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APPROXIMATIONS USING HILBERT TRANSFORM OF WAVELETS

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Abstract. Hilbert transform of wavelets has been used to approximate functions in $L^2(\mathbb{R})$. It is proved that Hilbert transform of wavelets with many vanishing moments does a good job in approximating smooth functions in $L^2(\mathbb{R})$. We also prove that Hölder continuity of a function helps in the decay of wavelet coefficients and thereby helps in approximating it. Finally, we give a result that relates the Hilbert transform of wavelet with dyadic scale differential operator and use it to decrease the wavelet coefficients.

1. Introduction

Approximations using trigonometric polynomials of functions lying in different classes of functions can be found in Zygmund [10]. In early 1950s, the finite element method proposed by engineers was found to be very close to the approximation theory. In 1964, Céa [1] proposed a lemma for proving error estimates for the finite element method as an application to elliptic partial differential equations which acts as an approximation problem in Sobolev spaces. The approximate representation of a function using splines significantly started in early 1970s. It has been observed in [2, 8] that if $\psi(t)$ is a real wavelet, then Hilbert transform of $\psi(t)$ i.e., $\mathscr{H}\psi(t)$ is also a real wavelet with same energy and admissibility coefficient of its generating wavelet, $\psi(t)$. Hilbert transform of Gabor and Wilson systems was studied by Jarrah and Panwar [5]. For various details related to Hilbert transform one may refer to [3, 6]. Walnut [9] gave the relationship between the vanishing moments of a wavelet and the decay of wavelet coefficients of a function. Holschneider and Tchamitchian [4] discussed that the uniform continuity of a function is reflected in its wavelet transform by the decrease of wavelet coefficients at small scale. Mallat [7] proved that "a wavelet with n vanishing moments" can be written as the n^{th} order derivative of a function θ and the resulting wavelet transform is a multiscale differential operator.

In the present paper, we study approximation of a function using Hilbert transform of wavelets and prove that vanishing moments play an important role in approximating smooth functions in $L^2(\mathbb{R})$. Also, we show how wavelet coefficients, induced by Hilbert transform of wavelets, can be reduced using the Hölder continuity of a function. At last, a result that relates the Hilbert transform of wavelets with dyadic scale differential operator and a relation between the decay of $|\langle f, \mathscr{H}\psi_{j,k}\rangle|$ and the uniform regularity of f is given.

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2. Main results

We begin this section with the following definition of Hilbert transform of a function given in [6]. The *Hilbert transform* of a function f on a real line is defined by

$$\mathscr{H}f(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|x-t|\geqslant\varepsilon}\frac{f(t)}{x-t}dt = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|t|\geqslant\varepsilon}\frac{f(x-t)}{t}dt,$$

provided that the limit exists in some sense.

Also recall from [6] that the *moment formula* for the Hilbert transform of f is given by

$$\mathscr{H}\{x^n f(x)\} = x^n\mathscr{H}f(x) - \frac{1}{\pi}\sum_{m=0}^{n-1}x^m\int_{\mathbb{R}}z^{n-1-m}f(z)\,dz, \quad n\geqslant 0.$$

Note that the above formula holds if $x^n f(x)\in L^p(\mathbb{R})$, $1<p<\infty$.

In the following result, we prove that the wavelet coefficients of a square integrable function decay fast as $j\to+\infty$ depending on the smoothness of f and the number of vanishing moments.

THEOREM 2.1. *Given $M\in\mathbb{N}$, suppose that the function $f\in L^2(\mathbb{R})$ is C^M on $\mathbb{R}$ and that $f^{(M)}\in L^\infty(\mathbb{R})$. Let $\psi\in L^2(\mathbb{R})$ be a function with compact support such that*

$$x^{M-1}\psi(x)\in L^2(\mathbb{R}), \tag{2.1}$$

and

$$\int_{\mathbb{R}}x^m\psi(x)dx = 0, \quad 0\leqslant m\leqslant M-2. \tag{2.2}$$

Then there exists a constant $K>0$ depending on M and $f(x)$ such that for every $j,k\in\mathbb{Z}$, $\left|\langle f,\mathscr{H}\psi_{j,k}\rangle\right|\leqslant K\,2^{-j(M+\frac{1}{2})}$, where $\mathscr{H}\psi_{j,k}$ denotes the Hilbert Transform of wavelet $\psi_{j,k}$ given by $\psi_{j,k}(x)=2^{\frac{j}{2}}\psi(2^jx-k)$, where $j,k\in\mathbb{Z}$.

Proof. Suppose that ψ is supported in the interval $\widetilde{J}=\widetilde{J}_{0,0}=[0,b]$ for $b>0$. It follows that the function $\psi_{j,k}(x)=2^{\frac{j}{2}}\psi(2^jx-k)$ is supported in the interval $\widetilde{J}_{j,k}=[2^{-j}k,2^{-j}(k+b)]$ and its length is $2^{-j}b$ denoted by $|\widetilde{J}_{j,k}|$. We denote the center of the interval $\widetilde{J}_{j,k}$ by $\widetilde{x}_{j,k}$ where $\widetilde{x}_{j,k}=2^{-(j+1)}b+2^{-j}k$. Using (2.1), (2.2) and moment formula for Hilbert transform, given any polynomial $p(x)$ of degree not greater than $M-1$ and for any $j,k\in\mathbb{Z}$, we get

$$\int_{\mathbb{R}}p(x)\overline{\mathscr{H}\psi_{j,k}(x)}dx = 0.$$

Since $f\in C^M(\mathbb{R})$, for each $j,k\in\mathbb{Z}$, we can apply the Taylor formula for $f(x)$ about the point $\widetilde{x}_{j,k}$. That is

$$f(x) = f(\widetilde{x}_{j,k})+(x-\widetilde{x}_{j,k})f^{(1)}(\widetilde{x}_{j,k})+\ldots+\frac{1}{(M-1)!}(x-\widetilde{x}_{j,k})^{M-1}f^{(M-1)}(\widetilde{x}_{j,k})+R_M(x),$$

where $R_M(x) = \frac{1}{M!}(x-\widetilde{x}_{j,k})^M f^{(M)}(\xi)$ for some ξ between $\widetilde{x}_{j,k}$ and x.

This gives

$$\mathscr{H} f(x) = \sum_{r=0}^{M-1} \frac{1}{r!} \mathscr{H}[(x-\widetilde{x}_{j,k})^r] f^{(r)}(\widetilde{x}_{j,k}) + \mathscr{H} R_M(x). \tag{2.3}$$

Also note that

$$\begin{aligned}\langle f, \mathscr{H}\psi_{j,k}\rangle &= \int_{\mathbb{R}} f(x)\overline{\mathscr{H}\psi_{j,k}(x)}dx \\ &= -\int_{\mathbb{R}} \mathscr{H} f(x)\overline{\psi_{j,k}(x)}dx. \end{aligned} \tag{2.4}$$

Using equations (2.3) and (2.4), we get

$$\begin{aligned}\langle f, \mathscr{H}\psi_{j,k}\rangle &= -\int_{\mathbb{R}} \left[\sum_{r=0}^{M-1} \frac{1}{r!} \mathscr{H}[(x-\widetilde{x}_{j,k})^r] f^{(r)}(\widetilde{x}_{j,k}) + \mathscr{H} R_M(x)\right] \overline{\psi_{j,k}(x)}dx \\ &= -\sum_{r=0}^{M-1} \frac{1}{r!} f^{(r)}(\widetilde{x}_{j,k}) \int_{\mathbb{R}} \mathscr{H}[(x-\widetilde{x}_{j,k})^r]\overline{\psi_{j,k}(x)}dx \\ &\quad -\int_{\mathbb{R}} \mathscr{H} R_M(x)\overline{\psi_{j,k}(x)}dx. \end{aligned} \tag{2.5}$$

Also $\int_{\widetilde{J_{j,k}}} |(x-\widetilde{x_{j,k}})^r|^2 \, dx < \infty$. So, we have

$$\begin{aligned}\int_{\mathbb{R}} \mathscr{H}[(x-\widetilde{x_{j,k}})^r] \, \overline{\psi_{j,k}(x)}dx &= -\int_{\mathbb{R}} (x-\widetilde{x_{j,k}})^r \, \overline{\mathscr{H}\psi_{j,k}(x)}dx \\ &= 0, \end{aligned} \tag{2.6}$$

where $r = 0, \ldots, M-1$.

From (2.5) and (2.6), we obtain

$$\begin{aligned}|\langle f, \mathscr{H}\psi_{j,k}\rangle| &= \left|-\int_{\mathbb{R}} \mathscr{H}[R_M(x)] \, \overline{\psi_{j,k}(x)}dx\right| \\ &= \frac{|f^{(M)}(\xi)|}{M!}\left|-\int_{\mathbb{R}} (x-\widetilde{x}_{j,k})^M \, \overline{\mathscr{H}[\psi_{j,k}(x)]}dx\right| \\ &= \frac{|f^{(M)}(\xi)|}{M!}\left|\int_{\mathbb{R}} \mathscr{F}[(x-\widetilde{x}_{j,k})^M](\gamma) \, \overline{\mathscr{F}\mathscr{H}[\psi_{j,k}(x)](\gamma)}d\gamma\right| \\ &= \frac{|f^{(M)}(\xi)|}{M!}\left|\int_{\mathbb{R}} \mathscr{F}[(x-\widetilde{x}_{j,k})^M](\gamma) \, \overline{\mathscr{F}[\psi_{j,k}(x)](\gamma)}d\gamma\right| \\ &= \frac{|f^{(M)}(\xi)|}{M!}\left|\int_{\mathbb{R}} (x-\widetilde{x}_{j,k})^M \, \overline{\psi_{j,k}(x)}dx\right| \\ &\leqslant \frac{1}{M!}\max_{x\in\widetilde{J_{j,k}}}|f^{(M)}(x)|\int_{\widetilde{J_{j,k}}}\left|(x-\widetilde{x}_{j,k})^M\overline{\psi_{j,k}(x)}\right|dx \\ &\leqslant \frac{1}{M!}\max_{x\in\widetilde{J_{j,k}}}|f^{(M)}(x)|\left[\int_{\widetilde{J_{j,k}}}\left|(x-\widetilde{x}_{j,k})^M\right|^2dx\right]^{\frac{1}{2}}\cdot\left[\int_{\widetilde{J_{j,k}}}|\psi_{j,k}(x)|^2dx\right]^{\frac{1}{2}}. \end{aligned}$$

Since $\psi \in L^2(\mathbb{R})$, there exists a positive constant C' such that

$$\left[\int_{\widetilde{J_{j,k}}} |\psi_{j,k}(x)|^2 dx\right]^{\frac{1}{2}} \leqslant C'.$$

Therefore,

$$\begin{aligned}
|\langle f, \mathscr{H}\psi_{j,k}\rangle| &\leqslant \frac{C'}{M!} \max_{x\in\widetilde{J_{j,k}}} |f^{(M)}(x)| \left[\int_{\widetilde{J_{j,k}}} 2^{-2M(j+1)}\, b^{2M} dx\right]^{\frac{1}{2}} \\
&\leqslant \frac{C'}{M!} \max_{x\in\widetilde{J_{j,k}}} |f^{(M)}(x)|\, 2^{-M(j+1)}\, b^M\, |\widetilde{J_{j,k}}|^{\frac{1}{2}} \\
&\leqslant \frac{C'}{M!} ||f^{(M)}||_\infty\, 2^{-M}\, b^{\frac{1}{2}+M}\, 2^{-jM}\, 2^{-\frac{j}{2}} \\
&= K\, 2^{-j(M+\frac{1}{2})},
\end{aligned}$$

where

$$K = \frac{C'}{M!}\, ||f^{(M)}||_\infty\, 2^{-M}\, b^{\frac{1}{2}+M}.$$

The wavelet coefficients of such a function will have rapid decay as $j \to +\infty$. □

The following example illustrates the above result.

EXAMPLE 2.2. Daubechies wavelets form an orthonormal basis through a multiresolution analysis.

Let ψ be the Daubechies wavelet with N vanishing moments that lead to an orthonormal basis of $L^2(\mathbb{R})$. It has a basic support equal to $[-N+1,N]$ and the support of the corresponding scaling function ϕ is $[0,2N+1]$.

For the Daubechies wavelet ψ of order N, we have

$$\int_{\mathbb{R}} x^n\ \psi(x)\ dx = 0, \quad n = 0,1,...,N-1.$$

We consider a smooth signal $f \in L^2(\mathbb{R})$ which is twice continuously differentiable and $f^{(2)} \in L^\infty(\mathbb{R})$ and take ψ to be Daubechies wavelet of order $N = 2$ supported on $[-1,2]$. Now using moment formula for Hilbert transform, we have

$$\int_{\mathbb{R}} x^n \mathscr{H}\{\psi(x)\}\ dx = 0, \quad n = 0,1,2.$$

Using these moments we approximate the wavelet coefficients for smooth signal f and proceeding as in Theorem 2.1, we conclude that

$$|\langle f, \mathscr{H}\psi_{j,k}\rangle| \leqslant C \cdot 2^{\frac{-5j}{2}}, \text{ where } C \text{ is a constant.} \quad \square$$

In the following result, uniform Hölder continuity of a function is used to obtain sufficient conditions that result in decreasing the wavelet coefficients of a function.

THEOREM 2.3. *Let* $f \in L^2(\mathbb{R})$ *is a Hölder continuous function with exponent* β*,* $0 < \beta < 1$ *and let* $\psi \in L^2(\mathbb{R})$ *be a wavelet such that*

$$x\psi(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \tag{2.7}$$

$$\int_{\mathbb{R}} x^p \psi(x)\, dx = 0, \ \ for \ \ p = 0, 1. \tag{2.8}$$

Then $|\langle f, \mathscr{H}\psi_{j,k}\rangle| \leqslant C\, 2^{-j(\beta+\frac{1}{2})}$.

Proof. Note that

$$\begin{aligned}\langle f, \mathscr{H}\psi_{j,k}\rangle &= 2^{\frac{j}{2}} \int_{\mathbb{R}} f(x)\overline{\mathscr{H}\psi(2^j x - k)}\, dx \\ &= 2^{\frac{j}{2}} \int_{\mathbb{R}} [f(x) - f(2^{-j}k)]\mathscr{H}\overline{\psi(2^j x - k)}\, dx.\end{aligned}$$

Since f is a Hölder continuous function with exponent β, we have

$$|\langle f, \mathscr{H}\psi_{j,k}\rangle| \leqslant 2^{\frac{j}{2}}\, C' \int_{\mathbb{R}} |x - 2^{-j}k|^{\beta} |\mathscr{H}\psi(2^j x - k)|\, dx.$$

Write $2^j x - k = u$, then

$$|\langle f, \mathscr{H}\psi_{j,k}\rangle| \leqslant C'\, 2^{\frac{j}{2}}\, 2^{-j\beta}\, 2^{-j} \int_{\mathbb{R}} |u^{\beta}\ \mathscr{H}\psi(u)|\, du.$$

Now using (2.7), (2.8) and moment formula for the Hilbert transform, we obtain

$$|\langle f, \mathscr{H}\psi_{j,k}\rangle| \leqslant C\, 2^{-j(\beta+\frac{1}{2})}. \quad \square$$

The next result is a general version of above result.

THEOREM 2.4. *Let* $f \in L^2(\mathbb{R})$ *be n-times continuously differentiable function such that* $f^{(n)}$ *is Hölder continuous with exponent* β *for* $0 < \beta < 1$ *and let* $\psi \in L^2(\mathbb{R})$ *be a wavelet satisfying the following conditions*

$$x^{n+1}\psi(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \tag{2.9}$$

$$\int_{\mathbb{R}} x^p \psi(x)\, dx = 0, \ \ for \ \ p = 0, 1, 2, ..., n+1. \tag{2.10}$$

Then $|\langle f, \mathscr{H}\psi_{j,k}\rangle| \leqslant C\, 2^{-j(n+\beta+\frac{1}{2})}$.

Proof. By hypothesis, we may write $f(x) = P_{b,n-1}(x) + R_{n-1}(x)$, where $R_{n-1}(x) = \frac{1}{(n-1)!}\int_b^x (x-t)^{n-1}\, f^{(n)}(t)\, dt$ and $P_{b,n-1}(x) = \sum_{j=0}^{n-1} \frac{(x-b)^j}{j!}\, f^{(j)}(b)$ is a polynomial of degree $(n-1)$.

This gives

$$|f(x) - P_{b,n}(x)| \leqslant \frac{1}{(n-1)!} \int_b^x |x-b|^{n-1} \, |f^{(n)}(t) - f^{(n)}(b)| \, dt.$$

Since $f^{(n)}$ is Hölder continuous with exponent β, there exists a constant C' and $0 < \beta < 1$ such that

$$|f^{(n)}(x) - f^{(n)}(b)| \leqslant C' \, |x-b|^{\beta}.$$

So, we have

$$|f(x) - P_{b,n}(x)| \leqslant C'' \, |x-b|^{n+\beta}, \quad \text{where} \quad C'' = \frac{C'}{(\beta+1)\,(n-1)!}.$$

Now $\langle f, \mathscr{H}\psi_{j,k}\rangle = \int_{\mathbb{R}} [f(x) - P_{b,n}(x)] \, 2^{\frac{j}{2}} \, \overline{\mathscr{H}\psi(2^j x - k)} \, dx.$
This gives $|\langle f, \mathscr{H}\psi_{j,k}\rangle| \leqslant \int_{\mathbb{R}} C'' \, |x-b|^{n+\beta} \, 2^{\frac{j}{2}} \, \overline{\mathscr{H}\psi(2^j x - k)} \, dx.$
Writing $b = 2^{-j}k$, we get

$$|\langle f, \mathscr{H}\psi_{j,k}\rangle| \leqslant C'' \, 2^{-j(n+\beta)} \, 2^{\frac{j}{2}} \int_{\mathbb{R}} |2^j x - k|^{n+\beta} \, \overline{\mathscr{H}\psi(2^j x - k)} \, dx.$$

If we write $2^j x - k = u$, we have

$$|\langle f, \mathscr{H}\psi_{j,k}\rangle| \leqslant C'' \, 2^{-j(n+\beta+\frac{1}{2})} \int_{\mathbb{R}} |u^{n+\beta} \, \mathscr{H}\psi(u)| \, du. \tag{2.11}$$

Using (2.9), (2.10) and moment formula for the Hilbert transform, (2.11) reduces to

$$|\langle f, \mathscr{H}\psi_{j,k}\rangle| \leqslant C \, 2^{-j(n+\beta+\frac{1}{2})}, \text{ where } C \text{ is a constant independent of } j. \quad \square$$

Recall from [7] that

A bounded function $f \in C^n(\mathbb{R})$ is said to have decay rate $m \in \mathbb{N}$ if there exists a constant C_m such that $|f^{(p)}(x)| \leqslant \frac{C_m}{1+|x|^m}$, $0 \leqslant p \leqslant n$, for all $x \in \mathbb{R}$.

In the following result, we obtain a relationship between the Hilbert transform of wavelets and dyadic scale differential operator in order to decrease the wavelet coefficients $\langle f, \mathscr{H}\psi_{j,k}\rangle$ and thereby to approximate the function $f \in C^n$ with bounded n^{th} order derivative.

THEOREM 2.5. *Let m and n be integers such that $m \geqslant n+2$. Suppose that a wavelet $\psi \in L^2(\mathbb{R})$ satisfies the conditions:*

(i) $x^n \psi(x) \in L^2(\mathbb{R})$,

(ii) ψ *has* $(n-1)$ *vanishing moments,*

(iii) $|\psi(t)| \leqslant \frac{C_m}{1+|t|^m}$ *and* $\mathscr{H}\psi$ *having decay rate* m.

Then there exists a bounded function μ such that $\mathscr{H}\mu$ is also bounded with decay rate m and satisfying

$$\mathscr{H}\psi(t) = (-i)^n\ \mathscr{H}\mu^{(n)}(t). \tag{2.12}$$

Moreover, if $f \in C^n$ with bounded n^{th} order derivative then

$$|\langle f, \mathscr{H}\psi_{j,k}\rangle| = O(2^{\frac{j(n+1)}{2}}).$$

Proof. Since $|\psi(t)| \leqslant \frac{C_m}{1+|t|^m}$ with $m \geqslant n+2$, we have

$$\int_{\mathbb{R}} |\psi(-\omega)|\ (1+|\omega|^{m-2})\ d\omega \leqslant \int_{\mathbb{R}} \frac{C_m}{1+|\omega|^m}(1+|\omega|^{m-2})\ d\omega < \infty.$$

Therefore, by Theorem 2.5 [7] $\hat{\psi} \in C^n(\mathbb{R})$ is bounded. Also by using the Fourier transform property $\widehat{t^n\psi(t)}(\omega) = i^n\widehat{\psi}^{(n)}(\omega)$, we have $\widehat{t^k\ \psi}(0) = 0$ for $0 \leqslant k < n$.

Thus, $\widehat{\psi}^{(k)}(0) = 0$ for $0 \leqslant k < n$. Hence, there exists a bounded function $\hat{\mu} \in C(\mathbb{R})$ such that $\hat{\psi}(\omega) = \omega^n\ \hat{\mu}(\omega)$. This gives

$$\widehat{\mathscr{H}\psi}(\omega) = \frac{1}{i^n}\ \mathscr{F}\left\{\frac{d^n}{dt^n}\mathscr{H}\mu(t)\right\}(\omega)$$

which yields $\mathscr{H}\psi(t) = (-i)^n\ \frac{d^n}{dt^n}\mathscr{H}\mu(t)$. This verifies (2.12).

Now, for $n = 1$, we have $\mathscr{H}\mu^{(1)}(t) = i\ \mathscr{H}\psi(t)$.
This gives $|\mathscr{H}\mu^{(1)}(t)| \leqslant \frac{C_m}{1+|t|^m}$.
Also, we have $\mathscr{H}\mu(t) = -i\ \int_t^\infty \mathscr{H}\psi(x)\ dx$.
For $t \geqslant 0$, we have $\int_t^\infty |\mathscr{H}\psi(x)|\ dx \leqslant C_m \int_t^\infty \frac{1}{1+x^m}dx$.
The function $t \mapsto \int_t^\infty \frac{1}{1+x^m}$ is continuous function which has finite value at $t = 0$. This means that this function is bounded for all $t \geqslant 0$. Therefore, we have

$$|\mathscr{H}\mu(t)| \leqslant C_m \int_t^\infty \frac{1}{1+x^m}dx \leqslant K'. \tag{2.13}$$

Similarly, we can obtain (2.13) for $t < 0$.

It is easy to verify that

$$|\mathscr{H}\mu^{(p)}(t)| \leqslant \frac{C_m}{1+|t|^m} \quad \text{for } 0 \leqslant p \leqslant n, \ \text{ for all } t \in \mathbb{R}.$$

Thus $\mathscr{H}\mu$ has decay rate m. Now

$$\begin{aligned}
\langle f, \mathscr{H}\psi_{j,k}\rangle &= \int_{\mathbb{R}} f(t)\overline{\mathscr{H}\psi_{j,k}(t)}\ dt \\
&= \int_{\mathbb{R}} f(t)\cdot 2^{\frac{-j}{2}}\overline{\mathscr{H}\psi(2^{-j}t-k)}\ dt. \\
&= \int_{\mathbb{R}} f(t)\ \overline{\mathscr{H}\psi'_{2^j}(2^jk-t)}\ dt, \ \text{ where } \mathscr{H}\psi'_{2^j}(t) = 2^{\frac{-j}{2}}\ \mathscr{H}\psi(-2^{-j}t) \\
&= f * \mathscr{H}\psi'_{2^j}(2^jk).
\end{aligned}$$

Now using (2.12), we have

$$\mathscr{H}\psi'_{2^j}(u) = i^n\, 2^{jn}\, \frac{d^n}{du^n}\mathscr{H}\mu'_{2^j}(u), \tag{2.14}$$

where $u = 2^j k$ and $\mathscr{H}\mu'_{2^j}(u) = 2^{\frac{-j}{2}}\mathscr{H}\mu(-2^{-j}u)$.
This gives $\langle f, \mathscr{H}\psi_{j,k}\rangle = i^n\, 2^{jn}\, \frac{d^n}{du^n}(f * \mathscr{H}\mu'_{2^j}(u))$.
Also we have $\int_{\mathbb{R}} t^n\, \mathscr{H}\psi(t)\, dt = i^n\, n!\int_{\mathbb{R}}\mathscr{H}\mu(t)\, dt$.

Now we find that $\mathscr{H}\psi$ has n vanishing moments which implies

$$\int_{\mathbb{R}} t^n\, \mathscr{H}\psi(t)\, dt \neq 0.$$

This gives $\int_{\mathbb{R}}\mathscr{H}\mu(t)\, dt \neq 0$.

Let $\int_{\mathbb{R}}\mathscr{H}\mu(t)\, dt = K$. Then, by using weak convergence of dirac delta function δ, we find that

$$\lim_{j\to-\infty}\frac{\mathscr{H}\mu'_{2^j}(u)}{2^{\frac{j}{2}}} = K\delta.$$

From (2.14), we have

$$\langle f, \mathscr{H}\psi_{j,k}\rangle = i^n\, 2^{jn}\left(\frac{d^n f}{dt^n} * \mathscr{H}\mu'_{2^j}(u)\right).$$

Thus,

$$\begin{aligned}\lim_{j\to-\infty}\frac{\langle f, \mathscr{H}\psi_{j,k}\rangle}{2^{j(n+\frac{1}{2})}} &= \lim_{j\to-\infty} i^n\, 2^{jn}\frac{\left(f^{(n)} * \mathscr{H}\mu'_{2^j}(u)\right)}{2^{j(n+\frac{1}{2})}}\\ &= i^n\left(f^{(n)}(u) * K\delta\right)\\ &= K\, i^n\, f^{(n)}(u).\end{aligned}$$

Since $f \in C^n$ with bounded n^{th} order derivative, we have

$$|\langle f, \mathscr{H}\psi_{j,k}\rangle| = O(2^{j(n+\frac{1}{2})}). \quad \square$$

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