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ESTIMATION FOR INITIAL COEFFICIENTS OF BI–UNIVALENT λ–CONVEX ANALYTIC FUNCTIONS IN THE UNIT DISC

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Abstract. In this paper, we find improved bounds on the modulii of the second and third coefficients of the Taylor-Maclaurin's series for bi-univalent λ-convex analytic functions in the unit disc of the complex plane. We also find an estimate on modulus of the fourth coefficient. The result on the fourth coefficient is new. Our bounds are obtained by refining well known estimates for the initial coefficients of the Carthéodory functions.

1. Introduction

Let $\mathscr{A}$ denote the class of analytic functions in the in the open unit disk of the complex plane:

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\},$$

which are represented by the following *normalized* Taylor-Maclaurin's series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.1}$$

The function $f \in \mathscr{A}$ is said to be *univalent* in $\mathbb{U}$ if $f(z)$ is one-to-one in $\mathbb{U}$. As usual we denote by $\mathscr{S}$, the subclass of functions in $\mathscr{A}$ which are univalent in $\mathbb{U}$. The function $f \in \mathscr{S}$ has a *compositional inverse* f^{-1}, defined by

$$f^{-1}(f(z)) = z, \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w, \quad (w \in \text{range of } f).$$

By Koebe's one-quarter theorem [5], the compositional inverse function $f^{-1}(w)$ is analytic in some disc $|w| < r_0(f)$, $r_0(f) \geqslant \frac{1}{4}$ for every function $f \in \mathscr{S}$. Also, $f^{-1}(w)$ has the Taylor-Maclaurin series expansion of the form:

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n, \quad (|w| < r_0(f))$$

where

$$b_n = \frac{(-1)^{n+1}}{n!}|A_{ij}|$$

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and $|A_{ij}|$ is the $(n-1)^{th}$ order determinant whose entries are defined, in terms of the coefficients of $f(z)$, by the following [6]:

$$|A_{ij}| = \begin{cases} [(i-j+1)n+j-1]a_{i-j+2} & \text{if } i+1 \geqslant j \\ 0 & \text{if } i+1 < j. \end{cases}$$

For initial values of n we, therefore, have:

$$b_2 = -a_2, \quad b_3 = 2a_2^2 - a_3, \quad b_4 = 5a_2a_3 - 5a_2^3 - a_4 \tag{1.2}$$

and so on. In two recent papers ([8], [17]) the coefficient estimate problem has been discussed for the functions f^{-1} when f is a member of the class $\mathscr{S}$ in addition to being a *starlike function of positive order*. For a discussion on $\mathscr{S}$ and the inverse functions see ([5], [6], [19]).

The function $f \in \mathscr{S}$ is said to be *bi-univalent* in $\mathbb{U}$ if $f^{-1}(w)$ has univalent analytic continuation to the unit disk $\mathbb{U}$. For example, the function

$$f(z) = e^z - 1$$

is bi-univalent in $\mathbb{U}$. For some more examples and basic properties see ([7], [12], [15], [16]). We denote by σ the class of analytic bi-univalent functions in $\mathbb{U}$. Seminal work on the class σ can be found in ([2], [3], [10], [13], [20]). Also see [9] for a brief history. Through out in this paper we denote by $g(w)$ the analytic continuation of the function $f^{-1}(w)$ to the unit disc $\mathbb{U}$.

Miller and Mocanu [11] studied the class of *univalent* λ*-convex* functions, denoted by $\mathscr{M}_\lambda$, $(0 \leqslant \lambda < \infty)$, which by definition, consists of functions $f \in \mathscr{A}$ satisfying the following conditions:

$$\frac{f(z)}{z} f'(z) \neq 0 \qquad (z \in \mathbb{U} \setminus \{0\})$$

and

$$\Re\left\{\lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\lambda)\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{1.3}$$

We know that [11] univalent λ-convex functions are *starlike* if $\lambda \geqslant 0$ and *convex* if $\lambda \geqslant 1$. More over, the class $\mathscr{M}_\lambda$ is nested in the sense that if $0 \leqslant \lambda_2 < \lambda_1$ then $\mathscr{M}_{\lambda_1} \subset \mathscr{M}_{\lambda_2}$.

Subclasses of bi-univalent functions analogous to the class $\mathscr{M}_\lambda$ have also been studied in the literature. See for example ([1], [9] and [22]). We thus have the following:

DEFINITION 1. The function $f \in \mathscr{A}$ is said to be in the class $\mathscr{M}_\sigma^\lambda(\beta)$ $(0 \leqslant \beta < 1, \lambda \geqslant 0)$ if the following conditions are satisfied:

$$f'(z)\frac{f(z)}{z} \neq 0, \qquad (z \in \mathbb{U} \setminus \{0\}),$$

$$f \in \sigma \text{ and } \quad \Re\left\{\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\lambda)\frac{zf'(z)}{f(z)}+\right\} > \beta \quad (z \in \mathbb{U}), \tag{1.4}$$

and

$$\Re\left\{\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\lambda)\frac{wg'(w)}{g(w)}\right\} > \beta \quad (w \in \mathbb{U}) \tag{1.5}$$

where g is the analytic continuation of f^{-1} to $\mathbb{U}$.

By taking $\lambda = 0$ and $\lambda = 1$ in Definition 1 we get the classes of *bi-starlike functions of order* β and *bi-convex functions of order* β respectively ([3] , [12], [20]).

We shall also need the class $\mathscr{P}$ consisting of functions $p(z)$ which are analytic in $\mathbb{U}$, satisfy $\Re(p(z)) > 0$ $(z \in \mathbb{U})$ and $p(0) = 1$. The Functions $p(z) \in \mathscr{P}$ are named after Carthéodory.

In a recent paper Srivastava, the first named author of this paper and Gochhayat [16] discussed the old unsolved problem of finding coefficient bounds for bi-univalent analytic functions. They obtained estimates for the second and third coefficients for functions in certain interesting subclasses of σ. At present there are about one hundred follow up research papers on the estimates of the second and third coefficients of functions in different subclasses of σ. In the present paper we extend a very recent result of Mishra and Soren [12] to find estimate on the fourth coefficient for $f \in \mathscr{M}_\sigma^\lambda(\beta)$. We also improve upon bounds of Ali et al. [1] and Zaprawa [22] on $|a_2|$ and $|a_3|$ for $f \in \mathscr{M}_\sigma^\lambda(\beta)$. As such, the well known bounds, obtained earlier by Brannan and Taha [3], on the second and third coefficients for bi-starlike functions are also improved. The methods adopted and developed in this paper are applicable for finding improved coefficient estimates for the several sub-classes of bi-univalent functions studied in the literature, for example in ([4], [18] and [14]).

2. Coefficient bounds for the function class $\mathscr{M}_\sigma^\lambda(\beta)$

We state and prove the following.

THEOREM 1. *Let the function* f *given by (1.1) be in the class* $\mathscr{M}_\sigma^\lambda(\beta)$ $(0 \leqslant \beta < 1,\ \lambda \geqslant 0)$. *Then*

$$|a_2| \leqslant \begin{cases} \sqrt{\frac{2(1-\beta)}{1+\lambda}} & \left(\frac{1+\lambda}{1-\beta} < 2\right) \\ \frac{2(1-\beta)}{1+\lambda} & \left(\frac{1+\lambda}{1-\beta} \geqslant 2\right) \end{cases} \tag{2.1}$$

and

$$|a_3| \leqslant \begin{cases} \frac{(1-\beta)\{2(1-\beta)(1+3\lambda)+(1+\lambda)^2\}}{(1+\lambda)^2(1+2\lambda)}, & \left(\frac{1+\lambda}{1-\beta} > 2\right) \\ \frac{2(1-\beta)}{(1+\lambda)}, & \left(\frac{1+\lambda}{1-\beta} \leqslant 2\right). \end{cases} \tag{2.2}$$

Furthermore, let λ_1 *be the unique positive root of the equation* $1-14\lambda-23\lambda^2=0$. *Then*

$$|a_4| \leqslant \frac{2(1-\beta)}{3(1+3\lambda)} \begin{cases} 1+\frac{4(1-\beta)(1-\lambda)}{(1+\lambda)^2} & (0 \leqslant \lambda \leqslant \lambda_1) \\ 1+\frac{3(1-\beta)(1+5\lambda)}{(1+\lambda)(1+2\lambda)} & (\lambda > \lambda_1). \end{cases} \tag{2.3}$$

Proof. Let the function f be a member of the class $\mathscr{M}_\sigma^\lambda(\beta)$. Then by Definition 1, we have the following:

$$\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\lambda)\frac{zf'(z)}{f(z)} = \beta+(1-\beta)p(z), \tag{2.4}$$

and

$$\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\lambda)\frac{wg'(w)}{g(w)} = \beta+(1-\beta)q(w), \tag{2.5}$$

where the functions p and q are members of the Carthéodory class $\mathscr{P}$ and have the forms:

$$p(z) = 1+c_1z+c_2z^2+c_3z^3+\ldots \quad (z \in \mathbb{U}) \tag{2.6}$$

and

$$q(w) = 1+l_1w+l_2w^2+l_3w^3+\ldots \quad (w \in \mathbb{U}). \tag{2.7}$$

Now, equating the coefficient of $\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\lambda)\frac{zf'(z)}{f(z)}$ with the coefficients of $\beta+(1-\beta)p(z)$ we have the following:

$$(1+\lambda)a_2 = (1-\beta)c_1, \tag{2.8}$$

$$2(1+2\lambda)a_3-(1+3\lambda)a_2^2 = (1-\beta)c_2, \tag{2.9}$$

$$3(1+3\lambda)a_4-(3+15\lambda)a_2a_3+(1+17\lambda)a_2^3 = (1-\beta)c_3, \tag{2.10}$$

Similarly, a comparison of both the side of (2.5) yields:

$$(1+\lambda)a_2 = -(1-\beta)l_1, \tag{2.11}$$

$$-2(1+2\lambda)a_3+(3+5\lambda)a_2^2 = (1-\beta)l_2, \tag{2.12}$$

and

$$-3(1+3\lambda)a_4+(12+30\lambda)a_2a_3-(10+22\lambda)a_2^3 = (1-\beta)l_3. \tag{2.13}$$

We shall first obtain refined estimates on $|c_1|$ and $|c_2+l_2|$ by using the above relations. For this purpose we add (2.9) and (2.12) and get the following:

$$(2+2\lambda)a_2^2 = (1-\beta)(c_2+l_2).$$

Substituting $a_2 = \frac{1-\beta}{1+\lambda}c_1$ from (2.8), in the above relation we get after simplification, the following relation involving c_1 and c_2+l_2:

$$c_1^2 = \frac{1+\lambda}{2(1-\beta)}(c_2+l_2). \tag{2.14}$$

The relation (2.14) also gives the following refined estimates:

$$|c_1| \leqslant \begin{cases} \sqrt{\frac{2(1+\lambda)}{1-\beta}} & \left(\frac{1+\lambda}{1-\beta} < 2\right) \\ 2 & \left(\frac{1+\lambda}{1-\beta} \geqslant 2\right), \end{cases} \tag{2.15}$$

and

$$|c_2 + l_2| \leqslant \begin{cases} \frac{8(1-\beta)}{1+\lambda} & \left(\frac{1+\lambda}{1-\beta} > 2\right) \\ 4 & \left(\frac{1+\lambda}{1-\beta} \leqslant 2\right). \end{cases} \tag{2.16}$$

Now using the estimate (2.15) in (2.8) we get:

$$|a_2| \leqslant \begin{cases} \sqrt{\frac{2(1-\beta)}{1+\lambda}} & \left(\frac{1+\lambda}{1-\beta} < 2\right) \\ \frac{2(1-\beta)}{1+\lambda} & \left(\frac{1+\lambda}{1-\beta} \geqslant 2\right) \end{cases} \tag{2.17}$$

We get the claimed bound of (2.1).

To find bounds on $|a_3|$, we multiply $(3+5\lambda)$ and $(1+3\lambda)$ to the relations (2.9) and (2.12) respectively and on adding them we obtain:

$$\begin{aligned} 4(1+\lambda)(1+2\lambda)a_3 &= (1-\beta)\{(3+5\lambda)c_2 + (1+3\lambda)l_2)\} \\ &= (1-\beta)\{(1+3\lambda)(c_2+l_2) + 2(1+\lambda)c_2\}. \end{aligned}$$

Now, using the estimate (2.16) in the range $\frac{1+\lambda}{(1-\beta)} > 2$ and the estimate $|c_2| \leqslant 2$, $|l_2| \leqslant 2$ otherwise, we get:

$$|a_3| \leqslant \begin{cases} \frac{(1-\beta)\{2(1-\beta)(1+3\lambda)+(1+\lambda)^2\}}{(1+\lambda)^2(1+2\lambda)}, & \left(\frac{1+\lambda}{1-\beta} > 2\right) \\ \frac{2(1-\beta)}{(1+\lambda)}, & \left(\frac{1+\lambda}{1-\beta} \leqslant 2\right). \end{cases}$$

This is precisely the assertion at (2.2).

By subtracting (2.12) form (2.9) we get:

$$a_3 = \frac{(1-\beta)^2}{(1+\lambda)^2}c_1^2 + \frac{(1-\beta)}{4(1+2\lambda)}(c_2 - l_2). \tag{2.18}$$

Similarly, we add the equations (2.13) with (2.10) and get:

$$(9+15\lambda)a_2a_3 - (9+5\lambda)a_2^3 = (1-\beta)(c_3+l_3). \tag{2.19}$$

We are now ready to find a bound for $|a_4|$. As in our consideration for $|a_3|$ in this case also we shall express a_4 in terms of the first three coefficients of $p(z)$ and $q(w)$. For this we subtract (2.13) from (2.10) and get

$$\begin{aligned} 6(1+3\lambda)a_4 &= (15+45\lambda)a_2a_3 - (11+39\lambda)a_2^3 + (1-\beta)(c_3-l_3) \\ &= (9+15\lambda)a_2a_3 - (9+5\lambda)a_2^3 + (6+30\lambda)a_2a_3 \\ &\quad - (2+34\lambda)a_2^3 + (1-\beta)(c_3-l_3). \end{aligned}$$

On replacing $(9+15\lambda)a_2a_3-(9+5\lambda)a_2^3$ by the right hand side of (2.19), the above reduces to

$$6(1+3\lambda)a_4 = 2(1-\beta)c_3+(6+30\lambda)a_2a_3-(2+34\lambda)a_2^3.$$

We substitute $a_3 = \frac{(1-\beta)^2}{(1+\lambda)^2}c_1^2+\frac{1-\beta}{4(1+2\lambda)}(c_2-l_2)$ from (2.18) and $a_2 = \frac{1-\beta}{1+\lambda}c_1$ in the above relation and get

$$\begin{aligned}6(1+3\lambda)a_4 &= 2(1-\beta)c_3+\frac{(6+30\lambda)(1-\beta)c_1}{1+\lambda}\left(\frac{(1-\beta)^2c_1^2}{(1+\lambda)^2}+\frac{1-\beta}{4(1+2\lambda)}(c_2-l_2)\right)\\ &\quad -\frac{(2+34\lambda)(1-\beta)^3}{(1+\lambda)^3}c_1^3,\\ &= 2(1-\beta)c_3+4\frac{(1-\beta)^3(1-\lambda)}{(1+\lambda)^3}c_1c_1^2+\frac{(1-\beta)^2(3+15\lambda)}{2(1+\lambda)(1+2\lambda)}c_1(c_2-l_2).\end{aligned}$$

On replacing $c_1^2 = \frac{1+\lambda}{2(1-\beta)}(c_2+l_2)$ from (2.14) the above relation reduces to the following:

$$\begin{aligned}6(1+3\lambda)a_4 &= 2(1-\beta)c_3+2\frac{(1-\beta)^2(1-\lambda)}{(1+\lambda)^2}c_1(c_2+l_2)\\ &\quad +\frac{(1-\beta)^2(3+15\lambda)}{2(1+\lambda)(1+2\lambda)}c_1(c_2-l_2)\\ &= 2(1-\beta)c_3+\left(\frac{2-2\lambda}{1+\lambda}+\frac{3+15\lambda}{2(1+2\lambda)}\right)\frac{(1-\beta)^2}{1+\lambda}c_1c_2\\ &\quad +\left(\frac{2-2\lambda}{1+\lambda}-\frac{3+15\lambda}{2(1+2\lambda)}\right)\frac{(1-\beta)^2}{1+\lambda}c_1l_2\\ &= 2(1-\beta)c_3+\frac{(1-\beta)^2}{2(1+\lambda)^2(1+2\lambda)}(7+22\lambda+7\lambda^2)c_1c_2\\ &\quad +\frac{(1-\beta)^2}{2(1+\lambda)^2(1+2\lambda)}(1-14\lambda-23\lambda^2)c_1l_2.\end{aligned} \tag{2.20}$$

Let λ_1 be the only positive root of the equation $1-14\lambda-23\lambda^2=0$. We observe that $0\leqslant\lambda_1<1$. More over, $1-14\lambda-23\lambda^2$ is nonnegative for $0\leqslant\lambda\leqslant\lambda_1$ and is strictly negative for $\lambda>\lambda_1$. By applying triangle inequality in (2.20) we get the following:

$$6(1+3\lambda)|a_4| \leqslant \begin{cases} 2(1-\beta)|c_3|+\frac{(1-\beta)^2}{2(1+\lambda)^2(1+2\lambda)}(7+22\lambda+7\lambda^2)|c_1||c_2| \\ +\frac{(1-\beta)^2}{2(1+\lambda)^2(1+2\lambda)}(1-14\lambda-23\lambda^2)|c_1||l_2| & (0\leqslant\lambda\leqslant\lambda_1) \\ \\ 2(1-\beta)|c_3|+\frac{(1-\beta)^2}{2(1+\lambda)^2(1+2\lambda)}(7+22\lambda+7\lambda^2)|c_1||c_2| \\ +\frac{(1-\beta)^2}{2(1+\lambda)^2(1+2\lambda)}(23\lambda^2+14\lambda-1)|c_1||l_2| & (\lambda>\lambda_1). \end{cases} \tag{2.21}$$

We apply the well known estimates $|c_1| \leqslant 2$, $|c_2| \leqslant 2$, $|c_3| \leqslant 2$, $|l_2| \leqslant 2$, and get the following after simplification:

$$|a_4| \leqslant \frac{2(1-\beta)}{3(1+3\lambda)} \begin{cases} 1+\frac{4(1-\beta)(1-\lambda)}{(1+\lambda)^2} & (0 \leqslant \lambda \leqslant \lambda_1) \\ 1+\frac{3(1-\beta)(1+5\lambda)}{(1+\lambda)(1+2\lambda)} & (\lambda > \lambda_1). \end{cases}$$

This is precisely our assertion at (2.3). This completes the proof of the Theorem 1. □

3. Concluding Remarks

REMARK 1. In the range $0 \leqslant \beta \leqslant \frac{1}{2}$ and $0 \leqslant \lambda < \min(\lambda_1, 1-2\beta)$, we can further refine the estimate for $|a_4|$ by using the estimate (2.15) for $|c_1|$ in the first bound of (2.21). Thus we shall get the following:

$$|a_4| \leqslant \frac{2(1-\beta)}{3(1+3\lambda)} \begin{cases} 1+\frac{2(1-\beta)(1-\lambda)}{(1+\lambda)^2}\sqrt{\frac{2(1+\lambda)}{1-\beta}} & (0 \leqslant \beta \leqslant \frac{1}{2},\ 0 \leqslant \lambda < \min(\lambda_1, 1-2\beta)) \\ 1+\frac{4(1-\beta)(1-\lambda)}{(1+\lambda)^2} & (0 \leqslant \beta < \frac{1}{2},\ 1-2\beta \leqslant \lambda \leqslant \lambda_1), \\ & \text{if } \min(\lambda_1, 1-2\beta) = 1-2\beta \\ 1+\frac{4(1-\beta)(1-\lambda)}{(1+\lambda)^2} & (\frac{1}{2} \leqslant \beta < 1,\ 0 \leqslant \lambda \leqslant \lambda_1) \\ 1+\frac{3(1-\beta)(1+5\lambda)}{(1+\lambda)(1+2\lambda)} & (\lambda > \lambda_1). \end{cases} \tag{3.1}$$

REMARK 2. In the particular case $\lambda = 0$, our estimate (3.1) gives the following recent result of Mishra and Soren [12].

Let f, given by (1.1), be a bi-starlike function of order β. Then

$$|a_4| \leqslant \frac{2(1-\beta)}{3} \begin{cases} 1+2\sqrt{2(1-\beta)} & (0 \leqslant \beta \leqslant \frac{1}{2}), \\ 1+4(1-\beta) & (\frac{1}{2} \leqslant \beta < 1. \end{cases} \tag{3.2}$$

REMARK 3. For the functions $f \in \mathscr{M}_\sigma^\lambda(\beta)$ and represented by the series (1.1), Ali et al. [1] obtained bounds for the modulii of the second and third coefficient as follows:

$$|a_2| \leqslant \sqrt{\frac{2(1-\beta)}{1+\lambda}} \qquad (\lambda \geqslant 0,\ 0 \leqslant \beta < 1) \tag{3.3}$$

and

$$|a_3| \leqslant \frac{2(1-\beta)}{1+\lambda} \qquad (\lambda \geqslant 0,\ 0 \leqslant \beta < 1). \tag{3.4}$$

Very recently, Zaprawa [22] also proved (3.4) while investigating the Fekete-Szegö problem for the class $\mathscr{M}_\sigma^\lambda(\beta)$. A comparison of (3.3) with our results (2.1) shows that we have improved the bounds of Ali et al. [1] on the second coefficient in the range

$\left(\frac{1+\lambda}{1-\beta} \geqslant 2\right)$. Similarly, by comparing (3.4) with (2.2) we get that the estimate of Ali et al. [1] and Zaprawa [22] on the third coefficient is improved in the range $\left(\frac{1+\lambda}{1-\beta} > 2\right)$. Our bound on $|a_4|$ is new.

REMARK 4. The following class of functions have also been studied in the literature.

DEFINITION 2. The function $f \in \mathscr{A}$ is said to be in the class $\mathscr{M}_\sigma^{\lambda,\alpha}$ $(0 < \alpha \leqslant 1, \lambda \geqslant 0)$ if the following conditions are satisfied:

$$f'(z)\frac{f(z)}{z} \neq 0 \quad (0 < |z| < 1), \qquad (z \in \mathbb{U}),$$

$$f \in \sigma \quad \text{and} \left|\arg\left\{\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\lambda)\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}), \tag{3.5}$$

and

$$\left|\arg\left\{\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\lambda)\frac{wg'(w)}{g(w)}\right\}\right| < \frac{\alpha\pi}{2} \quad (w \in \mathbb{U}), \tag{3.6}$$

where g is the analytic continuation of f^{-1} to $\mathbb{U}$.

We have also obtained estimates for the modulii of the second, third and fourth coefficients for functions in the class $\mathscr{M}_\sigma^{\lambda,\alpha}$ $(0 < \alpha \leqslant 1,\ \lambda \geqslant 0)$ following the techniques of our Theorem 1. However, the results are too technical. We, therefore, do not include them here.

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