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CONVERGENCE ESTIMATES OF CERTAIN q–BETA–SZÁSZ TYPE OPERATORS

RANI YADAV AND MAN SINGH BENIWAL

Abstract. The applications of q calculus in operator theory is an active area of research in the last two decades. Several new q-operators were introduced and their approximation behavior were discussed. This paper is extension of the paper [South. Asian Bul. Math. (36) (2012), 343–352]. We propose here the Stancu variant of q-Beta-Szász type operators. We estimate the moments and also establish direct results in terms of modulus of continuity.

1. Introduction

The q-calculus in approximation theory is an interesting area of research. In the year 1987, first q-analogue of classical Bernstein polynomials was given by A. Lupas [18]. The most important q-analogue of the Bernstein polynomials was introduced in 1997 by Phillips [19]. After that many researchers worked in this direction and proposed many q-operators and studied their different properties related to special function, number theory and convergence behavior. Gupta et al. [12] established the generating functions of some q-basis functions. Related to the number theory the main contribution is due to Kim and collaborators (see [15], [16], [17]). In the theory of approximation, the convergence is important in this context, we mention some of the results for convergence of q-discrete operators due to [1], [2] and [3] etc.

Discrete operators are not possible to approximate integrable functions. In the year 2008 Gupta [11] introduced an important q-analogue of the Bernstein Durrmeyer operators based on q-beta function of first kind. Later based on q-beta function of second kind Aral and Gupta [4] introduced q-Baskakov Durrmeyer operators. Recently Gupta and Yadav [14] introduced the Durrmeyer type mixed q-operators having Beta and Szász basis functions in summation and integration.

Also, very recently Buyukyazici and collaborators have proposed the Stancu variants of several well-known operators and estimated some direct results (see e.g. [6], [7], [8] etc). Actually the Stancu variant is based on two parameters and it generalizes the original operator. Motivated by the recent studies, we now propose the Stancu type generalization of the Beta-Szász operators. For $0<q<1$ and $0 \leqslant \alpha \leqslant \beta$, we propose here the q-Beta-Szász-Stancu operators as

$$B^{q}_{n,\alpha,\beta}(f,x) = \sum_{k=0}^{\infty} p^{q}_{n,k}(x) \int_{0}^{q/(1-q^n)} q^{-k-1} s^{q}_{n,k}(t) f\left(\frac{[n]_q t q^{-k-1}+\alpha}{[n]_q+\beta}\right) d_q t, \qquad (1.1)$$

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where $b_{n,k}(x)$ and $s_{n,k}(t)$ are Beta and Szász basis functions and

$$p_{n,k}^{q}(x) = \frac{q^{\frac{k(k-1)}{2}}}{B_q(k+1,n)} \frac{x^k}{(1+x)_q^{n+k+1}}$$

and

$$s_{n,k}^{q}(t) = E_q(-[n]_q t) \frac{([n]_q t)^k}{[k]_q!}.$$

As a special case when $\alpha = \beta = 0$ and $q = 1$, the above operators reduce to the Beta-Szász operators introduced by Gupta and Srivastava [13]. Also for $\alpha = \beta = 0$ and $q \in (0,1)$, we get the operators $B_{n,0,0}^{q}(f,x) = B_n^q(f,x)$ recently introduced and studied in [14]. Recently Aral-Gupta-Agarwal [5] compiled some of the important results in their recent book. We use the similar notations for q-calculus here, as used in [5]. For ready reference we give below some of the notations as:

$$(1+x)_q^n := \begin{cases} (1+x)(1+qx)\dots\left(1+q^{n-1}x\right), & n = 1,2,\dots \\ 1, & n = 0. \end{cases}$$

The q-integer and q-factorial are defined as

$$[n]_q := \frac{1-q^n}{1-q},$$

$$[n]_q! := \begin{cases} [n]_q [n-1]_q \cdots [1]_q, & n = 1,2,\dots \\ 1, & n = 0 \end{cases}$$

and the q-analogues of the exponential function is considered as

$$E_q(z) = \prod_{j=0}^{\infty}(1+(1-q)q^j z) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{z^k}{[k]_q!}, \quad |q| < 1.$$

In the present article, we estimate the moments of the q-Beta-Szász-Stancu operators and establish some direct results which include the error estimation in terms of modulus of continuity and the weighted approximation for these operators (1.1).

2. Moment estimations

In the sequel, we need the following moment estimates:

LEMMA 2.1. [14] *The following equalities hold:*

i) $B_n^q(1,x) = 1$,

ii) $B_n^q(t,x) = x\left(1 + \frac{1}{q[n]_q}\right) + \frac{1}{[n]_q}$,

iii) $B_n^q(t^2,x) = \frac{[n+1]_q [n+2]_q}{q^3 [n]_q^2} x^2 + \frac{[n+1]_q}{q^2 [n]_q^2}(1+2q+q^2)x + \frac{[2]_q}{[n]_q^2}.$

LEMMA 2.2. *For* $0 < q < 1$ *and* $0 \leqslant \alpha \leqslant \beta$, *we have*

$$\begin{aligned} B^q_{n,\alpha,\beta}(1,x) &= 1, \\ B^q_{n,\alpha,\beta}(t,x) &= \frac{x\,(q[n]_q+1)+q(1+\alpha)}{q\,([n]_q+\beta)}, \\ B^q_{n,\alpha,\beta}(t^2,x) &= \frac{[n+1]_q[n+2]_q x^2}{([n]_q+\beta)^2\, q^3} \\ &\quad + \frac{\left(q[n+1]_q(1+2q+q^2)+2\alpha([n]_q q^3+q^2)\right)x}{([n]_q+\beta)^2\, q^3} \\ &\quad + \frac{\left([2]_q+\alpha^2+1\right)q^3}{([n]_q+\beta)^2\, q^3}. \end{aligned}$$

Proof. By Lemma 2.1, it is obvious that

$$B^q_{n,\alpha,\beta}(1,x) = 1.$$

Further, we have

$$\begin{aligned} B^q_{n,\alpha,\beta}(t,x) &= \sum_{k=0}^{\infty} p^q_{n,k}(x) \int_0^{q/(1-q^n)} q^{-k-1} s^q_{n,k}(t) \left(\frac{[n]_q t q^{-k-1}+\alpha}{[n]_q+\beta} \right) d_q t \\ &= \frac{[n]_q}{[n]_q+\beta} B^q_n(t,x) + \frac{\alpha}{[n]_q+\beta} B^q_n(1,x) \\ &= \frac{[n]_q}{[n]_q+\beta} \left(x\left(1+\frac{1}{q[n]_q}\right) + \frac{1}{[n]_q} \right) + \frac{\alpha}{[n]_q+\beta} \\ &= \frac{x\,(q[n]_q+1)+q(1+\alpha)}{q\,([n]_q+\beta)}. \end{aligned}$$

Finally, we have

$$\begin{aligned} B^q_{n,\alpha,\beta}(t^2,x) &= \sum_{k=0}^{\infty} p^q_{n,k}(x) \int_0^{q/(1-q^n)} q^{-k-1} s^q_{n,k}(t) \left(\frac{[n]_q t q^{-k-1}+\alpha}{[n]_q+\beta} \right)^2 d_q t \\ &= \left(\frac{[n]_q}{[n]_q+\beta} \right)^2 B^q_n(t^2,x) + \left(\frac{2[n]_q\alpha}{([n]_q+\beta)^2} \right) B^q_n(t,x) + \left(\frac{\alpha}{[n]_q+\beta} \right)^2 B^q_n(1,x) \\ &= \left(\frac{[n]_q}{[n]_q+\beta} \right)^2 \left(\frac{[n+1]_q[n+2]_q}{q^3[n]_q^2} x^2 + \frac{[n+1]_q}{q^2[n]_q^2}(1+2q+q^2)x \frac{[2]_q}{[n]_q^2} \right) \\ &\quad + \left(\frac{2[n]_q\alpha}{([n]_q+\beta)^2} \right) \left(x\left(1+\frac{1}{q[n]_q}\right) + \frac{1}{[n]_q} \right) + \left(\frac{\alpha}{[n]_q+\beta} \right)^2 \\ &= x^2 \left[\frac{[n+1]_q[n+2]_q}{([n]_q+\beta)^2\, q^3} \right] \end{aligned}$$

$$+x\left[\frac{\left(q[n+1]_q(1+2q+q^2)+2\alpha([n]_q q^3+q^2)\right)}{([n]_q+\beta)^2 q^3}\right] + \frac{\left([2]_q+\alpha^2+1\right)q^3}{([n]_q+\beta)^2 q^3}. \quad \square$$

LEMMA 2.3. *For $x\in[0,\infty)$ and $q\in(0,1)$, the central moments are given as*

$$B_{n,\alpha,\beta}^{q}(t-x,x)=\frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)},$$

$$B_{n,\alpha,\beta}^{q}((t-x)^2,x)=x^2\left[\frac{[n+1]_q[n+2]_q}{q^3([n]_q+\beta)^2}-\frac{2(q[n]_q+1)}{q([n]_q+\beta)}+1\right] +x\left[\frac{[n+1]_q(1+2q+q^2)}{q^2([n]_q+\beta)^2}+\frac{2\alpha(q[n]_q+1)}{q([n]_q+\beta)^2}-\frac{2(1+\alpha)}{[n]_q+\beta}\right] +\frac{[2]_q+\alpha^2+2\alpha}{([n]_q+\beta)^2}.$$

As the operators $B_{n,\alpha,\beta}^{q}$ are linear, the proof of the above lemma is immediate, we omit the details.

3. Convergence estimates

By $C_B[0,\infty)$ we denote the space of real valued continuous bounded functions f on the interval $[0,\infty)$, the norm-$||.||$ on the space $C_B[0,\infty)$ is given by

$$||f||=\sup_{0\leqslant x<\infty}|f(x)|.$$

The Peetre's K-functional is defined by

$$K_2(f,\delta)=\inf\{||f-g||+\delta||g''||:g\in W^2\},$$

where $W^2=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}$. By [9], there exists a positive constant $C>0$ such that $K_2(f,\delta)\leqslant C\omega_2(f,\delta^{1/2})$, $\delta>0$ where the second order modulus of smoothness is given by

$$\omega_2(f,\delta)=\sup_{0<h\leqslant\sqrt{\delta}}\sup_{0\leqslant x<\infty}|f(x+2h)-2f(x+h)+f(x)|.$$

Also for $f\in C_B[0,\infty)$ the usual modulus of continuity is given by

$$\omega(f,\delta)=\sup_{0<h\leqslant\delta}\sup_{0\leqslant x<\infty}|f(x+h)-f(x)|.$$

THEOREM 3.1. *Let* $f \in C_B[0,\infty)$ *and* $0 < q < 1$. *Then for all* $x \in [0,\infty)$ *and* $n \in \mathbb{N}$, *there exists an absolute constant* $C > 0$ *such that*

$$|B^q_{n,\alpha,\beta}(f,x) - f(x)| \leqslant C\omega_2(f,\delta_n(x)) + \omega\left(f, \frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right),$$

where $\delta_n(x) = \left[B^q_{n,\alpha,\beta}\left((t-x)^2,x\right) + \left(\frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right)^2\right]^{1/2}$.

Proof. Let us introduce the auxiliary operators $\overline{B}^q_{n,\alpha,\beta}$ defined by

$$\overline{B}^q_{n,\alpha,\beta}(f,x) = B^q_{n,\alpha,\beta}(f,x) - f\left(x + \frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right) + f(x), \tag{3.1}$$

$x \in [0,\infty)$. The operators $\overline{B}^q_{n,\alpha,\beta}(f,x)$ are linear and preserve the linear functions:

$$\overline{B}^q_{n,\alpha,\beta}(t-x,x) = 0. \tag{3.2}$$

Let $g \in W^2$. From Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)\, g''(u)\, du, \quad t \in [0,\infty)$$

and (3.2), we get

$$\overline{B}^q_{n,\alpha,\beta}(g,x) \ = \ g(x) + \overline{B}^q_n\left(\int_x^t (t-u)\, g''(u)\, du, x\right).$$

Hence, by (3.1) one has

$$\begin{aligned}
|\overline{B}^q_{n,\alpha,\beta}(g,x) - g(x)| &\leqslant \left| B^q_{n,\alpha,\beta}\left(\int_x^t (t-u)\, g''(u)\, du, x\right)\right| \\
&\quad + \left|\int_x^{\frac{x(q[n]_q+1)+q(1+\alpha)}{q([n]_q+\beta)}} \left(\frac{x(q[n]_q+1)+q(1+\alpha)}{q([n]_q+\beta)} - u\right) g''(u)\, du\right| \\
&\leqslant B^q_{n,\alpha,\beta}\left(\left|\int_x^t |t-u|\, |g''(u)|\, du\right|, x\right) \\
&\quad + \int_x^{\frac{x(q[n]_q+1)+q(1+\alpha)}{q([n]_q+\beta)}} \left|\frac{x(q[n]_q+1)+q(1+\alpha)}{q([n]_q+\beta)} - u\right| |g''(u)| du \\
&\leqslant \left[B^q_{n,\alpha,\beta}\left((t-x)^2,x\right) + \left(\frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right)^2\right] \|g''\| \\
&= \delta_n^2(x)||g''||.
\end{aligned} \tag{3.3}$$

On the other hand, by (3.1), we have

$$|\overline{B}^q_{n,\alpha,\beta}(f,x)| \leqslant |B^q_{n,\alpha,\beta}(f,x)|+2||f|| \leqslant 3||f||. \tag{3.4}$$

Now (3.1), (3.3) and (3.4) imply

$$\begin{aligned}|B^q_{n,\alpha,\beta}(f,x)-f(x)| &\leqslant |\overline{B}^q_n(f-g,x)-(f-g)(x)|+|\overline{B}^q_{n,\alpha,\beta}(g,x)-g(x)| \\ &\quad + \left| f\left(x+\frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right)-f(x)\right| \\ &\leqslant 4\,\|f-g\|+\delta_n^2(x)\|g''\| \\ &\quad + \left| f\left(x+\frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right)-f(x)\right|.\end{aligned}$$

Hence taking infimum on the right hand side over all $g \in W^2$, we get

$$|B^q_{n,\alpha,\beta}(f,x)-f(x)| \leqslant CK_2\left(f,\delta_n^2(x)\right)+\omega\left(f,\frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right).$$

Using the property of K-functional

$$|B^q_{n,\alpha,\beta}(f,x)-f(x)| \leqslant C\omega_2\left(f,\delta_n(x)\right)+\omega\left(f,\frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right).$$

This completes the proof of the theorem. □

Let $H_{x^2}[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)| \leqslant M_f(1+x^2)$, where M_f is a constant depending only on f.

Also, let

$$C^*_{x^2}[0,\infty) := \left\{ f \in H_{x^2}[0,\infty) : \lim_{x\to\infty}\frac{f(x)}{1+x^2} < \infty \right\}.$$

The norm on $C^*_{x^2}[0,\infty)$ is defined as

$$||f||_{x^2} = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

We denote the modulus of continuity of f on closed interval $[0,a]$, $a>0$ as by

$$\omega_a(f,\delta) = \sup_{|t-x|\leqslant\delta}\ \sup_{x,t\in[0,a]} |f(t)-f(x)|.$$

We observe that for function $f \in C^*_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero, when $\delta \to 0$.

THEOREM 3.2. *Let $q=q_n$ satisfies $0<q_n<1$ and let $q_n \to 1$ as $n\to\infty$ for each $f \in C^*_{x^2}[0,\infty)$, we have*

$$\lim_{n\to\infty} ||B^{q_n}_{n,\alpha,\beta}(f,x)-f(x)||_{x^2} = 0$$

Proof. Using the Theorem in [10], we observe that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} ||B_{n,\alpha,\beta}^{q_n}(t^\nu,x)-x^\nu||_{x^2}=0, \quad \nu=0,1,2. \tag{3.5}$$

Since $B_{n,\alpha,\beta}^{q_n}(1,x)=1$, (3.5) holds for $\nu=0$.

$$\begin{aligned} ||B_{n,\alpha,\beta}^{q_n}(t,x)-x||_{x^2} &= \sup_{x\in[0,\infty)} \frac{x(1-q\beta)+q(1+\alpha)}{q\left([n]_q+\beta\right)}\frac{1}{1+x^2} \\ &\leqslant \frac{1-q\beta}{q\left([n]_q+\beta\right)}\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{q(1+\alpha)}{q\left([n]_q+\beta\right)}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}. \end{aligned}$$

Thus

$$\lim_{n\to\infty} ||B_{n,\alpha,\beta}^{q_n}(t,x)-x||_{x^2}=0.$$

$$\begin{aligned} ||B_{n,\alpha,\beta}^{q_n}(t^2,x)-x^2||_{x^2} \leqslant & \left(\frac{[n+1]_q[n+2]_q}{([n]_q+\beta)^2q^3}-1\right)\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2} \\ &+\left(\frac{q[n+1]_q(1+2q+q^2)+2\alpha([n]_qq^3+q^2)}{([n]_q+\beta)^2q^3}\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2} \\ &+\left(\frac{[2]_q+\alpha^2+q^3}{([n]_q+\beta)^2q^3}\right)\sup_{x\in[0,\infty)}\frac{1}{1+x^2} \end{aligned}$$

which implies that

$$\lim_{n\to\infty} ||B_{n,\alpha,\beta}^{q_n}(t^2,x)-x^2||_{x^2}=0. \quad \square$$

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Rani Yadav
Department of Applied Sciences, The Northcap University
(Formerly ITM University)
Sector 23-A Gurgaon-122017, India
e-mail: raniyadav23@gmail.com

Man Singh Beniwal
Department of Applied Sciences
Maharaja Surajmal Institute of Technology
New Delhi-110058, India
e-mail: man_s_2005@yahoo.co.in