alpha_2}\right) + N\left(r, \frac{1}{F - 1}\right)$$
$$+ N\left(r, \frac{1}{f - 2}\right) + N\left(r, \frac{1}{f - 1}\right) + N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g - \alpha_1}\right),$$
$$+ N\left(r, \frac{1}{g - \alpha_2}\right) + N\left(r, \frac{1}{G - 1}\right) + N\left(r, \frac{1}{g - 2}\right) + N\left(r, \frac{1}{g - 1}\right)$$
$$- N_0\left(r, \frac{1}{f'}\right) - N_0\left(r, \frac{1}{g'}\right) + S(r, f) + S(r, g)$$
$$\leq 2\left(N(r, f) + N(r, g)\right) + 2\left(N_2\left(r, \frac{1}{f - \alpha_1}\right) + N_2\left(r, \frac{1}{g - \alpha_1}\right)\right)$$
$$+ 2\left(N_2\left(r, \frac{1}{f - \alpha_2}\right) + N_2\left(r, \frac{1}{g - \alpha_2}\right)\right) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right)$$
$$+ 2\left(N\left(r, \frac{1}{f - 1}\right) + N\left(r, \frac{1}{g - 1}\right)\right) + S(r, f) + S(r, g)$$
$$\leq 7\left(T(r, f) + T(r, g)\right) + S(r, f) + S(r, g),$$

a contradiction.

Therefore, we have $H \equiv 0$, namely

$$\frac{F''}{F'} - \frac{2F'}{F - 1} \equiv \frac{G''}{G'} - \frac{2G'}{G - 1}.$$

Integrating both sides of the equality, we get

$$\frac{1}{F - 1} \equiv \frac{A}{G - 1} + B,$$
where \( A \neq 0, B \) are constants.

By Lemma 7, we have \( B = 0 \), so that \( \frac{1}{F - 1} \equiv \frac{A}{G - 1} \), we may write this as \( AF = G + (A - 1) \).

Applying Lemma 6, we obtain that either \( A = 1 \) or \( N\left(r, \frac{1}{g}\right) \neq S(r) \) and \( N\left(r, \frac{1}{f}\right) \neq S(r) \), when \( N\left(r, \frac{1}{g}\right) \neq S(r) \) and \( N\left(r, \frac{1}{f}\right) \neq S(r) \), there exists \( z_0 \) such that \( f(z_0) = g(z_0) = 0 \), and hence, we also get \( A = 1 \). Thus, by Lemma 5, we get \( f \equiv g \).

Similar to the proceeding of proof for \( n = 5 \), we can obtain \( f \equiv g \) for \( n \geq 6 \). Therefore, we complete the proof of Theorem 1. \( \Box \)

4. Further Remarks

In the direction to the question of Gross, Yi [17] also proved that there exists \( S_1 \) \((\sharp(S_1) = 2)\) and \( S_2 \) \((\sharp(S_2) = 9)\) such that any two meromorphic functions \( f \) and \( g \) satisfying \( E(S_j, f) = E(S_j, g) \) for \( j = 1, 2 \) must be identical.

**Theorem D.** [17] Let \( S_1 = \{a + b, a + bw, \cdots, a + bw^{n - 1}\} \) and let \( S_2 = \{c_1, c_2\} \), where \( n > 8 \), \((c_1 - a)^n \neq (c_2 - a)^n, (c_1 - a)^n(c_2 - a)^n \neq b^{2n}(k, j = 1, 2)\). If \( f \) and \( g \) are non-constant meromorphic functions such that \( E(S_j, f) = E(S_j, g) \) for \( j = 1, 2 \), then \( f \equiv g \).

Whereafter, Li and Yang [9], Yi [18] proved that there exists a set \( S \) \((\sharp(S) = 11)\) such that the conditions \( E(S, f) = E(S, g) \) and \( E(\infty, f) = E(\infty, g) \) imply \( f(z) \equiv g(z) \) for any pair of non-constant meromorphic functions \( f \) and \( g \).

In 1997, Fang and Guo [2] extended the result of Li and Yang [9], Yi [18], and proved that \( \sharp(S) = 9 \). Afterwards, I. Lahiri [8], Fang and Lahiri [3]. H. Yi and the present author [22] also obtained \( \sharp(S) = 8 \) or \( \sharp(S) = 7 \) under adding certain condition respectively.

By the polynomial \( P(z) \) defined as \((1)\), Yi proved the following result.

**Theorem E.** [20] Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions in \( \mathbb{C} \), and let \( S = \left\{ z \mid P(z) = 0 \right\} \), where \( a \neq 0, 2 \) is a constant. If \( E(S, f) = E(S, g) \), \( E(\infty, f) = E(\infty, g) \) and \( n \geq 8 \), then \( f \equiv g \).

From Theorem E we immediately obtain that there exist two sets \( S_1 \) \((\sharp(S_1) = 1)\) and \( S_2 \) \((\sharp(S_2) = 8)\) such that any two meromorphic functions \( f \) and \( g \) sharing \( S_j \) \((j = 1, 2)\) with the same multiplicities must be identical.

Noting that Lemma 5 and Lemma 7 are valid under the condition \( “E(\{0, 1\}, f) = E(\{0, 1\}, g)” \), i.e. \( \sharp(S_1) = 2 \) and \( n \geq 5 \), so we are natural to pose an open question.

**Open Question.** Whether can Theorem 1 hold under the condition \( E(\{0, 1\}, f) = E(\{0, 1\}, g) \)?

In recent years, along with the value distribution theory of meromorphic functions, many authors introduced and investigated the approximations of functions in various
fields [1]. Furthermore, V. N. Mishra constructed and investigated various properties on the approximations of functions in Banach spaces [11]. Naturally, we are interesting to know what happen on the subject of difference and $q$-difference under the sharing-set conditions of Theorem 1. Unfortunately, we do not find the effective method to resolve it.

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