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ON THE CONVOLUTION THEOREM FOR THE FOURIER TRANSFORM OF BV_0 FUNCTIONS

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Abstract. In this paper we prove the Convolution Theorem for the Fourier Integral transform over a subset of bounded variation functions which vanish at infinity. This subset is dense in $L^2(\mathbb{R})$. Moreover, it does not have inclusion relations with the space of Lebesgue integrable functions. We employ the Henstock-Kurzweil integral.

1. Introduction

Let $f,g:\mathbb{R}\longrightarrow\mathbb{R}$ be functions. The convolution of f with g is denoted as $f*g$, and the Fourier Integral transform of f is denoted by $\widehat{f}$. They are defined in $s,t\in\mathbb{R}$, whenever the integrals exist, as

$$(f*g)(t)=\int_{-\infty}^{\infty} f(t-x)g(x)dx \tag{1}$$

and

$$\widehat{f}(s)=\int_{-\infty}^{\infty} e^{-ist}f(t)dt. \tag{2}$$

Let us distinguish between "Fourier Integral transform" and "Fourier transform". The first will be that which has an expression as (2), and the second will be when it exists in another sense. For example that defined for the elements of L^p spaces, $1<p\leqslant 2$. In $L(\mathbb{R})$ the Fourier Integral transform fulfills the Convolution Theorem, that is: if $f,g\in L(\mathbb{R})$, then for every real number s we have

$$\widehat{f*g}(s)=\widehat{f}(s)\widehat{g}(s). \tag{3}$$

The convolution is defined in other spaces with respect to the Lebesgue integral. For example if $f\in L^p(\mathbb{R})$ and $g\in L^q(\mathbb{R})$, where p and q are conjugates. The case $p=q=2$ is important to argue the goal of our work because we know that the Fourier transform is extended unitarily onto all of $L^2(\mathbb{R})$ and the equality $\mathscr{F}(f*g)=\mathscr{F}(f)\mathscr{F}(g)$ is true. Here $\mathscr{F}$ denotes the Fourier transform in this case, see [4]. Since the Fourier transforms in $L^2(\mathbb{R})$ do not have necessarily a pointwise integral expression as (2), then it seems interesting to find classes of functions belonging

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to $L^2(\mathbb{R})\setminus L(\mathbb{R})$ for which their Fourier Integral transforms are defined, their convolution is expressed as (1), at the same time satisfying the relation (3).

Let us consider I a closed interval in $\mathbb{R}$. We will denote by $HK(I)$ the space of Henstock-Kurzweil integrable functions on I; as $HK_{loc}(I)$ the class of all Henstock-Kurzweil integrable functions on every compact interval contained in I; $BV(I)$ will be the space of bounded variation functions over I; and $BV_0(I)$ the space of bounded variation functions which vanish at infinity, this last case when I is not bounded. Over intervals, any Lebesgue integrable function and any Lebesgue integrable function in the improper sense are Henstock-Kurzweil integrable and their values coincide with the values from this last integral, see [2] and [9]. Further, $HK(I)$ is a semi-normed space with the Alexiewicz semi-norm definied as $||f||_A = \sup_{[c,d]\subset I}\left|\int_c^d f(t)dt\right|$.

An interesting sub-class of $HK(\mathbb{R})$ is $HK(\mathbb{R})\cap BV(\mathbb{R})$. This intersection is a dense subspace in $L^2(\mathbb{R})$. It is contained in $BV_0(\mathbb{R})$ but does not have inclusion relations with $L(\mathbb{R})$, see [7]. For example, the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(t) = \sin(t^{1/2})/t$ if $t\in[\pi^2,\infty)$ and 0 otherwise, belongs to $(HK(\mathbb{R})\cap BV(\mathbb{R}))\setminus L(\mathbb{R})$.

Taking into account the above facts and using the Henstock-Kurzweil integral, our main result consists of showing that: if $f,g\in HK(\mathbb{R})\cap BV(\mathbb{R})$ then their convolution has an expression as (1), the Fourier Integral transform of this convolution is defined for $s\neq 0$, and satisfies the equality (3).

2. Some fundamental theorems of the H-K integral

We state some of the fundamental theorems about the Henstock-Kurzweil integral that we will use frequently, see [2] and [9].

MULTIPLIER THEOREM. *Let $[a,b]$ be a bounded interval. If $f\in HK([a,b])$, $\varphi\in BV([a,b])$ and $F(x)=\int_a^x f(t)dt$, for $x\in[a,b]$, then $f\varphi\in HK([a,b])$ and*

$$\int_a^b f\varphi = F(b)\varphi(b) - \int_a^b F d\varphi. \tag{4}$$

If $a\in\mathbb{R}$ and $b=\infty$, then $f\varphi\in HK([a,\infty])$ and

$$\int_a^\infty f\varphi = \lim_{b\to\infty}\left[F(b)\varphi(b) - \int_a^b F d\varphi\right]. \tag{5}$$

The integrals on the right are Riemann-Stieltjes integrals. In (5), $\lim_{b\to\infty}\int_a^b F d\varphi$, which is denoted as $\int_a^\infty F d\varphi$, will be the improper Riemann-Stieltjes integral. If the integration is on the intervals $[-\infty,b]$ or $[-\infty,\infty]$, we take the respective limits in (5).

HAKE'S THEOREM. *For $I=[a,b]$. $f\in(I)$ iff $f\in HK(J)$ for every compact interval $J=[c,d]\subset(a,b)$ and*

$$\lim_{c\to a,d\to b}\int_c^d f(t)dt = \int_a^b f(t)dt.$$

This result is valid for infinite intervals.

According to Hake's Theorem, if a function $f:\mathbb{R}\to\mathbb{R}$ is locally Lebesgue (or Riemann) integrable and

$$\lim_{b\to\infty, a\to-\infty}\int_a^b f(t)dt$$

exists, then this limit and the Cauchy principal value is its integral in the Henstock-Kurzweil sense.

CHARTIER-DIRICHLET TEST. *Let $f,\ \varphi:[a,\infty]\to\mathbb{R}$ and suppose that:*

(a) $f\in HK([a,c])$ for each $c\geqslant a$, and $F(x)=\int_a^x f(t)dt$ is bounded on $[a,\infty)$; and

(b) φ is monotonous with $\lim_{x\to\infty}\varphi(x)=0$. Then $f\varphi\in HK([a,\infty])$.

As consequence from the Multiplier theorem, Hake's theorem and the Chartier-Dirichlet test we have that if: $\varphi\in BV_0([a,\infty]),\ f\in HK([a,b])$ for every $b>a$, and $F(t)=\int_a^t f dx$ is bounded on $[a,\infty)$. Then $f\varphi\in HK([a,\infty])$,

$$\int_a^\infty f\varphi dt=-\int_a^\infty F(t)d\varphi(t),$$

and

$$\left|\int_a^\infty f\varphi dt\right|\leqslant \sup_{a<t}|F(t)|V_\varphi([a,\infty].$$

Here $\|F\|_\infty=\sup_{a<t}|F(t)|$. *Similar results are valid for the cases $[-\infty,\infty]$ and $[-\infty,a]$.*

The following lemma is a slightly modified version of [3, Dirichlet's test]

LEMMA 1. *Let I be an interval, bounded or unbounded. Let $f,\varphi:[0,\infty)\times I\to\mathbb{R}$ be functions such that for every $t\in I$; $\varphi_t(x)=\varphi(x,t)$ is monotone, and $f(\cdot,t)\in HK_{loc}(\mathbb{R})$. Suppose that: (a) there exists $A>0$ such that for every $t\in I$ and for every interval $[\alpha,\beta]\subset[0,\infty)$:*

$$\left|\int_\alpha^\beta f(x,t)dx\right|\leqslant A.$$

(b) φ_t converges uniformly to 0, when $|x|\to\infty$, with respect to $t\in I$. Then the integral $\int_0^\beta f(x,t)\varphi(x,t)dx$ uniformly converges to $\int_0^\infty f(x,t)\varphi(x,t)dx$, with respect to $t\in I$, as $\beta\to\infty$.

Proof. Given $[\alpha,\beta]\subset[0,\infty)$, by Second Mean Value Theorem, there exists $\eta(t)\in[\alpha,\beta]$ such that

$$\int_a^b f(x,t)\varphi(x,t)dx=\varphi_t(a)\int_a^{\eta(t)} f(x,t)dx+\varphi_t(b)\int_{\eta(t)}^b f(x,t)dx. \tag{6}$$

According the uniform convergence of φ_t to zero, we have that given $\varepsilon>0$ there exists $N>0$ such that if $x\geqslant N$:

$$|\varphi_t(x)|<\varepsilon/2A, \tag{7}$$

for every $t \in I$. Then, assuming that $\beta > \alpha \geqslant N$, and by the expressions (6) and (7), we get

$$\left| \int_{\alpha}^{\beta} f(x,t)\varphi(x,t)dx \right| < \varepsilon.$$

By Cauchy Criterion [2, Theorem 16.6] the convergence is uniform respect to t. □

From the above lemma, we get the following corollary.

COROLLARY 1. *Let I be an interval, bounded or unbounded. Let $f : \mathbb{R} \times I \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be functions such that $f(\cdot,t) \in HK_{loc}(\mathbb{R})$ and $\varphi \in BV_0(\mathbb{R})$. Suppose that there exists $A > 0$ such that for every $t \in I$ and for every interval $[\alpha,\beta] \subset \mathbb{R}$:*

$$\left| \int_{\alpha}^{\beta} f(x,t)dx \right| \leqslant A.$$

Then $\int_{\alpha}^{\beta} f(x,t)\varphi(x)dx$ uniformly converges to $\int_{-\infty}^{\infty} f(x,t)\varphi(x)dx$, with respect to $t \in I$, as $\alpha \to -\infty$ y $\beta \to \infty$.

Proof. We provide the proof for the interval $[0,\infty)$. By the Charter-Dirichlet test, for every $t \in I$, the integral $\int_0^{\infty} f(x,t)\varphi(x)dx$ exists. On the other hand, through the Jordan's decomposition, there exist φ_1 and φ_2 increasing and bounded functions such that $\varphi = \varphi_1 - \varphi_2$.
Because of both functions converge when $x \to \infty$, and since $\varphi \in BV_0([0,\infty))$ then we can consider that

$$\lim_{x\to\infty} \varphi_1(x) = \lim_{x\to\infty} \varphi_2(x) = 0.$$

Thereby, by Lemma 1, $\int_0^b f(x,t)\varphi(x)dx$ uniformly converges to $\int_0^{\infty} f(x,t)\varphi(x)dx$. The proof in the case $(-\infty,0]$ is analogous. Therefore, we get the result. □

The nexts results are obtained in [6].

LEMMA 2. (Riemann-Lebesgue lemma) *If $f \in BV_0(\mathbb{R})$, then the Fourier Integral transform $\widehat{f}(s)$ exists on $\mathbb{R}\backslash\{0\}$ and $\widehat{f} \in C_0(\mathbb{R}\backslash\{0\})$.*

COROLLARY 2. *If $f \in HK(\mathbb{R}) \cap BV(\mathbb{R})$, then the Fourier Integral transform $\widehat{f}$ is defined for all $s \in \mathbb{R}$ and $\widehat{f} \in C_0(\mathbb{R}\backslash\{0\})$.*

3. The convolution on $HK(\mathbb{R}) \cap BV(\mathbb{R})$

It is possible that the convolution of two functions in $BV_0(\mathbb{R})$ does not exist in a set whose Lebesgue measure is positive, in contrast to what happens for functions in $L(\mathbb{R})$. For example, given $\gamma \in (0,1/2)$ fixed, let us define the following functions

$g(x) = 1/x^\gamma$ if $x \in [1,\infty)$, 0 otherwise, and $f(x) = g(-x)$. Let $b > 1$ and suppose that $t > 0$. Due to

$$\begin{aligned} \int_1^b f(t-x)g(x)dx &= \int_{-t+1}^{-t+b} \frac{1}{(tx+x^2)^\gamma} dx \\ &\geqslant \int_{-t+1}^{-t+b} \frac{1}{(t+x)^{2\gamma}} dx \\ &= \frac{1}{1-2\gamma}\left[(b)^{1-2\gamma} - 1\right], \end{aligned}$$

we get that $f * g$ is not defined in $(0,\infty)$. Considering this observation, it makes sense to focus on $HK(\mathbb{R}) \cap BV(\mathbb{R})$.

LEMMA 3. *Let f, g in $HK(\mathbb{R}) \cap BV(\mathbb{R})$. Then the convolution of f with g is defined for all t in $\mathbb{R}$ and $f * g \in C_0(\mathbb{R})$.*

Proof. For every $t \in \mathbb{R}$, $f_t(\cdot) = f(t - \cdot)$ is in $HK(\mathbb{R})$. Since $g \in BV(\mathbb{R})$, then $f_t \cdot g \in HK(\mathbb{R})$, therefore $f * g$ is defined for all $t \in \mathbb{R}$. The family $\{f_t : t \in \mathbb{R}\}$ is of uniform bounded variation, it means that for every $t \in \mathbb{R}$: $Var(f_t, \mathbb{R}) \leqslant Var(f, \mathbb{R})$. Due to $f \in HK(\mathbb{R}) \cap BV(\mathbb{R})$, by [8, Lemma 4.1], we have, for each $x \in \mathbb{R}$,

$$\lim_{|t|\to\infty} f_t(x) = 0 \tag{8}$$

and, for each $t \in \mathbb{R}$,

$$\lim_{|x|\to\infty} f_t(x) = 0. \tag{9}$$

By Multiplier Theorem and expression (9):

$$(f * g)(t) = \int_{-\infty}^{\infty} f_t(x)g(x)dx = -\int_{-\infty}^{\infty} G(x)df_t(x), \tag{10}$$

where $G(x) = \int_{-\infty}^{x} g(u)du$, is continuous in $\mathbb{R}$. Considering the expressions (8) and (10) let us apply the Helly's convergence Theorem, see in [5], thus we get:

$$\lim_{|t|\to\infty} (f * g)(t) = 0. \tag{11}$$

Because of $HK(\mathbb{R}) \cap BV(\mathbb{R}) \subset L^2(\mathbb{R})$, by Hölder's inequality,

$$\begin{aligned} |f * g(t+r) - f * g(t)| &\leqslant \left(\int_{-\infty}^{\infty} |f(t+r-x) - f(t-x)|^2 dx \right)^{1/2} ||g||_2 \\ &= ||f(t+r) - f(t)||_2 ||g||_2. \end{aligned}$$

Thus, the continuity of $f * g$ follows from the previous inequality and [1, Theorem 2.8.9]. □

COROLLARY 3. *Let* $f,g \in HK(\mathbb{R}) \cap BV(\mathbb{R})$. *Then for each* $[\alpha,\beta] \subset \mathbb{R}$; $f * g\chi_{[\alpha,\beta]}$ *is continuous at every real number and belongs to* $BV_0(\mathbb{R})$.

Proof. Let $\{t_1 < t_2 < \cdots < t_n\}$ be a finite partition in $\mathbb{R}$. Then

$$\begin{aligned}\sum_{i=1}^{n} \left| f * g\chi_{[\alpha,\beta]}(t_i) - f * g\chi_{[\alpha,\beta]}(t_{i-1}) \right| &\leqslant \sum_{i=1}^{n} \int_{\alpha}^{\beta} |f(t_i - x) - f(t_{i-1} - x)|\, |g(x)|\, dx \\ &= \int_{\alpha}^{\beta} \sum_{i=1}^{n} |f(t_i - x) - f(t_{i-1} - x)|\, |g(x)|\, dx \\ &\leqslant V_f(\mathbb{R})\, \|g\|_{1,[\alpha,\beta]}\,.\end{aligned}$$

Therefore, $f * g\chi_{[\alpha,\beta]} \in BV(\mathbb{R})$. Moreover, by Lemma 3, we conclude that $f * g\chi_{[\alpha,\beta]} \in C(\mathbb{R}) \cap BV_0(\mathbb{R})$. □

Given f and g, $\alpha > 0$, and $s \in \mathbb{R}$, the following notation will be used in the remainder of the article.

$$h(\alpha;t) = f * g(t)\chi_{[0,\alpha]}; \quad h(-\alpha;t) = f * g(t)\chi_{[-\alpha,0]}; \qquad h(-\alpha,\alpha;t) = f * g(t)\chi_{[-\alpha,\alpha]}.$$

LEMMA 4. *Assume that* $f,g \in HK(\mathbb{R}) \cap BV(\mathbb{R})$. *Then the convergence of* $h(\pm\alpha;t)$ *to* $f * g(t)\chi_{[0,\infty)}$ *and to* $f * g(t)\chi_{(-\infty,0]}$, *respectively, and of* $h(-\alpha,\alpha;t)$ *to* $f * g(t)$, *when* $\alpha \to \infty$, *is uniform with respect to* $t \in \mathbb{R}$.

Proof. Since $f \in HK(\mathbb{R})$, its Alexiewicz norm is an uniform upper bound, with respect to t, of all integrals of $f(t-x)$ with respect to x on bounded intervals. By Corollary 1 we get the result. □

THEOREM 1. *If* $f,g \in HK(\mathbb{R}) \cap BV(\mathbb{R})$ *and* $s \neq 0$ *is a fixed number. Then the double limit*

$$\lim_{b,\alpha\to\infty} \int_0^b e^{-ist} h(\alpha;t)dt$$

exists.

Proof. By Cauchy Criterion, [3, Lemma 33.2], given $\varepsilon > 0$ there exists $N > 0$ such that if $\alpha_2 > \alpha_1 > N$ then, for all $t \in \mathbb{R}$;

$$|h(\alpha_2;t) - h(\alpha_1;t)| = \left| \int_{\alpha_1}^{\alpha_2} f(t-x)g(x)dx \right| < \frac{\varepsilon\,|s|}{16}. \tag{12}$$

By Corollary 3: $h(\alpha_2;t) - h(\alpha_1;t) \in BV_0(\mathbb{R})$. Therefore, there exist increasing and bounded functions $h_1^*(t)$ and $h_2^*(t)$ such that

$$h(\alpha_2;t) - h(\alpha_1;t) = h_1^*(t) - h_2^*(t),$$

and if $t > N$, then

$$|h_1^*(t)|, \quad |h_2^*(t)| < \frac{\varepsilon\,|s|}{16}. \tag{13}$$

Furthermore, for every $b_1, b_2 \in \mathbb{R}$:

$$\left| \int_{b_1}^{b_2} e^{-ist} dt \right| \leqslant \frac{4}{|s|}. \tag{14}$$

By Second Mean Value Theorem, (see [2] and [3]) there exist $\xi_1, \xi_2 \in [a,b]$ such that

$$\int_a^b e^{-ist} [h(\alpha_2;t) - h(\alpha_1;t)] dt = h_1^*(a) \int_a^{\xi_1} e^{-ist} dt + h_1^*(b) \int_{\xi_1}^b e^{-ist} dt \\ - h_2^*(a) \int_a^{\xi_2} e^{-ist} dt - h_2^*(b) \int_{\xi_2}^b e^{-ist} dt.$$

Assuming that $\xi_1 < \xi_2$;

$$\int_a^b e^{-ist} [h(\alpha_2;t) - h(\alpha_1;t)] dt = [h_1^*(a) - h_2^*(a)] \int_a^{\xi_1} e^{-ist} dt - h_2^*(a) \int_{\xi_1}^{\xi_2} e^{-ist} dt \\ + [h_1^*(b) - h_2^*(b)] \int_{\xi_2}^b e^{-ist} dt + h_1^*(b) \int_{\xi_1}^{\xi_2} e^{-ist} dt.$$

Therefore, from expressions (12), (13) and (14), we have that if: $b > a > N$ and $\alpha_2 > \alpha_1 > N$, then

$$\left| \int_a^b e^{-ist} [h(\alpha_2;t) - h(\alpha_1;t)] dt \right| < \varepsilon.$$

By Cauchy criterion, [3, Theorem 19.5], we get the result. □

COROLLARY 4. *Let* $f, g \in HK(\mathbb{R}) \cap BV(\mathbb{R})$ *and* $s \neq 0$ *be a fixed number. Then*

$$\lim_{\alpha \to \infty, b \to \infty, a \to -\infty} \int_a^b e^{-ist} h(-\alpha, \alpha; t) dt = \lim_{b \to \infty, a \to -\infty} \int_a^b e^{-ist} f * g(t) dt \tag{15}$$

and

$$\lim_{\alpha \to \infty, b \to \infty, a \to -\infty} \int_a^b e^{-ist} h(-\alpha, \alpha; t) dt = \lim_{\alpha \to \infty} \int_{-\infty}^{\infty} e^{-ist} h(-\alpha, \alpha; t) dt. \tag{16}$$

Proof. By Theorem 1 the limit on the left exists. Because the convergence of $h(-\alpha, \alpha; t)$ to $f * g(t)$, is uniform, see Lemma 4, then for any $a < b$ in $\mathbb{R}$:

$$\lim_{\alpha \to \infty} \int_a^b e^{-ist} h(-\alpha, \alpha; t) dt = \int_a^b e^{-ist} \int_{-\infty}^{\infty} f(t-x) g(x) dx dt.$$

On the other hand, since, $h(-\alpha, \alpha; t) \in BV_0(\mathbb{R})$, then $\int_{-\infty}^{\infty} e^{-ist} h(-\alpha, \alpha; t) dt$ is well defined for each α, see Lemma 2. Moreover, by Hake's Theorem;

$$\lim_{b \to \infty, a \to -\infty} \int_a^b e^{-ist} h(-\alpha, \alpha; t) dt = \int_{-\infty}^{\infty} e^{-ist} h(-\alpha, \alpha; t) dt.$$

Therefore, the equalities (15) and (16) hold from [3, Theorem 19.6]. □

PROPOSITION 1. *Let* $f,g \in HK(\mathbb{R}) \cap BV(\mathbb{R})$, $a < b$, *and* $0 < \alpha$; *then for every* $s \in \mathbb{R}$:

$$\lim_{\alpha\to\infty} \lim_{b\to\infty, a\to-\infty} \int_a^b e^{-ist}\, h(-\alpha,\alpha;t)dt = \widehat{f}(s)\widehat{g}(s). \tag{17}$$

Proof. Since $f,\ g \in HK(\mathbb{R}) \cap BV(\mathbb{R})$, then $\widehat{f}$ and $\widehat{g}$ exist in $\mathbb{R}$. For any bounded intervals I,J; $f \in L(I)$, $g \in L(J)$ and $f(t-x)g(x) \in L(I\times J)$. By classical Fubini's Theorem, for $\alpha > 0$ and any two real numbers $a < b$:

$$\int_a^b e^{-ist} h(-\alpha,\alpha,t)dt = \int_{-\alpha}^{\alpha} g(x) \int_a^b e^{-ist} f(t-x)dtdx.$$

Besides, $e^{-is(\cdot)}f(\cdot - x)\chi_{[a,b]}(\cdot) \in HK(\mathbb{R})$. Let us define $\widehat{f_s}(x;a,b) = \int_a^b e^{-ist} f(t-x)dt$. Since $\widehat{f}(s)$ exists and $\widehat{f_s}(x;a,b) = e^{-isx}\int_{a-x}^{b-x} f(u)e^{-isu}du$, we have that

$$\lim_{b\to\infty, a\to-\infty} g(x)\widehat{f_s}(x;a,b) = \widehat{f}(s)e^{-isx}g(x)$$

for all $x \in \mathbb{R}$, in particular for every $x \in [-\alpha,\alpha]$. Separating $\widehat{f_s}(x;a,b)$ as a sum of its real and imaginary parts, we get that

$$\left|g(x)\widehat{f_s}(x;a,b)\right| \leqslant 2\,\|f(\cdot)\|_A\,\|g\|_\infty.$$

By the Convergence Dominated Theorem,

$$\lim_{b\to\infty, a\to-\infty} \int_{-\alpha}^{\alpha} g(x) \int_a^b e^{-ist} f(t-x)dtdx = \widehat{f}(s) \int_{-\alpha}^{\alpha} g(x)e^{-isx}dx.$$

Since $\widehat{g}(s)$ exists. We have that

$$\lim_{\alpha\to\infty} \lim_{b\to\infty, a\to-\infty} \int_{-\alpha}^{\alpha} g(x) \int_{-a}^{b} e^{-ist} f(t-x)dtdx = \widehat{f}(s)\widehat{g}(s). \qquad \square$$

THEOREM 2. *Let* $f,g \in HK(\mathbb{R}) \cap BV(\mathbb{R})$. *Then the Fourier Integral transform of* $f * g$ *exists for each* $s \in \mathbb{R}\backslash\{0\}$ *and*

$$\widehat{(f*g)}(s) = \widehat{f}(s)\ \widehat{g}(s).$$

Proof. By Corollary 4, $\widehat{(f*g)}(s)$ exists for each $s \in \mathbb{R}\backslash\{0\}$. From the equality (16) and Proposition 1 the result is obtained. $\square$

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