Journal of Classical Analysis
Volume 7, Number 2 (2015), 99–112 doi:10.7153/jca-07-10

INTEGRAL REPRESENTATIONS OF PRODUCTS OF AIRY FUNCTIONS RELATED TO FRACTIONAL CALCULUS

LEILA MOSLEHI AND ALIREZA ANSARI

Abstract. In this paper, using the inverse Laplace transform of multi-valued functions we show new identities for the powers of Airy functions in terms of the Riemann-Liouville and Weyl fractional integrals of order α. In this sense, we get new integral representations for the special functions including trigonometric functions in terms of the M-Wright function and Hilbert transform. Also, we get the Hadamard fractional integrals of Airy functions in terms of the Widder potential and Mellin transforms of the Volterra functions.

1. Introduction

The Airy functions of first and second kinds are given as the two linear independent solutions of differential equation $y'' - xy = 0$, [5, 6, 30]

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(xt + \frac{t^3}{3}\right)dt, \tag{1.1}$$

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \left(e^{xt-\frac{t^3}{3}} + \sin\left(xt + \frac{t^3}{3}\right)\right)dt. \tag{1.2}$$

The problem of determining integral representations for the products of Airy functions was stated by Reid in his papers [22], [23], [24]. He used the associated differential equations of these products and applied the Laplace integrals (along with their asymptotic analysis) to show the integral representations for quadratic, cubic and quartic products of Airy functions in terms of the Bessel functions of first and second kinds J_ν and Y_ν

$$Ai^2(x) = \frac{1}{2\pi^{\frac{3}{2}}}\int_0^\infty t^{-\frac{1}{2}}\cos\left(xt + \frac{t^3}{12} + \frac{\pi}{4}\right)dt, \tag{1.3}$$

$$Ai^3(x) = \frac{1}{(18\pi^3)^{\frac{1}{2}}}\left(\int_0^\infty t^{\frac{1}{2}}J_{-\frac{1}{6}}\left(\frac{4}{27}t^3\right)\cos\left(\frac{5}{27}t^3 + xt\right)dt - \int_0^\infty t^{\frac{1}{2}}J_{\frac{1}{6}}\left(\frac{4}{27}t^3\right)\sin\left(\frac{5}{27}t^3 + xt\right)dt\right), \tag{1.4}$$

$$Ai^4(x) = -\frac{1}{16\pi^2}\left(3\int_0^\infty J_0\left(\frac{1}{32}t^3\right)\sin\left(\frac{5}{96}t^3 + xt\right)dt + \int_0^\infty Y_0\left(\frac{1}{32}t^3\right)\cos\left(\frac{5}{96}t^3 + xt\right)dt\right), \tag{1.5}$$

Mathematics subject classification (2010): 44A10, 26A33, 33C10.
Keywords and phrases: Laplace transform, Airy function, Riemann-Liouville fractional integral, Bessel functions.

© Element, Zagreb
Paper JCA-07-10

$$Ai(x)Bi(x) = \frac{1}{2\pi^{\frac{3}{2}}} \int_0^\infty t^{-\frac{1}{2}} \sin\left(xt + \frac{t^3}{12} + \frac{\pi}{4}\right) dt, \tag{1.6}$$

$$\begin{aligned} Ai^2(x)Bi(x) = \frac{2}{(6\pi^3)^{\frac{1}{2}}} \bigg(& \int_0^\infty t^{\frac{1}{2}} J_{-\frac{1}{6}}\left(\frac{4}{27}t^3\right) \cos\left(\frac{5}{27}t^3 + xt\right) dt \\ & + \int_0^\infty t^{\frac{1}{2}} J_{\frac{1}{6}}\left(\frac{4}{27}t^3\right) \sin\left(\frac{5}{27}t^3 + xt\right) dt \bigg), \end{aligned} \tag{1.7}$$

$$Ai^3(x)Bi(x) = \frac{1}{8\pi^2} \int_0^\infty J_0\left(\frac{1}{32}t^3\right) \cos\left(\frac{5}{96}t^3 + xt\right) dt. \tag{1.8}$$

Later, using the integral representations for products of Airy functions, Varlamov obtained the Riesz fractional derivative of Airy functions and established some relations for conservation laws of KdV-type equations such as Ostrovsky equation [31, 32, 33, 34, 35]. See also [28], [29].

From another general point of view, in this paper we intend to motivate our investigation of new integral representations of products of Airy functions related to the fractional calculus. The theory of fractional calculus (integral and derivative of fractional orders) has been distinguished in different fields of applied mathematics and many investigations have been stated as modeling, existence of solutions and various methods for solving fractional differential equations [13], [17], [20], [25]. We concern with the Riemann-Liouville, Weyl and Hadamard fractional integrals of functions $Ai^n(x)$ and $Ai^n(x)Bi(x)$ in terms of the Bessel and Struve functions [7].

For this purpose, in Section 2 we state a fundamental theorem for the inverse Laplace transform of functions involving expression $(s^2+a^2)^\alpha$, $0<\alpha<1$, and show different integral representations for this inversion on interval $(0,\infty)$. These inversions are written in terms of the Fourier sine and cosine transforms to construct new integral representations for special functions including trigonometric functions.

In Section 3, we show the functions $Ai^n(x)$, $n=1,2,3,4$, and $Ai^n(x)Bi(x)$, $n=1,2,3$, in terms of the Abel integral equations (or the associated Riemann-Liouville and Weyl fractional integrals) and obtain the corresponding integral representations for these functions in terms of the Bessel functions. Also, we divide the functions $Ai^n(x)Bi(x)$ by $Ai^n(x)$ to get new identities for the ratio of Airy functions and the fractional powers of Airy functions. Moreover, for showing the flexibility of proposed method, we get the Riemann-Liouville and Weyl fractional integrals of the generalized Airy functions and Levy stable functions.

In Section 4, we extend our method to obtain other integral representations for Airy functions in terms of the M-Wright function and Hilbert transform.

In Section 5, as a complementary work on fractional calculus of Airy functions, we get the Hadamard fractional integrals of Airy functions. Finally, the main conclusions are set.

First, we recall the required definitions in fractional calculus.

DEFINITION 1. For $n-1<\alpha<n$, $n\in\mathbb{N}$ and $f\in L_1(a,b)$, the Riemann-Liouville fractional integral and derivative are defined as [13, 20]

$$(I_{a^+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) ds, \tag{1.9}$$

$$(D^{\alpha}_{a^+}f)(x) = D^n(I^{n-\alpha}_{a^+}f)(x), \tag{1.10}$$

where $D = \frac{d}{dx}$. In similar way, for $n-1<\alpha<n$, $n\in\mathbb{N}$ and $f\in L_1(\mathbb{R})$, the Weyl fractional integral and derivative are defined as

$$(I^{\alpha}_{+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty}(x-s)^{\alpha-1}f(s)ds, \tag{1.11}$$

$$(W^{\alpha}_{+}f)(x) = D^n(I^{n-\alpha}_{+}f)(x). \tag{1.12}$$

Also, the Hadamard fractional integral and fractional derivative of order α are defined as [13]

$$(\mathscr{J}^{\alpha}_{-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty}\frac{1}{s}\ln^{\alpha-1}\left(\frac{s}{x}\right)f(s)ds, \tag{1.13}$$

$$(\mathscr{D}^{\alpha}_{-}f)(x) = \big(-xD\big)^n(\mathscr{J}^{n-\alpha}_{-}f)(x). \tag{1.14}$$

2. Main theorem

THEOREM 1. *Let $F(s)$ be an analytic function for $\Re(s)>c$ and has two conjugate branch points $\pm ai$ on the imaginary axis. Furthermore, on the real negative semi-axis we have $F(re^{-i\pi}) = F(re^{i\pi})$, and the following properties are held for any sector $|arg(s)|<\pi-\eta$ where $0<\eta<\pi$,* [18]

$$F(s) = O(1), \qquad |s|\to\infty,$$
$$F(s) = O\left(\frac{1}{|s|}\right), \qquad |s|\to 0.$$

Then, the inverse Laplace transform of $F(s)$ can be written as two different integral representations in the forms

$$f(t) = \mathscr{L}^{-1}\{F(s);t\} = -\frac{2}{\pi}\int_a^{\infty}\sin(rt)\Im(F(re^{i\frac{\pi}{2}}))dr, \quad t>0, \tag{2.1}$$

$$f(t) = \mathscr{L}^{-1}\{F(s);t\} = \frac{2}{\pi}\int_0^{a}\cos(rt)\Re(F(re^{i\frac{\pi}{2}}))dr. \tag{2.2}$$

COROLLARY 1. *In view of the Hilbert transform*

$$\mathscr{H}\{f(x);y\} = \frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{f(x)}{x-y}dx, \tag{2.3}$$

of trigonometric functions, i.e., $\mathscr{H}\{\sin(ax);y\} = \cos(ay)$ and $\mathscr{H}\{\cos(ax);y\} = -\sin(ay)$, and using the relations (2.1) *and* (2.2)*, we get new integral representation as follows*

$$\mathscr{H}\{\text{sgn}(t)f(|t|);y\} = -\frac{2}{\pi}\int_a^{\infty}\cos(ry)\Im(F(re^{i\frac{\pi}{2}}))dr, \tag{2.4}$$

$$\mathscr{H}\{f(|t|);y\} = -\frac{2}{\pi}\int_0^{a}\sin(ry)\Re(F(re^{i\frac{\pi}{2}}))dr. \tag{2.5}$$

EXAMPLE 1. Using the fact that [21]

$$\mathscr{L}\{t^{\nu}J_{\nu}(at);s\}=\frac{(2a)^{\nu}\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}(s^2+a^2)^{\nu+\frac{1}{2}}},\quad \Re(\nu)>-\frac{1}{2}, \tag{2.6}$$

for the Bessel function of first kind and zero order, we have the following integral representations for $-\frac{1}{2}<\Re(\nu)<\frac{1}{2}$, [12]

$$J_{\nu}(at)=\frac{2a^{\nu}(\frac{t}{2})^{-\nu}}{\sqrt{\pi}\Gamma(\frac{1}{2}-\nu)}\int_a^{\infty}\frac{\sin(rt)}{(r^2-a^2)^{\nu+\frac{1}{2}}}dr,\quad t>0, \tag{2.7}$$

$$J_{\nu}(at)=\frac{2(\frac{t}{2a})^{\nu}}{\sqrt{\pi}\Gamma(\frac{1}{2}+\nu)}\int_0^{a}\cos(rt)(a^2-r^2)^{\nu-\frac{1}{2}}dr. \tag{2.8}$$

Also, using the relations (2.4) and (2.5), we have the following integral representations for the Neumann and Struve functions

$$Y_{\nu}(ay)=-\frac{2a^{\nu}(\frac{y}{2})^{-\nu}}{\sqrt{\pi}\Gamma(\frac{1}{2}-\nu)}\int_a^{\infty}\frac{\cos(ry)}{(r^2-a^2)^{\nu+\frac{1}{2}}}dr,\quad y>0, \tag{2.9}$$

$$\mathbf{H}_{\nu}(ay)=\frac{2(\frac{y}{2a})^{\nu}}{\sqrt{\pi}\Gamma(\frac{1}{2}+\nu)}\int_0^{a}\sin(ry)(a^2-r^2)^{\nu-\frac{1}{2}}dr, \tag{2.10}$$

where the Struve function of order ν is given by [1]

$$\mathbf{H}_{\nu}(z)=\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{z}{2})^{2k+\nu+1}}{\Gamma(k+\frac{3}{2})\Gamma(k+\nu+\frac{3}{2})}. \tag{2.11}$$

3. The Riemann-Liouville and Weyl fractional integrals of Airy functions

In this section, we intend to show new integral representations for the Airy functions $Ai^n(x)$ and $Ai^n(x)Bi(x)$ on interval $(0,\infty)$, or in general view, for the special functions which are expressed by the trigonometric functions. The main approach is constructed to the relations (2.1) and (2.2) and their Hilbert transforms (the relations (2.4) and (2.5)). The proofs of the stated propositions are similar and only we consider a proof of one case. Other proofs have the same procedures and are omitted.

3.1. Fractional integrals of $Ai^n(x)$, $n=1,2,3,4$

THEOREM 2. *For $0<\Re(\alpha)<1$ and $x>0$, the following Riemann-Liouville fractional integrals and derivatives hold for the powers of Airy functions*

$$f_n(\sqrt{x})=\frac{\Gamma(1-\alpha)}{2}I_0^{1-\alpha}\frac{Ai^n(\sqrt{x})}{\sqrt{x}},\quad Ai^n(\sqrt{x})=\frac{2\sqrt{x}}{\Gamma(1-\alpha)}D_0^{1-\alpha}f_n(\sqrt{x}),\quad n=1,2,3,4, \tag{3.1}$$

where

$$f_1(x) = \frac{\Gamma(1-\alpha)}{\sqrt{\pi}2^{\alpha+\frac{1}{2}}x^{\frac{1}{2}-\alpha}} \int_0^\infty \left(\xi^{\alpha-\frac{1}{2}} J_{\frac{1}{2}-\alpha}(\xi x)\cos\left(\frac{\xi^3}{3}\right) - \xi^{\alpha-\frac{1}{2}} \boldsymbol{H}_{\frac{1}{2}-\alpha}(\xi x)\sin\left(\frac{\xi^3}{3}\right)\right) d\xi, \tag{3.2}$$

$$f_2(x) = \frac{\Gamma(1-\alpha)}{\pi 2^{\alpha+2}x^{\frac{1}{2}-\alpha}} \int_0^\infty \xi^{\alpha-1} \left(J_{\frac{1}{2}-\alpha}(\xi x)\left(\cos\left(\frac{\xi^3}{12}\right) - \sin\left(\frac{\xi^3}{12}\right)\right) - \boldsymbol{H}_{\frac{1}{2}-\alpha}(\xi x)\left(\cos\left(\frac{\xi^3}{12}\right) + \sin\left(\frac{\xi^3}{12}\right)\right)\right) d\xi, \tag{3.3}$$

$$\begin{aligned} f_3(x) = {} & \frac{\Gamma(1-\alpha)}{3\sqrt{2}\pi 2^{\alpha+\frac{1}{2}}x^{\frac{1}{2}-\alpha}} \int_0^\infty \xi^{\alpha} \left(J_{\frac{1}{2}-\alpha}(\xi x) J_{-\frac{1}{6}}\left(\frac{4}{27}\xi^3\right)\cos\left(\frac{5\xi^3}{27}\right) \right. \\ & \left. -\boldsymbol{H}_{\frac{1}{2}-\alpha}(\xi x) J_{-\frac{1}{6}}\left(\frac{4}{27}\xi^3\right)\sin\left(\frac{5\xi^3}{27}\right)\right) d\xi \\ & - \frac{\Gamma(1-\alpha)}{3\sqrt{2}\pi 2^{\alpha+\frac{1}{2}}x^{\frac{1}{2}-\alpha}} \int_0^\infty \xi^{\alpha} \left(J_{\frac{1}{2}-\alpha}(\xi x) J_{\frac{1}{6}}\left(\frac{4}{27}\xi^3\right)\sin\left(\frac{5\xi^3}{27}\right) \right. \\ & \left. +\boldsymbol{H}_{\frac{1}{2}-\alpha}(\xi x) J_{\frac{1}{6}}\left(\frac{4}{27}\xi^3\right)\cos\left(\frac{5\xi^3}{27}\right)\right) d\xi, \end{aligned} \tag{3.4}$$

$$\begin{aligned} f_4(x) = {} & -\frac{\Gamma(1-\alpha)}{16\pi^{\frac{3}{2}} 2^{\alpha+\frac{1}{2}}x^{\frac{1}{2}-\alpha}} \int_0^\infty 3\xi^{\alpha-\frac{1}{2}} \left(J_{\frac{1}{2}-\alpha}(\xi x) J_0\left(\frac{1}{32}\xi^3\right)\sin\left(\frac{5\xi^3}{96}\right) \right. \\ & \left. +\boldsymbol{H}_{\frac{1}{2}-\alpha}(\xi x) J_0\left(\frac{1}{32}\xi^3\right)\cos\left(\frac{5\xi^3}{96}\right)\right) d\xi \\ & - \frac{\Gamma(1-\alpha)}{16\pi^{\frac{3}{2}} 2^{\alpha+\frac{1}{2}}x^{\frac{1}{2}-\alpha}} \int_0^\infty \xi^{\alpha-\frac{1}{2}} \left(J_{\frac{1}{2}-\alpha}(\xi x) Y_0\left(\frac{1}{32}\xi^3\right)\cos\left(\frac{5\xi^3}{96}\right) \right. \\ & \left. -\boldsymbol{H}_{\frac{1}{2}-\alpha}(\xi x) Y_0\left(\frac{1}{32}\xi^3\right)\sin\left(\frac{5\xi^3}{96}\right)\right) d\xi. \end{aligned} \tag{3.5}$$

Proof. We use the relations (2.8) and (2.10) to construct the Airy function $Ai(x)$ in terms of its addition formula in integral (1.1), i.e.,

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos(xt)\cos\left(\frac{t^3}{3}\right)dt - \frac{1}{\pi}\int_0^\infty \sin(xt)\sin\left(\frac{t^3}{3}\right)dt. \tag{3.6}$$

After a little algebra and setting $\alpha = \frac{1}{2} - \nu$, we get the following Abel integral equation

$$\begin{aligned} & \int_0^\infty \xi^{\alpha-\frac{1}{2}} \left(J_{\frac{1}{2}-\alpha}(\xi x)\cos\left(\frac{\xi^3}{3}\right) - \mathbf{H}_{\frac{1}{2}-\alpha}(\xi x)\sin\left(\frac{\xi^3}{3}\right)\right) d\xi \\ & \qquad = \frac{2^{\alpha+\frac{1}{2}}x^{\alpha-\frac{1}{2}}}{\sqrt{\pi}\Gamma(1-\alpha)} \int_0^x \frac{Ai(\xi)}{(\xi^2-x^2)^\alpha} d\xi, \end{aligned} \tag{3.7}$$

which by applying a suitable change of variables, we obtain the relation (3.1) for $n = 1$. In the same procedures, we can show other representations for $Ai^n(x)$, $n = 2,3,4$. □

THEOREM 3. *For $0<\Re(\alpha)<1$ and $x>0$, the following Weyl fractional integrals and derivatives hold for the powers of Airy functions*

$$g_n(\sqrt{x})=\frac{\Gamma(1-\alpha)}{2}I_{+}^{1-\alpha}\frac{Ai^n(\sqrt{x})}{\sqrt{x}},\quad Ai^n(\sqrt{x})=\frac{2\sqrt{x}}{\Gamma(1-\alpha)}W_{+}^{1-\alpha}g_n(\sqrt{x}),\quad n=1,2,3,4, \tag{3.8}$$

where $0<\Re(\alpha)<1$ and

$$g_1(x)=-\frac{\Gamma(1-\alpha)}{\sqrt{\pi}2^{\alpha+\frac{1}{2}}x^{\alpha-\frac{1}{2}}}\int_0^\infty\left(\xi^{\alpha-\frac{1}{2}}Y_{\alpha-\frac{1}{2}}(\xi x)\cos\left(\frac{\xi^3}{3}\right)\right.$$
$$\left.+\xi^{\alpha-\frac{1}{2}}J_{\alpha-\frac{1}{2}}(\xi x)\sin\left(\frac{\xi^3}{3}\right)\right)d\xi, \tag{3.9}$$

$$g_2(x)=\frac{\Gamma(1-\alpha)}{\pi 2^{\alpha+1}x^{\alpha-\frac{1}{2}}}\int_0^\infty\xi^{\alpha-1}\left(Y_{\alpha-\frac{1}{2}}(\xi x)\left(\sin\left(\frac{\xi^3}{12}\right)-\cos\left(\frac{\xi^3}{12}\right)\right)\right.$$
$$\left.-J_{\alpha-\frac{1}{2}}(\xi x)\left(\cos\left(\frac{\xi^3}{12}\right)+\sin\left(\frac{\xi^3}{12}\right)\right)\right)d\xi, \tag{3.10}$$

$$g_3(x)=-\frac{\Gamma(1-\alpha)}{3\pi 2^{\alpha+1}x^{\alpha-\frac{1}{2}}}\int_0^\infty\xi^{\alpha}\left(J_{\alpha-\frac{1}{2}}(\xi x)J_{-\frac{1}{6}}\left(\frac{4}{27}\xi^3\right)\sin\left(\frac{5\xi^3}{27}\right)\right.$$
$$\left.+Y_{\alpha-\frac{1}{2}}(\xi x)J_{-\frac{1}{6}}\left(\frac{4}{27}\xi^3\right)\cos\left(\frac{5\xi^3}{27}\right)\right)d\xi$$
$$-\frac{\Gamma(1-\alpha)}{3\pi 2^{\alpha+1}x^{\alpha-\frac{1}{2}}}\int_0^\infty\xi^{\alpha}\left(J_{\alpha-\frac{1}{2}}(\xi x)J_{\frac{1}{6}}\left(\frac{4}{27}\xi^3\right)\cos\left(\frac{5\xi^3}{27}\right)\right.$$
$$\left.-Y_{\alpha-\frac{1}{2}}(\xi x)J_{\frac{1}{6}}\left(\frac{4}{27}\xi^3\right)\sin\left(\frac{5\xi^3}{27}\right)\right)d\xi, \tag{3.11}$$

$$g_4(x)=-\frac{\Gamma(1-\alpha)}{16\pi^{\frac{3}{2}}2^{\alpha+\frac{1}{2}}x^{\alpha-\frac{1}{2}}}\int_0^\infty 3\xi^{\alpha-\frac{1}{2}}\left(J_{\alpha-\frac{1}{2}}(\xi x)J_0\left(\frac{1}{32}\xi^3\right)\cos\left(\frac{5\xi^3}{96}\right)\right.$$
$$\left.-Y_{\alpha-\frac{1}{2}}(\xi x)J_0\left(\frac{1}{32}\xi^3\right)\sin\left(\frac{5\xi^3}{96}\right)\right)d\xi$$
$$+\frac{\Gamma(1-\alpha)}{16\pi^{\frac{3}{2}}2^{\alpha+\frac{1}{2}}x^{\alpha-\frac{1}{2}}}\int_0^\infty\xi^{\alpha-\frac{1}{2}}\left(J_{\alpha-\frac{1}{2}}(\xi x)Y_0v\left(\frac{1}{32}\xi^3\right)\sin\left(\frac{5\xi^3}{96}\right)\right.$$
$$\left.+Y_{\alpha-\frac{1}{2}}(\xi x)Y_0\left(\frac{1}{32}\xi^3\right)\cos\left(\frac{5\xi^3}{96}\right)\right)d\xi. \tag{3.12}$$

3.2. Fractional integrals of $Ai^n(x)Bi(x)$, $n=1,2,3$

THEOREM 4. *For $0<\Re(\alpha)<1$ and $x>0$, the following Riemann-Liouville fractional integrals and derivatives hold for the products of Airy functions*

$$p_n(\sqrt{x})=\frac{\Gamma(1-\alpha)}{2}I_0^{1-\alpha}\frac{Ai^n(\sqrt{x})Bi(\sqrt{x})}{\sqrt{x}},\quad Ai^n(\sqrt{x})Bi(\sqrt{x})=\frac{2\sqrt{x}}{\Gamma(1-\alpha)}D_0^{1-\alpha}p_n(\sqrt{x}),$$
$$n=1,2,3, \tag{3.13}$$

where

$$p_1(x) = \frac{\Gamma(1-\alpha)}{2\pi 2^{\alpha+\frac{1}{2}} x^{\frac{1}{2}-\alpha}} \int_0^\infty \xi^{\alpha-1} \Big(J_{\frac{1}{2}-\alpha}(\xi x)\Big(\cos\Big(\frac{\xi^3}{12}\Big)+\sin\Big(\frac{\xi^3}{12}\Big)\Big)$$
$$+\boldsymbol{H}_{\frac{1}{2}-\alpha}(\xi x)\Big(\cos\Big(\frac{\xi^3}{12}\Big)-\sin\Big(\frac{\xi^3}{12}\Big)\Big)\Big)\, d\xi, \tag{3.14}$$

$$p_2(x) = \frac{2\Gamma(1-\alpha)}{\sqrt{6}\pi 2^{\alpha+\frac{1}{2}} x^{\frac{1}{2}-\alpha}} \int_0^\infty \xi^{\alpha} \Big(J_{\frac{1}{2}-\alpha}(\xi x) J_{-\frac{1}{6}}\Big(\frac{4}{27}\xi^3\Big)\cos\Big(\frac{5\xi^3}{27}\Big)$$
$$-\boldsymbol{H}_{\frac{1}{2}-\alpha}(\xi x) J_{-\frac{1}{6}}\Big(\frac{4}{27}\xi^3\Big)\sin\Big(\frac{5\xi^3}{27}\Big)\Big)\, d\xi$$
$$+\frac{2\Gamma(1-\alpha)}{\sqrt{6}\pi 2^{\alpha+\frac{1}{2}} x^{\frac{1}{2}-\alpha}} \int_0^\infty \xi^{\alpha} \Big(J_{\frac{1}{2}-\alpha}(\xi x) J_{\frac{1}{6}}\Big(\frac{4}{27}\xi^3\Big)\sin\Big(\frac{5\xi^3}{27}\Big)$$
$$+\boldsymbol{H}_{\frac{1}{2}-\alpha}(\xi x) J_{\frac{1}{6}}\Big(\frac{4}{27}\xi^3\Big)\cos\Big(\frac{5\xi^3}{27}\Big)\Big)\, d\xi, \tag{3.15}$$

$$p_3(x) = \frac{\Gamma(1-\alpha)}{8\pi^{\frac{3}{2}} 2^{\alpha+\frac{1}{2}} x^{\frac{1}{2}-\alpha}} \int_0^\infty \xi^{\alpha-\frac{1}{2}} \Big(J_{\frac{1}{2}-\alpha}(\xi x) J_0\Big(\frac{1}{32}\xi^3\Big)\cos\Big(\frac{5\xi^3}{96}\Big)$$
$$+\boldsymbol{H}_{\frac{1}{2}-\alpha}(\xi x) J_0\Big(\frac{1}{32}\xi^3\Big)\sin\Big(\frac{5\xi^3}{96}\Big)\Big)\, d\xi. \tag{3.16}$$

THEOREM 5. *For $0<\Re(\alpha)<1$ and $x>0$, the following Weyl fractional integral and derivative hold for the products of Airy functions*

$$q_n(\sqrt{x}) = \frac{\Gamma(1-\alpha)}{2} I_+^{1-\alpha} \frac{Ai^n(\sqrt{x})Bi(\sqrt{x})}{\sqrt{x}}, \quad Ai^n(\sqrt{x})Bi(\sqrt{x}) = \frac{2\sqrt{x}}{\Gamma(1-\alpha)} W_+^{1-\alpha} q_n(\sqrt{x}),$$
$$n=1,2,3, \tag{3.17}$$

where

$$q_1(x) = -\frac{\Gamma(1-\alpha)}{\pi 2^{\alpha+2} x^{\alpha-\frac{1}{2}}} \int_0^\infty \xi^{\alpha-1} \Big(Y_{\alpha-\frac{1}{2}}(\xi x)\Big(\sin\Big(\frac{\xi^3}{12}\Big)+\cos\Big(\frac{\xi^3}{12}\Big)\Big)$$
$$-J_{\alpha-\frac{1}{2}}(\xi x)\Big(\cos\Big(\frac{\xi^3}{12}\Big)-\sin\Big(\frac{\xi^3}{12}\Big)\Big)\Big)\, d\xi, \tag{3.18}$$

$$q_2(x) = -\frac{2\Gamma(1-\alpha)}{\sqrt{6}\pi 2^{\alpha+\frac{1}{2}} x^{\alpha-\frac{1}{2}}} \int_0^\infty \xi^{\alpha} \Big(J_{\alpha-\frac{1}{2}}(\xi x) J_{-\frac{1}{6}}\Big(\frac{4}{27}\xi^3\Big)\sin\Big(\frac{5\xi^3}{27}\Big)$$
$$+Y_{\alpha-\frac{1}{2}}(\xi x) J_{-\frac{1}{6}}\Big(\frac{4}{27}\xi^3\Big)\cos\Big(\frac{5\xi^3}{27}\Big)\Big)\, d\xi$$
$$+\frac{2\Gamma(1-\alpha)}{\sqrt{6}\pi 2^{\alpha+\frac{1}{2}} x^{\alpha-\frac{1}{2}}} \int_0^\infty \xi^{\alpha} \Big(J_{\alpha-\frac{1}{2}}(\xi x) J_{\frac{1}{6}}\Big(\frac{4}{27}\xi^3\Big)\cos\Big(\frac{5\xi^3}{27}\Big)$$
$$-Y_{\alpha-\frac{1}{2}}(\xi x) J_{\frac{1}{6}}\Big(\frac{4}{27}\xi^3\Big)\sin\Big(\frac{5\xi^3}{27}\Big)\Big)\, d\xi, \tag{3.19}$$

$$q_3(x) = -\frac{\Gamma(1-\alpha)}{8\pi^{\frac{3}{2}}2^{\alpha+\frac{1}{2}}x^{\alpha-\frac{1}{2}}}\int_0^{\infty}\xi^{\alpha-\frac{1}{2}}\left(J_{\alpha-\frac{1}{2}}(\xi x)J_0\left(\frac{1}{32}\xi^3\right)\sin\left(\frac{5\xi^3}{96}\right)\right.$$
$$\left.-Y_{\alpha-\frac{1}{2}}(\xi x)J_0\left(\frac{1}{32}\xi^3\right)\cos\left(\frac{5\xi^3}{96}\right)\right)d\xi. \tag{3.20}$$

3.3. Identities for $Ai^{\gamma}(x)$ and $Bi^{\gamma}(x)$, $\gamma > 0$

THEOREM 6. *For $\gamma > 0$, the following identities hold for the fractional powers of Airy functions and their ratios*

$$Ai^{\gamma}(\sqrt{x}) = \left(\frac{D_0^{1-\alpha}f_{n+1}(\sqrt{x})}{D_0^{1-\alpha}f_n(\sqrt{x})}\right)^{\gamma}, \quad Ai^{\gamma}(\sqrt{x}) = \left(\frac{W_+^{1-\alpha}g_{n+1}(\sqrt{x})}{W_+^{1-\alpha}g_n(\sqrt{x})}\right)^{\gamma}, \quad n = 1,2,3, \tag{3.21}$$

$$Bi^{\gamma}(\sqrt{x}) = \left(\frac{D_0^{1-\alpha}p_n(\sqrt{x})}{D_0^{1-\alpha}f_n(\sqrt{x})}\right)^{\gamma}, \quad Bi^{\gamma}(\sqrt{x}) = \left(\frac{W_+^{1-\alpha}q_n(\sqrt{x})}{W_+^{1-\alpha}g_n(\sqrt{x})}\right)^{\gamma}, \quad n = 1,2,3, \tag{3.22}$$

$$\frac{Bi^{\gamma}(\sqrt{x})}{Ai^{\gamma}(\sqrt{x})} = \left(\frac{D_0^{1-\alpha}p_n(\sqrt{x})}{D_0^{1-\alpha}f_{n+1}(\sqrt{x})}\right)^{\gamma}, \quad \frac{Bi^{\gamma}(\sqrt{x})}{Ai^{\gamma}(\sqrt{x})} = \left(\frac{W_+^{1-\alpha}q_n(\sqrt{x})}{W_+^{1-\alpha}g_{n+1}(\sqrt{x})}\right)^{\gamma}, \quad n = 2,3. \tag{3.23}$$

3.4. Fractional integrals of generalized Airy and Levy stable functions

In this section, we intend to obtain the Riemann-Liouville and Weyl fractional integrals of functions which can be written in terms of the trigonometric functions. Here, we choose the generalized Airy functions

$$\mathscr{A}_{2m+1}(x) = \frac{1}{\pi}\int_0^{\infty}\cos\left(tx + (-1)^{m+1}\frac{t^{2m+1}}{2m+1}\right)dt, \quad m = 1,2,\cdots, \tag{3.24}$$

and Levy stable functions

$$L_m(x) = \frac{1}{\pi}\int_0^{\infty}\cos(tx)e^{-t^{2m}}dt, \quad m = 1,2,\cdots, \tag{3.25}$$

which are considered as the fundamental solutions of the generalized heat equation $\frac{\partial u}{\partial t} = k_m\frac{\partial^m u}{\partial x^m}$, $k_m = \pm 1$, with initial condition $u(x,0) = \delta(x)$. For more details in properties of these functions, see [4], [11], [19] and references therein.

THEOREM 7. *For $0 < \Re(\alpha) < 1$ and $x > 0$, the following Riemann-Liouville and Weyl fractional integrals and derivatives hold for the generalized Airy functions*

$$f_m(\sqrt{x}) = \frac{\Gamma(1-\alpha)}{2}I_0^{1-\alpha}\frac{\mathscr{A}_{2m+1}(\sqrt{x})}{\sqrt{x}}, \quad \mathscr{A}_{2m+1}(\sqrt{x}) = \frac{2\sqrt{x}}{\Gamma(1-\alpha)}D_0^{1-\alpha}f_m(\sqrt{x}),$$
$$m = 1,2,\cdots, \tag{3.26}$$

$$g_m(\sqrt{x}) = \frac{\Gamma(1-\alpha)}{2} I_+^{1-\alpha} \frac{\mathscr{A}_{2m+1}(\sqrt{x})}{\sqrt{x}}, \quad \mathscr{A}_{2m+1}(\sqrt{x}) = \frac{2\sqrt{x}}{\Gamma(1-\alpha)} W_+^{1-\alpha} g_m(\sqrt{x}),$$
$$m = 1,2,\cdots, \tag{3.27}$$

where

$$f_m(x) = \frac{\Gamma(1-\alpha)}{\sqrt{\pi}2^{\alpha+\frac{1}{2}} x^{\frac{1}{2}-\alpha}} \int_0^\infty \xi^{\alpha-\frac{1}{2}} \Big(\boldsymbol{H}_{\frac{1}{2}-\alpha}(\xi x) \sin\Big((-1)^{m+1} \frac{\xi^{2m+1}}{2m+1} \Big)$$
$$-J_{\frac{1}{2}-\alpha}(\xi x) \cos\Big((-1)^{m+1} \frac{\xi^{2m+1}}{2m+1} \Big) \Big) d\xi, \tag{3.28}$$
$$g_m(x) = -\frac{\Gamma(1-\alpha)}{\sqrt{\pi}2^{\alpha+\frac{1}{2}} x^{\alpha-\frac{1}{2}}} \int_0^\infty \xi^{\alpha-\frac{1}{2}} \Big(Y_{\alpha-\frac{1}{2}}(\xi x) \cos\Big((-1)^{m+1} \frac{\xi^{2m+1}}{2m+1} \Big)$$
$$+J_{\alpha-\frac{1}{2}}(\xi x) \sin\Big((-1)^{m+1} \frac{\xi^{2m+1}}{2m+1} \Big) \Big) d\xi. \tag{3.29}$$

THEOREM 8. *For $0 < \Re(\alpha) < 1$ and $x > 0$, the following Riemann-Liouville and Weyl fractional integrals and derivatives hold for the Levy stable functions*

$$f_m(\sqrt{x}) = \frac{\Gamma(1-\alpha)}{2} I_0^{1-\alpha} \frac{L_m(\sqrt{x})}{\sqrt{x}}, \quad L_m(\sqrt{x}) = \frac{2\sqrt{x}}{\Gamma(1-\alpha)} D_0^{1-\alpha} f_m(\sqrt{x}), \quad m = 1,2,\cdots, \tag{3.30}$$
$$g_m(\sqrt{x}) = \frac{\Gamma(1-\alpha)}{2} I_+^{1-\alpha} \frac{L_m(\sqrt{x})}{\sqrt{x}}, \quad L_m(\sqrt{x}) = \frac{2\sqrt{x}}{\Gamma(1-\alpha)} W_+^{1-\alpha} g_m(\sqrt{x}), \quad m = 1,2,\cdots, \tag{3.31}$$

where

$$f_m(x) = \frac{\Gamma(1-\alpha)}{\sqrt{\pi}2^{\alpha+\frac{1}{2}} x^{\frac{1}{2}-\alpha}} \int_0^\infty e^{-\xi^{2m}} \xi^{\alpha-\frac{1}{2}} J_{\frac{1}{2}-\alpha}(\xi x) d\xi, \tag{3.32}$$

$$g_m(x) = -\frac{\Gamma(1-\alpha)}{\sqrt{\pi}2^{\alpha+\frac{1}{2}} x^{\alpha-\frac{1}{2}}} \int_0^\infty e^{-\xi^{2m}} \xi^{\alpha-\frac{1}{2}} Y_{\alpha-\frac{1}{2}}(\xi x) d\xi. \tag{3.33}$$

4. Other integral representations

In this section, we extend the previous results to obtain general classes of integral representations for the combinations of Airy functions. First, we state a theorem for the inverse Laplace transform of functions $F((s^2+a^2)^\alpha)$, $0 < \alpha < 1$, using the inverse Laplace transform of $F((s^2+a^2)^{\frac{1}{2}})$.

LEMMA 1. *Let $\mathscr{L}^{-1}\{F(s);x\} = f(x)$, then, the inverse Laplace transform of $F((s^2+a^2)^{\frac{1}{2}})$ is given by*

$$\mathscr{L}^{-1}\{F((s^2+a^2)^{\frac{1}{2}});x\} = f(x) - a\int_0^x u \frac{J_1(a\sqrt{x^2-u^2})}{\sqrt{x^2-u^2}} f(u) du, \quad \Re(s) > |\Im(a)|. \tag{4.1}$$

Proof. We begin with the well-known Sonine integral for $|\arg(z)| < \frac{\pi}{2}$, [12]

$$\int_0^\infty x^{\nu+1} J_\nu(ax) \frac{K_\mu(s\sqrt{x^2+z^2})}{(x^2+z^2)^{\frac{\mu}{2}}} dx = \left(\frac{a}{s}\right)^\nu \frac{(s^2+a^2)^{\frac{\mu-\nu-1}{2}}}{z^{\mu-\nu-1}} K_{\mu-\nu-1}(z\sqrt{s^2+a^2}),$$
$$a,s>0,\ \Re(\nu) > -1, \tag{4.2}$$

which in special case $\nu = 0, \mu = \frac{1}{2}$ is reduced to

$$\int_0^\infty xJ_0(ax) \frac{e^{-s\sqrt{x^2+z^2}}}{(x^2+z^2)^{\frac{1}{2}}} dx = \frac{e^{-z\sqrt{s^2+a^2}}}{(s^2+a^2)^{\frac{1}{2}}}. \tag{4.3}$$

At this point, using the suitable change of variables and the Leibniz integral rule we get

$$e^{-sz} - a\int_z^\infty ze^{-su} \frac{J_1(a\sqrt{u^2-z^2})}{\sqrt{u^2-z^2}} du = e^{-z\sqrt{s^2+a^2}}. \tag{4.4}$$

Now, by the definition of $F((s^2+a^2)^{\frac{1}{2}})$ in terms of the Laplace transform

$$F((s^2+a^2)^{\frac{1}{2}}) = \int_0^\infty e^{-z\sqrt{s^2+a^2}} f(z) dz, \tag{4.5}$$

and replacing the exponential function by relation (4.4), we obtain the result.

At this point, using the well-known M-Wright function $M_\alpha(x)$ in fractional calculus [16]

$$M_\alpha(x) = \sum_{k=0}^\infty \frac{(-x)^k}{k!\Gamma(-\alpha k + 1 - \alpha)}, \tag{4.6}$$

we intend to get the inverse Laplace transform of function $F((s^2+a^2)^{\frac{\alpha}{2}})$, $0 < \alpha < 1$. This function is a fundamental function in the theory of fractional diffusion equation [14, 15], and in the theory of the coupled Langevin equations in presence of one-sided stable Levy noises [8, 9, 26]. □

THEOREM 9. *Let* $\mathscr{L}^{-1}\{F(s);x\} = f(x)$, *then, the inverse Laplace transform of* $F((s^2+a^2)^{\frac{\alpha}{2}})$, $0 < \alpha < 1$, *is given by*

$$f^*(x) = \mathscr{L}^{-1}\{F((s^2+a^2)^{\frac{\alpha}{2}});x\} \tag{4.7}$$
$$= \frac{\alpha}{x^{\alpha+1}} \int_0^\infty uf(u) \left(M_\alpha(ux^{-\alpha}) - a\int_u^\infty \tau M_\alpha(\tau x^{-\alpha}) \frac{J_1(a\sqrt{\tau^2-u^2})}{\sqrt{\tau^2-u^2}} d\tau \right) du.$$

Proof. Using the Schouten-Vanderpol theorem for the Laplace transform [10]

$$g(x) = \mathscr{L}^{-1}\{F(\phi(s));x\} = \int_0^\infty f(\tau)\mathscr{L}^{-1}\{e^{-\tau\phi(s)};x\}\, d\tau, \tag{4.8}$$

in terms of the M-Wright function (case $\phi(s) = s^\alpha$)

$$\frac{\alpha\tau}{x^{\alpha+1}} M_\alpha(\tau x^{-\alpha}) = \mathscr{L}^{-1}\{e^{-\tau s^\alpha};x\}, \tag{4.9}$$

and combining with the relation (4.1), we get the result. □

THEOREM 10. *The following integral representations hold for the Airy function*

$$\int_0^\infty \left(\mathscr{H}\{\mathrm{sgn}(x)f^*(|x|);t\}\cos\left(\frac{t^3}{3}\right) - f^*(t)\sin\left(\frac{t^3}{3}\right)\right)dt \tag{4.10}$$
$$= -\frac{2}{\pi}\int_a^\infty Ai(r)\Im(F(i(r^2-a^2))^{\frac{\alpha}{2}})dr,$$

$$\int_0^\infty \left(\mathscr{H}\{f^*(|x|);t\}\sin\left(\frac{t^3}{3}\right) + f^*(t)\cos\left(\frac{t^3}{3}\right)\right)dt = \frac{2}{\pi}\int_0^a Ai(r)\Re(F((a^2-r^2)^{\frac{\alpha}{2}}))dr, \tag{4.11}$$

where $f^*(t)$ *is given by the relation* (4.7).

5. The Hadamard fractional integrals of Airy functions

In this section, we intend to get the Hadamard fractional integrals of Airy functions. For this purpose, we recall the definitions of Volterra function ν [2, 3]

$$\nu(z,\alpha) = \int_0^\infty \frac{z^{u+\alpha}}{\Gamma(u+\alpha+1)}du, \tag{5.1}$$

and Widder potential and Mellin transforms

$$\mathscr{P}\{f(x);y\} = \int_0^\infty \frac{xf(x)}{x^2+y^2}dx, \tag{5.2}$$

$$\mathscr{M}\{f(x);y\} = \int_0^\infty x^{y-1}f(x)dx. \tag{5.3}$$

The Volterra function (5.1) can be applied in representation of the mean square displacement of a free particle which dynamics is governed by the generalized Langevin equation with distributed order noise [27]. In order to avoid much computations, we mention that some Hadamard fractional integrals of Airy functions are shown. Other Hadamard fractional integrals of products of Airy functions, can be obtained in similar manner.

THEOREM 11. *For* $0 < \Re(\alpha) < 1$ *and* $x > 0$, *the following Hadamard fractional integrals hold for the Airy functions, where the notations* $\mathscr{P}$ *and* $\mathscr{M}$ *are the Widder potential and Mellin transforms, respectively*

$$\begin{aligned}
&\mathscr{J}_-^\alpha Ai(x)\\
&= \frac{1}{\pi\Gamma(\alpha)\Gamma(-\alpha)}\int_0^\infty \cos\left(\frac{\xi^3}{3}\right)\cos(x\xi)\mathscr{P}[\mathscr{M}(\nu(s,y)e^{-s}, y\to-\alpha), s\to x^2\xi^2]d\xi\\
&\quad -\frac{x}{\pi\Gamma(\alpha)\Gamma(-\alpha)}\int_0^\infty \xi\cos\left(\frac{\xi^3}{3}\right)\sin(x\xi)\mathscr{P}\left[\mathscr{M}\left(\nu(s,y)\frac{e^{-s}}{s},-\alpha\right),x^2\xi^2\right]d\xi\\
&\quad -\frac{1}{\pi\Gamma(\alpha)\Gamma(-\alpha)}\int_0^\infty \sin\left(\frac{\xi^3}{3}\right)\sin(x\xi)\mathscr{P}[\mathscr{M}(\nu(s,y)e^{-s}d\xi,-\alpha),x^2\xi^2]d\xi\\
&\quad -\frac{x}{\pi\Gamma(\alpha)\Gamma(-\alpha)}\int_0^\infty \xi\sin\left(\frac{\xi^3}{3}\right)\cos(x\xi)\mathscr{P}\left[\mathscr{M}\left(\nu(s,y)\frac{e^{-s}}{s}d\xi,-\alpha\right),x^2\xi^2\right]d\xi,
\end{aligned} \tag{5.4}$$

$$
\begin{aligned}
&\mathscr{J}_{-}^{\alpha} Ai^2(x)\\
&= \frac{\sqrt{2}}{4\pi^{\frac{3}{2}}\Gamma(\alpha)\Gamma(-\alpha)} \int_0^\infty \xi^{\frac{-1}{2}} \cos\left(\frac{\xi^3}{12}\right) \cos(x\xi) \mathscr{P}[\mathscr{M}(\nu(s,y)e^{-s},-\alpha),x^2\xi^2] d\xi\\
&\quad - \frac{\sqrt{2}x}{4\pi^{\frac{3}{2}}\Gamma(\alpha)\Gamma(-\alpha)} \int_0^\infty \xi^{\frac{1}{2}} \cos\left(\frac{\xi^3}{12}\right) \sin(x\xi) \mathscr{P}\left[\mathscr{M}\left(\nu(s,y)\frac{e^{-s}}{s}d\xi,-\alpha\right),x^2\xi^2\right] d\xi\\
&\quad - \frac{\sqrt{2}}{4\pi^{\frac{3}{2}}\Gamma(\alpha)\Gamma(-\alpha)} \int_0^\infty \xi^{\frac{-1}{2}} \cos\left(\frac{\xi^3}{12}\right) \sin(x\xi) \mathscr{P}[\mathscr{M}(\nu(s,y)e^{-s},-\alpha),x^2\xi^2] d\xi\\
&\quad - \frac{\sqrt{2}x}{4\pi^{\frac{3}{2}}\Gamma(\alpha)\Gamma(-\alpha)} \int_0^\infty \xi^{\frac{1}{2}} \cos\left(\frac{\xi^3}{12}\right) \cos(x\xi) \mathscr{P}\left[\mathscr{M}\left(\nu(s,y)\frac{e^{-s}}{s},-\alpha\right),x^2\xi^2\right] d\xi\\
&\quad - \frac{\sqrt{2}}{4\pi^{\frac{3}{2}}\Gamma(\alpha)\Gamma(-\alpha)} \int_0^\infty \xi^{\frac{-1}{2}} \sin\left(\frac{\xi^3}{12}\right) \sin(x\xi) \mathscr{P}[\mathscr{M}(\nu(s,y)e^{-s},-\alpha),x^2\xi^2] d\xi\\
&\quad - \frac{\sqrt{2}x}{4\pi^{\frac{3}{2}}\Gamma(\alpha)\Gamma(-\alpha)} \int_0^\infty \xi^{\frac{1}{2}} \sin\left(\frac{\xi^3}{12}\right) \cos(x\xi) \mathscr{P}\left[\mathscr{M}\left(\nu(s,y)\frac{e^{-s}}{s},-\alpha\right),x^2\xi^2\right] d\xi\\
&\quad - \frac{\sqrt{2}}{4\pi^{\frac{3}{2}}\Gamma(\alpha)\Gamma(-\alpha)} \int_0^\infty \xi^{\frac{-1}{2}} \sin\left(\frac{\xi^3}{12}\right) \cos(x\xi) \mathscr{P}[\mathscr{M}(\nu(s,y)e^{-s},-\alpha),x^2\xi^2] d\xi\\
&\quad + \frac{\sqrt{2}x}{4\pi^{\frac{3}{2}}\Gamma(\alpha)\Gamma(-\alpha)} \int_0^\infty \xi^{\frac{1}{2}} \sin\left(\frac{\xi^3}{12}\right) \sin(x\xi) \mathscr{P}\left[\mathscr{M}\left(\nu(s,y)\frac{e^{-s}}{s},-\alpha\right),x^2\xi^2\right] d\xi.
\end{aligned}
\tag{5.5}
$$

Proof. Using the following integral representation for logarithmic function in terms of the Volterra function (5.1), [2, page 162]

$$
\int_0^\infty \int_0^\infty e^{-px} y^{q-1} \nu(x,y) dx dy = \frac{\Gamma(q)}{p \ln^{q+1}(p)}, \quad \Re(p) > -1,\ \Re(q) > -1, \tag{5.6}
$$

and applying the Hadamard fractional integral (1.13) to the Airy function

$$
\mathscr{J}_{-}^{\alpha} Ai(x) = \frac{1}{\pi\Gamma(\alpha)} \int_x^\infty \ln^{\alpha-1}\left(\frac{t}{x}\right) \int_0^\infty \cos\left(t\xi + \frac{\xi^3}{3}\right) d\xi \frac{dt}{t}, \tag{5.7}
$$

we can get the relation (5.4). In this sense, we used the definitions of the Widder potential and Mellin transforms (5.2) and (5.3), and the following relations for simplification

$$
\int_x^\infty e^{-ct} \cos(t\xi) dt = e^{-cx} \frac{c\cos(x\xi) - \xi \sin(x\xi)}{\xi^2 + c^2}, \quad c > 0, \tag{5.8}
$$

$$
\int_x^\infty e^{-ct} \sin(t\xi) dt = e^{-cx} \frac{c\sin(x\xi) + \xi \cos(x\xi)}{\xi^2 + c^2}, \quad c > 0. \tag{5.9}
$$

In the same procedure, we can obtain the relation (5.5). □

6. Concluding remarks

This paper provides new results in the inverse Laplace transform of multi-valued functions involving two conjugate branch points. These results enabled us to construct the Riemann-Liouville and Weyl fractional integrals of Airy functions and their products. Using the proposed method, we obtained different representations for the inversion of a suitable multi-valued function. Applying these inversions, we could show many integral representations and identities for functions including the trigonometric functions.

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics, 9th printing, Dover, New York, 1972.

[2] A. APELBLAT, *Integral Transforms and Volterra Functions*, Nova Science Publishers, New York, 2010.

[3] A. APELBLAT, *Volterra Functions*, Nova Science Publishers, New York, 2008.

[4] A. ANSARI AND H. ASKARI, *On fractional calculus of $\mathscr{A}_{2n+1}(x)$ function*, Applied Mathematics and Computation, **232** (2014), 487–497.

[5] A. ANSARI AND M. R. MASOMI, *Some results involving integral transforms of Airy functions*, Integral Transforms and Special Functions, **26** (7) (2015), 539–549.

[6] A. ANSARI *The fractional airy transform*, Hacettepe Journal of Mathematics and Statistics, **44** (4) (2015), 761–766.

[7] N. L. BALAZS, H. C. PAULI AND O. B. DABBOUSI, *Tables of Weyl fractional integrals for the Airy function*, Mathematics of Computation, **33** (145) (1979), 353–358.

[8] A. BAULE AND R. FRIEDRICH, *Joint probability distributions for a class of non- Markovian processes*, Physical Review E., **71**, 026101 (2005), 321–331.

[9] A. V. CHECHKIN, M. HOFMANN AND I. M. SOKOLOV, *Continuous-time random walk with correlated waiting times*, Physical Review E., **80**, 031112 (2009).

[10] D. G. DUFFY, *Transform Methods for Solving Partial Differential Equations*, National Bureau of Standards, CRC Press, New York, 2004.

[11] K. GÓRSKA, A. HORZELA, K. A. PENSON AND G. DATTOLI, *The higher-order heat-type equations via signed Lévy stable and generalized Airy functions*, Journal of Physics A: Mathematical and Theoretical, **46**, (2013), 16 pages.

[12] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series, and Products*, Elsevier Academic Press, Seventh Edition, New York, 2007.

[13] A. A. KILBAS, H. M. SRIVASTAVA AND J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, Elsevier Amsterdam, 2006.

[14] J. KLAFTER, S. C. LIM AND R. METZLER, *Fractional Dynamics: Recent Advances*, World Scientific, Singapore, 2012.

[15] F. MAINARDI, G. PAGNINI AND R. K. SAXENA, *Fox H-function in fractional diffusion*, Messenger Math., **178**, (1–2) (2005), 321–331.

[16] F. MAINARDI, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press., London, 2010.

[17] K. S. MILLER AND B. ROSS, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, NY, USA, 1993.

[18] L. MOSLEHI AND A. ANSARI, *Some remarks on inverse Laplace transforms involving conjugate branch points with applications*, Scientific Bulletin, Series A, Applied Mathematics and Physics, (2015), To appear.

[19] E. ORSINGHER AND M. D'OVIDIO, *Probabilistic representation of fundamental solutions to $\frac{\partial u}{\partial t} = k_m \frac{\partial^m u}{\partial x^m}$*, Electronic Communications in Probability, **17** (2012), 1–12.

[20] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[21] A. P. PRUDNIKOV, Y. A. BRYCHKOV AND O. I. MARICHEV, *Integrals and Series, Direct Laplace Transforms*, Gordon and Breach, Vol 4, Amsterdam, 1992.

[22] W. H. REID, *Integral representations for products of Airy functions*, Zeitschrift fur angewandte Mathematik und Physik, **46**, (1995), 159–170.

[23] W. H. REID, *Integral representations for products of Airy functions Part 2: Cubic products*, Zeitschrift fur angewandte Mathematik und Physik, **48**, (1997), 646–655.

[24] W. H. REID, *Integral representations for products of Airy functions, Part 3: Quartic products*, Zeitschrift fur angewandte Mathematik und Physik, **48**, (1997), 656–664.

[25] S. G. SAMKO, A. A. KILBAS AND O. I. MARICHEV, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, Switzerland, 1993.

[26] T. SANDEV, A. V. CHECHKIN, N. KORABEL, H. KANTZ, I. M. SOKOLOV AND R. METZLER, *Distributed-order diffusion equations and multifractality: models and solutions*, Physical Review E., **92**, 042117 (2015).

[27] T. SANDEV AND Z. TOMOVSKI, *Langevin equation for a free particle driven by power law type of noises*, Physics Letters A., **378** (2014), 1–9.

[28] C. SCHUBERTA AND V. VARLAMOV, *Integrals containing products of Airy functions and planar photon propagation in a magnetic field*, Mathematical Methods in the Applied Sciences, **34**, (2011), 1638–1648.

[29] N. M. TEMME AND V. VARLAMOV, *Asymptotic expansions for Riesz fractional derivatives of Airy functions and applications*, Journal of Computational and Applied Mathematics, **232**, (2009), 201–215.

[30] O. VALLEE AND M. SOARES, *Airy Functions and Applications to Physics*, Imperial College Press, London, 2004.

[31] V. VARLAMOV, *Semi-integer derivatives of the Airy functions and related properties of the Korteweg-de Vries-type equations*, Zeitschrift fur angewandte Mathematik und Physik, **59**, (2008), 381–399.

[32] V. VARLAMOV, *Fractional derivatives of products of Airy functions*, Journal of Mathematical Analysis and Applications, **337**, (2008), 667–685.

[33] V. VARLAMOV, *Integrals involving products of Airy functions, their derivatives and Bessel functions*, Journal of Mathematical Analysis and Applications, **370**, (2010), 687–702.

[34] V. VARLAMOV, *Differential and integral relations involving fractional derivatives of Airy functions and applications*, Journal of Mathematical Analysis and Applications, **348**, (2008), 101–115.

[35] V. VARLAMOV, *Riesz fractional derivatives of the product of Airy transforms*, Physica Scripta, T136 (2009) 014004 (5pp).

(Received August 16, 2015)

Leila Moslehi
Department of Applied Mathematics
Faculty of Mathematical Sciences
Shahrekord University
P. O. Box 115, Shahrekord, Iran
e-mail: leilamoslehi@yahoo.com

Alireza Ansari
Department of Applied Mathematics
Faculty of Mathematical Sciences
Shahrekord University
P. O. Box 115, Shahrekord, Iran
e-mail: alireza_1038@yahoo.com